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On Cantor's singular moments

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Abstract

We evaluate a constant explicitly, thereby answering a question raised in [1].

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In problem **10621** *of the American Mathematical Monthly, Cantor's singular moments* J_n *were to be computed. In the published answers* [1] *they come out as*

$$J_n = \frac{2}{3(n+1)} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{3 \cdot 2^{j-1} - 1} \quad \textit{for} \quad n \geq 1$$

and $J_0 = 1$, *with Bernoulli numbers* B_n.

The editor asked, whether it is possible to compute

$$J_{-1} = \sum_{n \geq 0} J_n$$

exactly.

The purpose of this note is to do that. In [5] *we considered a similar problem, and the gentle reader is invited to consult this paper for more background about the technique, as well as* [6, 4] *for more information about the Cantor distribution.*

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Following the method described in [3], *we can write* J_n *as a contour integral viz.*

$$J_n = \frac{2}{3(n+1)} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+2)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1}\, ds$$
$$= \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+2-s)} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1}\, ds.$$

Therefore

$$\sum_{n=0}^{N} J_n = J_0 + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(N+2)\Gamma(1-s)}{\Gamma(N+2-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1}\, ds$$
$$- \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(2)\Gamma(1-s)}{\Gamma(2-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1}\, ds.$$

From this form, one could even compute the asympotics as $N \to \infty$*. However, here, we only have to note that the first integral is of order* $N^{1-\log_2 3}$*, which means that it goes to zero. Consequently*

$$\sum_{n \geq 0} J_n = 1 - \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{(1-s)s} \frac{\zeta(1-s)}{3 \cdot 2^{s-1} - 1}\, ds$$
$$= \frac{4}{3} + \frac{2}{3} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \frac{\zeta(s)}{3 \cdot 2^{-s} - 1}\, ds$$
$$= \frac{4}{3} + \frac{2}{3} \sum_{k,m \geq 1} 3^{-k} \cdot \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} \Big(\frac{2^k}{m}\Big)^s\, ds.$$

The last step was by using the Dirichlet series for $\zeta(s)$ *and the geometric series, both valid for* $\Re s = \frac{3}{2}$*. A simple application of residue calculus, as it is often used in the context of the Mellin–Perron summation formula (see* [7, 2]*) evaluates the integrals inside the summation:*

$$\frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{s(s-1)} t^s\, ds = t - 1 for\ t \geq 1 0 for\ t < 1\ .$$

Therefore

$$\sum_{n \geq 0} J_n = 1 + \frac{2}{3} \sum_{k \geq 1} \sum_{1 \leq m \leq 2^k} 3^{-k} \Big(\frac{2^k}{m} - 1\Big)$$
$$= 1 + \frac{2}{3} \sum_{k \geq 1} \Big(\frac{2}{3}\Big)^k H_{2^k} - \frac{2}{3} \sum_{k \geq 1} \Big(\frac{2}{3}\Big)^k$$
$$= -\frac{1}{3} + \frac{2}{3} \sum_{k \geq 1} \Big(\frac{2}{3}\Big)^k H_{2^k} = 3.36465\ 07281\ 00925\ 16083\ 89349\ 6289\ldots,$$

with harmonic numbers $H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$.

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