Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 July 2000, pp. 16 – 21
Submitted: March 22, 1999. Published: July 15, 2000

Fixed Point Theorem by Altering Distance between the Points

M. El Amrani and A. B. Mbarki

Abstract

In this note we establish a fixed point theorem in complete metric spaces which generalizes some results in [2].

Subject Classification Codes: 47H10
Key Words: Fixed Points, Complete Metric Spaces, Generalized Contractive Conditions

1 Introduction

Delbosco [1] and Skof [3] have established fixed point theorems for self-maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i) $\phi : [0, \infty) \to [0, \infty)$ is continuous and stricly increasing in $\Re^+$;
(ii)$\phi(t) = 0$ iff $t = 0$;
(iii)$\phi(t) \geq Mt^u$ for every $t > 0, M, u > 0$ are constants.

Department of Mathematics, Faculty of Science, Oujda University, Morocco
E-mail: elamrani@sciences.univ-oujda.ac.ma & mbarki@sciences.univ-oujda.ac.ma

In 1977, F. skof [3] gave the following theorem

Theorem 1.1 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying the conditions (i), (ii) and (iii). Suppose that T satisfies the following condition:*

$$(A.1)\ \phi(d(Tx, Ty)) \leq a\phi(d(x, y)) + b\phi(d(x, Tx)) + c\phi(d(y, Ty)),$$

where $x, y \in X$, a, b and c are three nonnegative constants satisfying $a + b + c < 1$. Then T has a unique fixed point.

In [1], the author has considered functions ϕ satisfying the conditions (i),(ii) and (iii) such that $\phi(t) = t^{n+1}$, $n \in N$ for every $t \geq 0$. Clearly, $\phi \circ d$ is not in general, a metric.

In 1984, M. S. Khan et al. [2] generalized Theorem 1.1 by using a much extensive condition than (A. 1) and removed the hypothesis (iii) which seems superfluous. They proved the following theorem:

Theorem 1.2 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b, c be three decreasing functions from $(0, \infty)$ into $[0, 1)$ such that*

$$(B.1)\quad a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$.

Suppose that T satisfies the following condition:

$$\begin{aligned}(B.2)\ \phi(d(Tx, Ty)) &\leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))[\phi(d(x, Tx)) + \phi(d(y, Ty))] \\ &+ c(d(x, y))min\{\phi(d(y, Tx)), \phi(d(x, Ty))\}.\end{aligned}$$

Where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In this note we shall extend this last theorem by using a much extensive hypothesis than (B. 1) and a, b and c are three decreasing functions.

2 Main result

We shall make frequent use of the following definition:

Definition 2.1 *A function ψ will be said to have property (P) if $\psi : (0, \infty) \to [0, 1)$ and if for every $t > 0$ there exists $\eta(t) > 0$ such that $0 \leq s - t < \eta(t)$ implies $\psi(s) \leq \gamma(t) < 1$. (that is, $\limsup_{s \to t^+} \psi(s) < 1$).*

We establish the following:

Theorem 2.1 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b, c be three functions from $(0, \infty)$ into $[0, 1)$ such that $a + 2b$ and $a + c$ have both property (P). Suppose that T satisfies the condition (B. 2). Then T has a unique fixed point.*

Proof. Let x be some point in X, define

$$x_n = T^n x, \ n = 0, 1, 2, ...$$

Let $\tau_n = d(x_n, x_{n+1})$. We have for $n \geq 1$ (We may assume that $\tau_n > 0$)

$$\phi(\tau_n) = \phi(d(x_n, x_{n+1})) = \phi(d(T^n x, T^{n+1} x)) \leq A_1 + A_2 + A_3$$

Where

$$\begin{aligned}
A_1 &= a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)), \\
A_2 &= b(d(x_{n-1}, x_n))[\phi(d(x_{n-1}, x_n)) + \phi(d(x_n, x_{n+1}))], \\
A_3 &= c(d(x_{n-1}, x_n))min\{\phi(d(x_{n-1}, x_{n+1})), \phi(d(x_n, x_n))\}.
\end{aligned}$$

So

$$\phi(\tau_n) \leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)]$$

Hence we have

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \ (1)$$

Since ϕ is increasing, $\{\tau_n\}$ is a decreasing sequence. Let $lim_n \tau_n = \tau$, assume that $\tau > 0$. Since $a + 2b$ has property (P), then for $0 \leq s - \tau < \eta(\tau)$ we have $a(s) + 2b(s) \leq \gamma(\tau) < 1$, which implies

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \frac{\gamma(\tau) - b(s)}{1 - b(s)}.$$

It follows that for these s

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \gamma(\tau). \ (2)$$

Since

$$\tau_n \geq \tau, \quad \lim_n \tau_n = \tau \quad \text{and} \quad \lim_n \frac{\phi(\tau_n)}{\phi(\tau_{n-1})} = 1,$$

there exists an integer $p \geq 1$ such that

$$0 \leq \tau_{p-1} - \tau < \eta(\tau) \quad (3) \quad \text{while} \quad \gamma(\tau) < \frac{\phi(\tau_p)}{\phi(\tau_{p-1})},$$

the last inequality with (1) imply that

$$\gamma(\tau) < \frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})}$$

and (2) with (3) imply that

$$\frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})} \leq \gamma(\tau),$$

which is a cantradiction. Consequently $\tau = 0$.
If $\{x_n\}$ is not cauchy sequence, there exists $\varepsilon > 0$, and two sequences of natural numbers $\{m(i)\}, \{n(i)\}, n(i) > m(i), m(i) \to \infty$ as $i \to \infty$ such that

$$d(x_{n(i)}, x_{m(i)}) > \varepsilon \text{ while } \ d(x_{n(i)-1}, x_{m(i)}) \leq \varepsilon.$$

Then we have

$$\begin{aligned} \varepsilon < d_i = d(x_{n(i)}, x_{m(i)}) &\leq d(x_{n(i)}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{m(i)}) \\ &\leq \tau_{n(i)-1} + \varepsilon, \end{aligned}$$

since $\{\tau_n\}$ converges to 0, $d_i \to \varepsilon$. Futhermore, by triangular inequality, it follows that

$$d_i - \tau_{n(i)} - \tau_{m(i)} \leq d(x_{n(i)+1}, x_{m(i)+1}) \leq d_i + \tau_{n(i)} + \tau_{m(i)}$$

and therefore the sequence $\{d(x_{n(i)+1}, x_{m(i)+1})\}$ converges to ε
From (B. 2), we also deduce:

$$\begin{aligned} &\phi(d(x_{n(i)+1}, x_{m(i)+1})) = \phi(d(T^{n(i)+1}x, T^{m(i)+1}x)) \\ &\quad \leq C_1 + C_2 + C_3 \\ &\quad \leq C_4 + C_5 + C_6 \quad (4) \end{aligned}$$

Where

$$\begin{aligned}
C_1 &= a(d_i)\phi(d_i),\\
C_2 &= b(d_i)[\phi(\tau_{n(i)}) + \phi(\tau_{m(i)}],\\
C_3 &= c(d_i)min\{\phi(d(x_{n(i)}, x_{m(i)+1}), \phi(d(x_{n(i)+1}, x_{m(i)})\},\\
C_4 &= a(d_i)\phi(d_i + \tau_{n(i)}),\\
C_5 &= [\phi(\tau_{n(i)}) + \phi(\tau_{m(i)})],\\
C_6 &= c(d_i)\phi(d_i + \tau_{n(i)}).
\end{aligned}$$

Since $d_i > \varepsilon$, $d_i \to \varepsilon$ and $a+c$ has property (P), there exists an integer $q \geq 1$ such that

$$0 \leq d_i - \varepsilon < \eta(\varepsilon), \text{ implies } \ a(d_i) + c(d_i) \leq \gamma(\varepsilon) < 1$$

for all $i \geq q$

From the last inequality and (4) we have

$$\phi(d(x_{n(i)+1}, x_{m(i)+1}) \leq \gamma(\varepsilon)\phi(d_i + \tau_{n(i)}) + \phi(\tau_{n(i)}) + \phi(\tau_{m(i)}),$$

for all $i \geq q$

As $i \to \infty$, we obtain:

$$\phi(\varepsilon) \leq \gamma(\varepsilon)\phi(\varepsilon) < \phi(\varepsilon).$$

This is a contradiction. Hence $\{x_n\}$ is a cauchy sequence in a complete metric space X, then there exists a point $\tilde{x} \in X$ such that $x_n \to \tilde{x}$ as $n \to \infty$. Now we shall show that $T\tilde{x} = \tilde{x}$.

¿From (B.2), we have :

$$\begin{aligned}
&\phi(d(x_n, T\tilde{x})) = \phi(d(T^n x, T\tilde{x}))\\
&\quad \leq a(d(x_{n-1}, \tilde{x}))\phi((d(x_{n-1}, \tilde{x}))\\
&\quad +b((d(x_{n-1}, \tilde{x})))[\phi(d(x_{n-1}, x_n) + \phi(d(T\tilde{x}, \tilde{x}))]\\
&\quad +c((d(x_{n-1}, \tilde{x})))min\{\phi((d(x_{n-1}, T\tilde{x})), \phi(d(x_n, \tilde{x}))\} \ (5)
\end{aligned}$$

Using $b < \frac{1}{2}$ and letting $n \to \infty$ in (5), we have

$$\phi(d(\tilde{x}, T\tilde{x})) \leq \frac{1}{2}\phi(d(\tilde{x}, T\tilde{x})).$$

That is $\phi(d(\tilde{x}, T\tilde{x})) = 0$ and therefore $d(\tilde{x}, T\tilde{x}) = 0$, .i.e $\tilde{x} = T\tilde{x}$. We claim that $\tilde{x}$ is unique fixed point of T. For this, we suppose that y ($\tilde{x} \neq y$) is another fixed point of T . Then

$$\begin{aligned}
\phi(d(y, \tilde{x})) &= \phi(d(Ty, T\tilde{x})) \\
&\leq a(d(\tilde{x}, y))\phi(d(\tilde{x}, y)) + b(d(\tilde{x}, y))[\phi(d(\tilde{x}, T\tilde{x})) + \phi(d(y, Ty))] \\
&+ c(d(\tilde{x}, y)) min\{\phi(d(\tilde{x}, Ty)), \phi(d(y, T\tilde{x}))\} \\
&\leq [a(d(\tilde{x}, y)) + c(d(\tilde{x}, y))]\phi(d(\tilde{x}, y)) < \phi(d(\tilde{x}, y)),
\end{aligned}$$

a cantradiction. This completes the proof of the Theorem.

We now present some consequences of Theorem 2.1

Corollary 2.1 *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and(ii). Furthermore,let a, b, c be three functions from (0 ,∞) into [0,1) satisfying at least one of the following conditions:*
(1) $a + 2b + c$ is decreasing function.
(2) $a + 2b + c$ has property (P).
(3)(4) $a + 2b$ and $a + c : (0, \infty) \to [0, 1)$ are decreasing functions.
(4) $a + 2b$ has property (P) and $a + c : (0, \infty) \to [0, 1)$ is decreasing function.
(5) $a + c$ has property (P) and $a + 2b : (0, \infty) \to [0, 1)$ is decreasing function.
Suppose that T satisfies the condition (B. 2). Then T has a unique fixed point.

Remark. We first note that Theorem 1.2 is a special case of each result of the above Corollary. Also, it is obvious that in cases (3), (4) and (5) of the Corollary, the function $a + 2b + c$ can take values larger than 1.

REFERENCES

1. D. Delbosco, Un estensione di un teorema sul punto fisso di S. Reich, Rend. Sem. Mat. Univers. Politecn. Torino **35** (1976-77), 233-239.
2. M. S. Khan, Fixed point theorems by altering distance between the points, Bull. Austral. Math. Soc. **30** (1984), 1-9.
3. F. Skof, Teorema di punti fisso per applicazioni negli spazi metrici, Atti. Accad. Aci. Torino **111** (1977), 323-329.