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AN EXAMPLE FOR MAPPINGS RELATED TO SEMI-CONFLUENCE

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Abstract. A semi-confluence of a mapping between topological spaces can be defined in several ways. J.J. Charatonik asked if two definitions of the semi-confluence, using components and quasi-components respectively, are equivalent for surjective mappings with compact point inverses. We give a negative answer to this question in Example 2.1.

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1. Introduction.

We recall from [1] the notation. All *mappings* considered in this paper are continuous. A mapping $f : X \to Y$ between metric continua X and Y is called *semi-confluent* provided for each subcontinuum Q of Y and for any two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

A general topological space admits several definitions of semi-confluent mappings. We recall from [1] three definitions using connectedness.

A topological space X is said to be *connected between two its subsets A and B* provided there is no closed and open subset in X that contains A and is disjoint from B. Clearly, connectedness of a space X between points is an equivalence relation on X. The equivalence classes of this relation are called *quasi-components.* In other words, a *quasi-component* of a space X containing a point $p \in X$ is the intersection of all closed and open subsets of X containing p.

A semi-confluence of a mapping $f : X \to Y$ between topological spaces X and Y can be defined (see [1]) by the following conditions :

(1) (SC1) For each connected closed nonempty subset Q of Y and for any two *components* C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

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(2) (SC2) For each connected closed nonempty subset Q of Y and for any two *quasi-components* C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

(3) (SC3) For each connected closed nonempty subset Q of Y and points x_1 and x_2 in $f^{-1}(Q)$ the set $f^{-1}(Q)$ is either *connected between* $\{x_1\}$ *and* $f^{-1}(f(x_2))$ or *connected between* $\{x_2\}$ and $f^{-1}(f(x_1))$.

J.J. Charatonik proved in [1], p.90 the following results.

Theorem 1.1. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (SC1) $\Rightarrow$ (SC2) $\Rightarrow$ (SC3).

(b) *conditions* (SC1) , (SC2) *and* (SC3) *are equivalent for compact Hausdorff spaces X and Y.*

(b) *each condition* (SC1) , (SC2) *and* (SC3) *is equivalent to semi-confluence for continua X and Y.*

J.J. Charatonik asked in [1], Question 3.4, the following question.

Question 1.3. *Assume a mapping $f : X \to Y$ between topological spaces X and Y has compact point inverses. Does* (SC2) *imply* (SC1) *?*

We give a negative answer in Example 2.1.

2. Counterexample.

The following example is a simple modification of Example 2.1 in [2].

Example 2.1. *There exist topological spaces X and Y and a surjective $f : X \to Y$ with compact point inverses such that* (SC2) *is satisfied and* (SC1) *is not satisfied.*

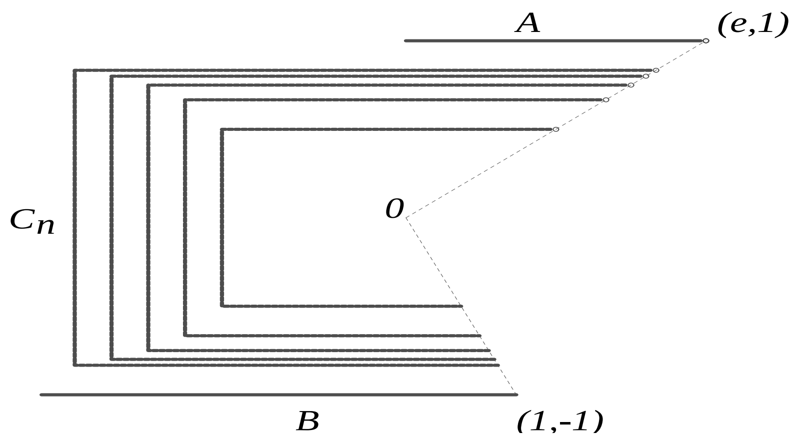

FIGURE 1 (EXAMPLE 2.1)

Proof. Let

$$X = A \cup B \cup \bigcup_{n=1}^{\infty} C_n \quad , \quad Y = A \cup \{(e,1)\} \quad ,$$

where $A = \{(x,1) \in \mathbb{R}^2 : 0 \le x < e\}$, $B = \{(x,-1) \in \mathbb{R}^2 : x \le 1\}$ and

$$\begin{aligned} C_n = &\{(x,y) \in \mathbb{R}^2 : y = 1 - 1/n, -n \le x < e(1-1/n)\} \\ &\cup\{(x,y) \in \mathbb{R}^2 : y = -1 + 1/n, -n \le x \le 1 - 1/n\} \\ &\cup\{(x,y) \in \mathbb{R}^2 : |y| \le 1 - 1/n, x = -n\} \end{aligned}$$

as in Figure 1. (X is a simple modification of the *nested rectangles* topological space in [3], p.137.) Let both X and Y inherit the topology from the Euclidean plane.

For the definition of the mapping $f : X \to Y$ we need some notation:

For any $(x,1) \in Y$ denote by a_x the plane segment joining $(x,1)$ with the origin and by b_x the plane segment joining $(\log x, -1) \in B$ with the origin.

If $0 \le x < e$ denote $x_n^A = a_x \cap C_n$ and $x_0^A = a_x \cap A$ (we have $x_0^A = (x,1)$). For convenience let $A_n = a_0 \cap C_n$.

If $0 < x \le e$ denote $x_0^B = b_x \cap B$. If $1/n \le x \le e$ denote $x_n^B = b_x \cap C_n$. For convenience let $B_n = b_{1/n} \cap C_n$.

For any two points $S, T \in C_n$ denote by ST the arc in C_n joining S with T and by $d(S,T)$ the linear measure of the arc ST.

If $0 < x < 1/n$ denote by x_n^B the point of A_nB_n such that $d(A_n, x_n^B)/d(A_n, B_n) = nx$.

Now we define $f : X \to Y$ by the formula $f(x_j^A) = f(x_j^B) = (x,1) \in Y$ for $j \ge 0$. See Figure 2.

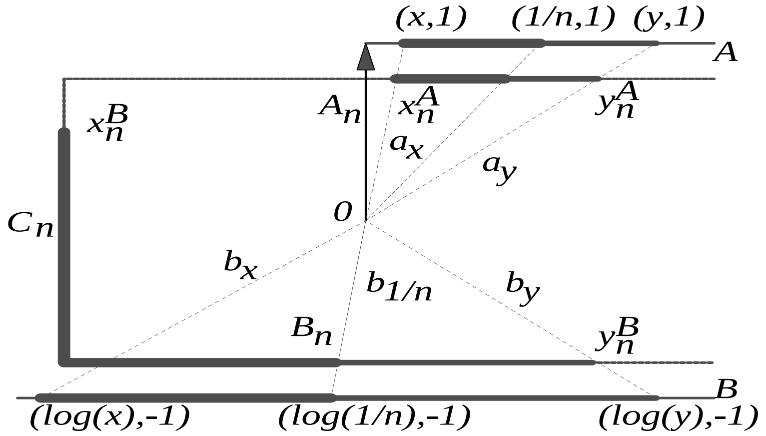

FIGURE 2 (PROOF OF EXAMPLE 2.1)

We claim the following properties of f :

(i) f *is continuous on* X.

(Proof. X is a subset of the space X in Example 2.1 in [2] and f is defined by the same way. ♡)

(ii) f *has compact point inverses.*

(Proof. Clearly the point inverse of $(e,1) \in Y$ is the sequence $\{e_j^B\}_0^\infty$ and all points e_j^B belong to the segment b_e. The sequence $\{e_j^B\}_0^\infty$ is converging to $e_0^B = (1,-1) \in B$. Hence the point inverse set of $(e,1) \in Y$ is compact in X. For all remaining points see [2]. ♡)

(iii) *For $f : X \to Y$ the condition* (SC2) *holds.*

(Proof. Let Q be a closed nonempty connected subset of Y. We have 4 cases :

Case (1) : When $(0,1) \notin Q$ and $(e,1) \notin Q$.

Then $f^{-1}(Q) \cap C_n$ is the set consisting of just two disjoint arcwise connected sets $Q_n^A = \{x_n^A : (x,1) \in Q\}$ and $Q_n^B = \{x_n^B : (x,1) \in Q\}$, both of them being closed and open in $f^{-1}(Q)$. Hence both Q_n^A and Q_n^B are quasi-components of $f^{-1}(Q)$ both being mapped onto Q under f.

Similarly, $f^{-1}(Q) \cap A$ is the set $Q^A = \{x_0^A : (x,1) \in Q\} = Q$. Clearly, Q^A is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^A in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^A is mapped onto Q under f.

Similarly, $f^{-1}(Q) \cap B$ is the set $Q^B = \{x_0^B : (x,1) \in Q\}$. Clearly, Q^B is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^B in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^B is mapped onto Q under f. See Figure 2.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds.

Case (2) : When $(0,1) \in Q$ and $(e,1) \notin Q$.

The set $f^{-1}(Q) \cap C_n$ contains all arcs joining x_n^A with x_n^B in C_n for any $(x,1) \in Q$. It is again a quasi-component of $f^{-1}(Q)$ and is mapped onto Q under f.

The set $Q^A \cup Q^B$ is a quasi-component of $f^{-1}(Q)$ because no closed and open set of $f^{-1}(Q)$ containing the arcwise connected set Q^A can be disjoint from the arcwise connected set Q^B due to the fact that all arcs joining x_n^A with x_n^B in C_n for any $(x,1) \in Q$ are in $f^{-1}(Q)$, see Figure 1 and Figure 2. Moreover, the set $Q^A \cup Q^B$ is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds.

Case (3) : When $(0,1) \notin Q$ and $(e,1) \in Q$.

Again $f^{-1}(Q) \cap C_n$ is the set consisting of just two disjoint arcwise connected sets $Q_n^A = \{x_n^A : (x,1) \in Q\}$ and $Q_n^B = \{x_n^B : (x,1) \in Q\}$, both of them being closed and open in $f^{-1}(Q)$. Hence both Q_n^A and Q_n^B are quasi-components of $f^{-1}(Q)$. We see that Q_n^A is mapped onto $Q \setminus \{(e,1)\}$ and Q_n^B is mapped onto Q.

Clearly, Q^A is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^A in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^A is mapped onto $Q \setminus \{(e,1)\}$ under f.

Clearly, Q^B is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^B in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover, Q^B is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped either onto Q or onto $Q \setminus \{(e,1)\}$, hence their images differ at most at one point and (SC2) holds.

Case (4) : When $(0,1) \in Q$ and $(e,1) \in Q$.

We have $Q = Y$, $C_n = f^{-1}(Q) \cap C_n$, $f^{-1}(Q) \cap A = A$ and $f^{-1}(Q) \cap B = B$ in this case.

The set C_n is again a quasi-component of $f^{-1}(Q)$ and is mapped onto Q under f.

The set $A \cup B$ is a quasi-component of $f^{-1}(Q)$ because no closed and open set of $f^{-1}(Q)$ containing the arcwise connected set A can be disjoint from the arcwise connected set B due to the fact that all arcs joining x_n^A with x_n^B in C_n for any

$(x,1) \in Q$ are in $f^{-1}(Q)$, see Figure 2 and Figure 1. Moreover, the set $A \cup B$ is mapped onto Q under f.

All quasi-components of $f^{-1}(Q)$ are mapped onto Q, hence their images are equal and (SC2) holds. ♡)

(iv) *For* $f : X \to Y$ *the condition* (SC1) does not hold.

(Proof. If $Q = Y$, then the set $Q^B = B$ is a component of $f^{-1}(Q)$ because it is arcwise connected and no larger subset of $f^{-1}(Q)$ containing Q^B is connected. We see that $(0,1) \notin f(Q^B)$ and $f(Q^B) = Y \setminus \{(0,1)\}$. Similarly $Q^A = A$ is a component of $f^{-1}(Q)$ because it is arcwise connected and no larger subset of $f^{-1}(Q)$ containing Q^A is connected. We see that $(e,1) \notin f(Q^A)$ and $f(Q^A) = Y \setminus \{(e,1)\}$. Now we have two components Q^A and Q^B of $f^{-1}(Q)$ such that neither $f(Q^A) \subset f(Q^B)$ nor $f(Q^B) \subset f(Q^A)$. The condition (SC1) does not hold. See Figure 1. ♡)

The proof is finished. □

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