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On the Solution of an Integral Equation Appearing in Displacement Problems

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Abstract

In this paper we are interested in solving an integral equation appearing in the study of two kinds of displacement problems. A simple way is used to obtain a solution of this equation.

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1. Introduction

It is known, in general, that a Fredholm integral equation of the first kind is difficult to solve. However, in this paper we present a simple way to solve an integral equation of the first kind arising in the study of two kinds of displacement problems. Let us consider the Fredholm integral equations of the first kind

$$\int_{-a}^{a} k(x,\ y)\ \phi(y)\ dy = \pi\ f(x). \tag{1.1}$$

Here, $f(x)$ and $k(x,\ y)$ are given functions, and $k(x,\ y)$ is the kernel of the integral equation (1.1) which may have a singularity while the unknown function $\phi(x)$ is to be determined.

Mkhitarian and Abdou, [6], applied Krein's method to obtain the basic formula of the potential function when $k(y) = -\ln(|y|)$ and $f(x) \in C^1[-a, a]$. Also, Popov in his work [9] solved Eq. (1.1) under certain conditions on the kernel using orthogonal polynomials method. The crucial achievement in Eq.(1.1) is Muskhelishvili's result, [7], for solving singular integral equations. Abdou [1] established Eq.(1.1) from three-dimensional contact problems in the theory of elasticity with potential kernel where the coefficients bed of the compressible materials are neglected. More recently, Fahmy and others [3] obtained the solution of Eq.(1.1) with Gauss's hypergeometric kernel in the half-plane using Abel's theorem.

In this paper, we consider the integral equation

$$-\int_{-a}^{a} K\left(\frac{y-x}{\lambda}\right)\phi(y)\ dy = \pi\ f(x), \quad |x| < a, \tag{1.2}$$

under the conditions

$$\int_{-a}^{a} \phi(y)\ dy = P < \infty, \tag{1.3}$$

and

$$K(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh\ \eta}{\eta} e^{i\eta\xi}\ d\eta. \tag{1.4}$$

Here, the given function f is assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $\phi \in C[-a, a]$ (see [8]) and λ is a parameter.

We are interested in solving Eq.(1.2), under the conditions (1.3) and (1.4), since it arises in the study of two kinds of displacement problems, see Section 3. In [2], the solution of Eqs.(1.2)- (1.4) is given by using Krein's method.

In the following section, we present a simple way to solve the integral equations (1.2)-(1.4).

2. Solution of the problem

Firstly, the second condition, (1.4), may be written in the form, [10, p. 32],

$$K(\xi) = -\ln\left|\tanh\frac{\pi\xi}{4}\right|. \tag{2.1}$$

If $\lambda \to \infty$ and ξ is sufficiently small, i.e., tanh $\theta \approx \theta$, we have that

$$\ln\left|\tanh\frac{\pi(y-x)}{4\lambda}\right| = \ln|y-x| - c,\ c = \ln\frac{4\lambda}{\pi}. \tag{2.2}$$

Then Eq.(1.2) becomes

$$\int_{-a}^{a}\left[c - \ln|y-x|\right]\phi(y)\,dy = -\pi f(x), \tag{2.3}$$

where $\phi(y)$ satisfies (1.3). Secondly, rewrite Eq.(2.3) in the form

$$\int_{-a}^{a}\ln|y-x|\ \phi(y)\,dy = B + \pi f(x), \tag{2.4}$$

where

$$B = c\int_{-a}^{a}\phi(y)\,dy. \tag{2.5}$$

But the solution of Eq.(2.4) is given by, [5, pp. 209],

$$\begin{aligned}(2.6)\ \phi(x) \;&=\; \frac{1}{\pi}\int_{-a}^{a}\sqrt{\frac{a^2-y^2}{a^2-x^2}}\,\frac{f'(y)}{y-x}\,dy+ \\ &+\frac{1}{\pi\ \ln(a/2)\sqrt{a^2-x^2}}\left\{B+\int_{-a}^{a}\frac{f(y)}{\sqrt{a^2-y^2}}\,dy\right\}.\end{aligned}$$

Now, it remains to find the unknown constant B in Eq.(2.6). Substitute Eq.(2.6) in Eq.(2.5) we obtain

$$B = \frac{c}{\ln(a/2)-c}\int_{-a}^{a}\frac{f(y)}{\sqrt{a^2-y^2}}\,dy, \tag{2.7}$$

where we used the fact that

$$\int_{-\alpha}^{\alpha} \frac{du}{\sqrt{\alpha^2 - u^2}} = \pi$$

and, from the complex analysis, the fact that

$$\int_{-\alpha}^{\alpha} \frac{1}{\sqrt{\alpha^2 - u^2}} \frac{du}{v - u} = 0, \quad |v| < 1.$$

Finally, substituting the value of B from (2.7) into (2.6) we have the the solution of Eq.(1.2) under the conditions (1.3) and (1.4) as

$$(2.8)\quad \phi(x) = \frac{1}{\pi\sqrt{a^2 - x^2}} \int_{-a}^{a} \sqrt{a^2 - y^2} \frac{f'(y)}{y - x}\, dy + \frac{P}{\pi\sqrt{a^2 - x^2}}, \quad |x| < a$$

and

$$(2.9)\qquad P = \frac{1}{\ln(a/2) - c} \int_{-a}^{a} \frac{f(y)}{\sqrt{a^2 - y^2}}\, dy.$$

This result is in accordance with the results obtained in [2, 6].

3. Discussion

The displacement problems of anti-plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h, can be formulated as, see [2] and and reference therein,

$$(3.1)\qquad -\int_{-a}^{a} \phi(\tau)\, \ln\left|\tanh\, \frac{\pi\,(\tau - x)}{4h}\right|\, d\tau = D,$$

where $\phi(\tau)$ is the unknown displacement stress and D is the displacement stress.

The problems of infinite rigid strip, with width $2a$ immersed in a viscous liquid layer of thickness h, where the strip has a velocity resulting from the impulsive force $V = V_0\, e^{-i\omega t}$. Such problems are represented by the Fredholm integral equation, see [3] and reference therein,

$$(3.2)\qquad -\int_{-a}^{a} \phi(\tau)\, \ln\left|\tanh\, \frac{\pi\,(\tau - x)}{4h}\right|\, d\tau = \pi\mu\, V,$$

where V_0 is the constant velocity, ω is the angular velocity resulting from rotating the strip about $z - axis$ and μ is the viscosity coefficient.

The depth of the liquid or the thickness of the elastic material becomes infinite as $h \to \infty$.

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