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Transient Analysis in Discrete Time of Markovian Queues with Quadratic Rates

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Abstract

Transient system size probabilities in discrete time are obtained in exact closed form for queueing models with constant, linear and quadratic arrival and service rates. This is achieved by using some interesting identities of tridiagonal determinants. Equilibrium probabilities are derived.

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1 Introduction

In the recent years, there has been a resurgence of interest in the analysis of discrete time queueing models due to their applications in communication systems and other related areas. These models arise naturally in the study of time sharing computing systems, where messages from a collection of terminals are assigned by different multiplexing methods to a central computer [4]. In such systems, all events occur only at definite regularly spaced time points. Due to its importance in computer communication systems, transient analysis in discrete time is also receiving considerable attention [2, 7]. One of the main reasons for the discrete time analysis is that continuous time analysis cannot accurately give the performance measures of discrete time models, e.g., in computer communication systems the elementary units are the machine cycle time of a processor, the bit or byte duration of signals on a channel. Moreover, the continuous time results can be derived from the discrete analysis as a limiting case.

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In this paper we consider Markovian queueing models with finite capacity. Such queueing models have applications in production and inventory problems, e.g., to optimize the size of the storage space, to determine the trade-off between throughput and inventory (or waiting time) and to exhibit the propagation of blockage. The performance of the produce-to-stock manufacturing facility can be obtained from the performance of the finite queueing systems [3] . The situation of a network of queues with finite buffers occurs widely in computer and telecommunications systems [1, 5, 14]. The phenomena of blocking, starvation and server breakdowns can severely restrict throughput and response time [12]. Data sources in computer networks are generally bursty in nature and it is appropriate to consider these peak arrival rates to be quadratic [6, 11, 15]. In order to process the messages rapidly to avoid congestion, service rates must also be quadratic. Hence the queueing models considered in this paper may be useful in computer networks with quadratic arrivals and service rates.

In the solution procedure, suitable probability generating functions are used to convert the system of difference equations of probabilities into a system of difference equations of generating functions. Using Cramer's rule, the required solution is represented by a ratio of two tridiagonal determinants , that is, ratio of two polynomials defined recursively by three-term recurrence relations. The roots of the denominator polynomial is obtained by reducing the order of the associated tridiagonal determinant by one and comparing the resultant determinant to the identities of tridiagonal determinants [9, 10]. Making use of these roots, the solution is obtained by partial fraction technique. These roots are calculated analytically in closed form irrespective of the order of the system capacity N which is in contrast with the computational difficulties encountered in finding these roots numerically for large value of N. Steady state probabilities are deduced.

2 Notations and solution procedure

We consider queueing models with finite capacity N where arrivals occur according to a Bernoulli process with the probability of an arrival being λ_n and service completions occur according to geometric distribution with probability of a service completion being μ_n during any time slot when there are n customers in the system and they are defined as follows:

a) constant rates

$$\begin{cases} \lambda_n = \lambda, & n = 0, 1, 2, \ldots, N-1, \\ \mu_n = \mu, & n = 1, 2, 3, \ldots, N; \end{cases} \tag{2.1}$$

b) linear rates

$$\begin{cases} \lambda_n = (N-n)a, & n = 0, 1, \ldots, N-1, \\ \mu_n = nb, & n = 1, 2, \ldots, N, \qquad a, b > 0; \end{cases} \tag{2.2}$$

c) quadratic rates

$$\begin{cases} \lambda_n = (N-n)(b-(n-1)c), & n = 0,1,\ldots,N-1, \\ \mu_n = (N+n+1)(b+(n+2)c), & n = 1,2,\ldots,N, \end{cases} \tag{2.3}$$

with $b > 0$, $-b/(N+2) < c < 0$ and

d)
$$\begin{cases} \lambda_n = (N-n)(b+(n-1)d), & n = 0,1,\ldots,N-1, \\ \mu_n = n(c-(n-1)d), & n = 1,2,\ldots,N, \end{cases} \tag{2.4}$$

where $b, c > 0$ and

$$\begin{cases} d \leq 0, & \text{if } N = 1 \text{ or } 2, \\ \frac{-b}{N-2} < d \leq 0, & \text{if } N > 2. \end{cases}$$

The conditions imposed on the parameters ensure the positivity of the rates.

Let X_r be the random variable denoting the number of customers in the system at discrete time epoch r. Let

$$P_r(n,m) = P(X_r = n | X_0 = m), \quad n,m = 0,1,\ldots,N.$$

For notational convenience we make the second argument of $P_r(n,m)$ implicit. The function $P_r(n)$ satisfies the following difference equations:

$$\begin{aligned} P_{r+1}(0) &= (1-\lambda_0)P_r(0) + \mu_1 P_r(1) \\ P_{r+1}(n) &= \lambda_{n-1}P_r(n-1) + (1-\lambda_n-\mu_n)P_r(n) + \mu_{n+1}P_r(n+1), \; n = 1,2,3,\ldots,N-1 \\ P_{r+1}(N) &= \lambda_{N-1}P_r(N-1) + (1-\mu_N)P_r(N). \end{aligned}$$

If $G_z(n) = \sum_{r=0}^{\infty} P_r(n) z^r$; $|z| \leq 1$, then the above system can be written as

$$\begin{aligned} \left[\frac{1}{z} - (1-\lambda_0)\right] G_z(0) - \mu_1 G_z(1) &= \frac{P_0(0)}{z} \\ -\lambda_{r-1}G_z(r-1) + \left[\frac{1}{z} - (1-\lambda_r-\mu_r)\right] G_z(r) - \mu_{r+1}G_z(r+1) &= \frac{P_0(r)}{z}, \\ & \quad r = 1,2,3,\ldots,N-1 \\ -\lambda_{N-1}G_z(N-1) + \left[\frac{1}{z} - (1-\mu_N)\right] G_z(N) &= \frac{P_0(N)}{z} \end{aligned}$$

where $P_0(j) = \delta_{mj}$, the Kronecker delta.

If $e_m = [\delta_{m0}, \delta_{m1}, \ldots, \delta_{mN}]'$, is a column vector with 1 in the mth place and 0 elsewhere, then the above system can be written in matrix form as follows:

$$M(z)\,[G_z(0), G_z(1), \ldots, G_z(N)]' = \frac{e_i}{z} \ ,$$

where

$$M(z) = \begin{bmatrix} \frac{1}{z}-1+\lambda_0 & -\mu_1 & & & \\ -\lambda_0 & \frac{1}{z}-1+\lambda_1+\mu_1 & -\mu_2 & & \\ & & & \ddots & \\ & & & -\lambda_{N-1} & \frac{1}{z}-1+\lambda_N+\mu_N \end{bmatrix}_{(N+1)\times(N+1)}.$$

For the sake of simplicity we use the transformation $s = \dfrac{1}{z} - 1$ and denote $M(z)$ by $M(s)$ and $G_z(N)$ by $G_s(N)$. Hence

$$M(s) = \begin{bmatrix} s+\lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & s+\lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & s+\lambda_2+\mu_2 & -\mu_3 & & \\ & & & & \ddots & \\ & & & & -\lambda_{N-1} & s+\mu_N \end{bmatrix}_{(N+1)\times(N+1)} \tag{2.5}$$

Using Cramer's rule we have

$$G_s(j) = \frac{\frac{1}{z}Q_{m,j}(s)}{|M(s)|}, \tag{2.6}$$

where $\frac{1}{z}Q_{m,j}(s)$ is obtained from $|M(s)|$ by replacing column j by $\frac{e_m}{z}$ which can be given explicitly by

$$Q_{m,j}(s) = c_{m,j}A_{N-max(m,j)}(s)B_{min(m,j)}(s) \tag{2.7}$$

where

$$\begin{aligned} B_0(s) &= 1 \\ B_1(s) &= s+\lambda_0 \\ B_n(s) &= (s+\lambda_{n-1}+\mu_{n-1})B_{n-1}(s) - \lambda_{n-2}\mu_{n-1}B_{n-2}(s), \quad n = 2,3,\ldots,N \\ B_{N+1}(s) &= (s+\mu_N)B_N(s) - \lambda_{N-1}\mu_N B_{N-1}(s), \end{aligned} \tag{2.8}$$

$$\begin{aligned} A_0(s) &= 1 \\ A_1(s) &= s+\mu_N \\ A_n(s) &= (s+\lambda_{N+1-n}+\mu_{N+1-n})A_{n-1}(s) - \lambda_{N+1-n}\mu_{n-2}A_{n-2}(s), \quad n = 2,3,\ldots,N+1. \end{aligned}$$

and

$$c_{m,j} = \begin{cases} \lambda_m\lambda_{m+1}\ldots\lambda_{j-1}, & m < j \\ 1, & m = j \\ \mu_{j+1}\mu_{j+2}\ldots\mu_m, & m > j \end{cases}$$

Derivation of (2.7) can be had from [13].

To analyze (2.6), the roots of $|M(s)|$ are to be computed. We observe that $|M(s)|$ is zero when $-s$ is an eigenvalue of the matrix

$$\begin{bmatrix} \lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & \lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & \lambda_2+\mu_2 & \mu_3 & & \\ & & & \ddots & & \\ & & & & -\lambda_{N-1} & \mu_N \end{bmatrix}_{(N+1)\times(N+1)}.$$

This matrix can be transformed into a real symmetric positive definite tridiagonal matrix with non-zero sub-diagonal elements and therefore the eigenvalues are real and distinct [16]. Suppose $s_0, s_1, s_2, \ldots, s_N$ are the roots of $|B(s)|$ with $s_0 < s_1 < \ldots < s_N$. From (2.6) we have,

$$G_s(j) = \frac{\frac{1}{z}Q_{m,j}(s)}{\prod_{k=0}^{N}(s+s_k)}$$

Using partial fraction technique and replacing s by $\frac{1}{z} - 1$ we have,

$$G_z(j) = \sum_{k=0}^{N} \frac{\frac{1}{z}Q_{m,j}(s_k)}{\left(\frac{1}{z} - 1 - s_k\right)\prod_{j=0,j\neq k}^{N}(s_k - s_j)}$$

Expanding binomially in terms of z and comparing the coefficients of z^r we have,

$$P_r(j) = \sum_{k=0}^{N} \frac{Q_{m,j}(s_k)}{\prod_{j=0,j\neq k}^{N}(s_k - s_j)}(1+s_k)^r$$

At the roots of $|M(s)|$, $Q_{m,j}(s)$ given by (2.7) can be written as

$$Q_{m,j}(s_k) = c_{m,N}c_{N,j}\frac{B_m(s_k)B_j(s_k)}{B_N(s_k)}, \quad k = 0, 1, \ldots, N.$$

Therefore,

$$P_r(j) = c_{m,N}c_{N,j}\sum_{k=0}^{N} \frac{B_m(s_k)B_j(s_k)}{B_N(s_k)\prod_{j=0,j\neq k}^{N}(s_k - s_j)}(1+s_k)^r. \tag{2.9}$$

Using the following two identities of tridiagonal determinants

$$\begin{vmatrix} a_1 & b_1 & & \\ c_1 & a_2 & b_2 & \\ & c_2 & a_3 & \\ & & & \ddots \end{vmatrix}_{N\times N} = \begin{vmatrix} a_1 & 1 & & \\ b_1c_1 & a_2 & 1 & \\ & b_2c_2 & a_3 & \\ & & & \ddots \end{vmatrix}_{N\times N}.$$

and

$$\begin{vmatrix} \eta+a_0 & a_0 & & & & \\ b_1 & \eta+a_1+b_1 & a_1 & & & \\ & b_2 & \eta+a_2+b_2 & a_2 & & \\ & & & \ddots & & \\ & & & & \eta+a_{N-1}+b_{N-1} & a_{N-1} \\ & & & & b_N & \eta+b_N \end{vmatrix}_{(N+1)\times(N+1)}$$

$$= \eta \times \begin{vmatrix} \eta+a_0+b_1 & b_1 & & & \\ a_1 & \eta+a_1+b_2 & b_2 & & \\ & & \ddots & & \\ & & & \eta+a_{N-2}+b_{N-1} & b_{N-1} \\ & & & a_{N-1} & \eta+a_{N-1}+b_N \end{vmatrix}_{N\times N}.$$

the determinant of the matrix $|M(s)|$ can be written as

$$M_{N+1}(s) = s \times \begin{vmatrix} s+\lambda_0+\mu_1 & \lambda_1 & & & & \\ \mu_1 & s+\lambda_1+\mu_2 & \lambda_2 & & & \\ & \mu_2 & s+\lambda_2+\mu_3 & \lambda_3 & & \\ & & & \ddots & & \\ & & & & \mu_{N-1} & s+\lambda_{N-1}+\mu_N \end{vmatrix}_{N\times N} \tag{2.10}$$

Equation (2.10) shows that one of the roots of $|M(s)|$ is zero(let $s_0 = 0$) and hence we can calculate steady state probabilities $p(j),\ j = 0, 1, \dots, N$. When r goes to infinity, we have from (2.9),

$$p(j) = (-1)^N c_{m,N} c_{N,j} \frac{B_m(0) B_j(0)}{B_N(0) \prod_{j=1, j\neq k}^{N} (s_j)}$$

Using

$$B_k(0) = \prod_{i=0}^{k-1} \lambda_i, \quad k = 0, 1, \dots, N \tag{2.11}$$

we have,

$$p(j) = \frac{\left(\prod_{k=j+1}^{N} \mu_k\right)\left(\prod_{k=0}^{j-1} \lambda_k\right)}{\prod_{k=1}^{N} (-s_k)}, \quad j = 0, 1, \dots, N. \tag{2.12}$$

Therefore (2.9) can be written as

$$P_r(j) = p(j) + c_{m,N} c_{N,j} \sum_{k=1}^{N} \frac{B_m(s_k) B_j(s_k)}{B_N(s_k) \prod\limits_{j=0, j\neq k}^{N} (s_k - s_j)} (1+s_k)^r. \tag{2.13}$$

The length of time required for the system to reach steady state is called relaxation time [8] and is given by

$$T = \frac{1}{\min\limits_{1\leq k\leq N} (-s_k)}. \tag{2.14}$$

3 Queueing model with constant rates

In this section we find the roots of $M_{N+1}(s)$ for the queueing model with rates (2.1).

Theorem 1 *For the queueing model with constant λ_n and μ_n given by (2.1) the roots of $M_{N+1}(s)$ are given by $s_0 = 0, \quad s_k = \; -\lambda - \mu + 2\sqrt{\lambda\mu}\ \cos\frac{k\pi}{N+1}, k = 1, 2, ..., N.$*

Proof We use the following identity to prove this theorem.
Identity

$$\begin{vmatrix} \eta & a & . & . & . & . & . & . \\ b & \eta & a & . & . & . & . & . \\ . & b & \eta & a & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & b & \eta & a \\ . & . & . & . & . & . & b & \eta \end{vmatrix}_{N\times N} = \prod_{j=1}^{N} \left(\eta - 2\sqrt{ab}\ \cos\left(\frac{j\pi}{N+1}\right)\right).$$

From (2.10) we have

$$M_{N+1}(s) = s \times \begin{vmatrix} s+\lambda+\mu & \lambda & & & \\ \mu & s+\lambda+\mu & \lambda & & \\ & \mu & s+\lambda+\mu & \lambda & \\ & & & \ddots & \\ & & & \mu & s+\lambda+\mu \end{vmatrix}_{N\times N}.$$

On comparing the above determinant with the determinant in the identity we get

$$M_{N+1}(s) = s\prod_{i=1}^{N} \left(s + \lambda + \mu - 2\sqrt{\lambda\mu}\ \cos\frac{i\pi}{N+1}\right).$$

Therefore the roots of $M_{N+1}(s)$ are given by

$$0, \quad -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{\pi}{N+1}, \ -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{2\pi}{N+1}, \ldots, -\lambda-\mu+2\sqrt{\lambda\mu}\ \cos\frac{N\pi}{N+1}$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{\lambda+\mu-2(\lambda\mu)^{1/2}\cos\frac{\pi}{N+1}}.$$

In the following theorem we find the explicit expression for $B_i(s)$ defined by (2.8) at the roots of $M_{N+1}(s)$.

Theorem 2 *For the queueing model with constant λ_n and μ_n given by (2.1)*

$$B_i(s_k) = \begin{cases} \lambda^i & \textit{for } k=0 \\ & \textit{for } i=0,1,\ldots,N, \\ (\lambda\mu)^{i/2}\,\textit{cosec}\left(\frac{k\pi}{N+1}\right)\left\{\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)\right\}, & \textit{for } k=1,2,\ldots,N. \end{cases}$$

Proof The result for the case $k=0$ follows from (2.11). We assume $k>0$.

$B_i(s)$ defined recursively in (2.8) can be represented in tridiagonal determinant form as follows

$$B_i(s) = \begin{vmatrix} s+\lambda_0 & 1 & & & \\ \lambda_0\mu_1 & s+\lambda_1+\mu_1 & 1 & & \\ & \lambda_1\mu_2 & s+\lambda_2+\mu_2 & 1 & \\ & & & \ddots & \\ & & & \lambda_{i-2}\mu_{i-1} & s+\lambda_{i-1}+\mu_{i-1} \end{vmatrix}_{i\times i}. \tag{3.1}$$

By substituting $\lambda_n=\lambda$ and $\mu_n=\mu$ in the above determinant and using the identity given in theorem 1 we get

$$B_i(s) = \prod_{j=1}^{i}\left\{s+\lambda+\mu-2\sqrt{\lambda\mu}\ \cos\frac{j\pi}{i+1}\right\} - \mu\prod_{j=1}^{i-1}\left\{s+\lambda+\mu-2\sqrt{\lambda\mu}\ \cos\frac{j\pi}{i}\right\}.$$

With $s_k = -\lambda-\mu+2\sqrt{(\lambda\mu)}\cos\left(\frac{k\pi}{N+1}\right)$,

$$\begin{aligned} B_i(s_k) &= 2^i(\lambda\mu)^{i/2}\prod_{j=1}^{i}\left\{\cos\left(\frac{k\pi}{N+1}\right)-\cos\left(\frac{j\pi}{i+1}\right)\right\} \\ &\quad -2^{i-1}\mu(\lambda\mu)^{(i-1)/2}\prod_{j=1}^{i-1}\left\{\cos\left(\frac{k\pi}{N+1}\right)-\cos\left(\frac{j\pi}{i}\right)\right\} \end{aligned}$$

Using the identity

$$\prod_{r=1}^{n-1}\left\{\cos\left(\frac{k\pi}{n+1}\right)-\cos\left(\frac{r\pi}{n}\right)\right\}=2^{-(n-1)}\sin\left(\frac{nk\pi}{n+1}\right)\operatorname{cosec}\left(\frac{k\pi}{n+1}\right)$$

we have

$$B_i(s_k)=(\lambda\mu)^{i/2}\operatorname{cosec}\left(\frac{k\pi}{N+1}\right)\left\{\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)\right\}.$$

□

Using the results of theorems 1, 2 and (2.13) the transient system size probabilities are obtained after considerable simplification and are given in the following theorem.

Theorem 3 *For the queueing model with constant λ_n and μ_n given by (2.1) the transient system size probabilities are given by*

$$\begin{aligned} P_r(j) &= \frac{\rho^{N-n}}{\prod_{j=1}^{N}\left\{1+\rho-2\sqrt{\rho}\cos\left(\frac{j\pi}{N+1}\right)\right\}} \\ &+ (-1)^{N+1}\rho^{m-j}\lambda\mu\sum_{k=1}^{N}\frac{Y_{mk}Y_{jk}\left[1-(\lambda+\mu-2\sqrt{(\lambda\mu)}\cos\frac{k\pi}{N+1})\right]}{(N+1)\left(1+\rho-2\sqrt{\rho}\cos\left(\frac{k\pi}{N+1}\right)\right)} \end{aligned}$$

$$j=0,1,2,\ldots,N,$$

where $Y_{ik}=\sin\left(\frac{(i+1)k\pi}{N+1}\right)-\sqrt{\frac{\mu}{\lambda}}\sin\left(\frac{ik\pi}{N+1}\right)$, $\rho=\frac{\lambda}{\mu}$ and $m(\geq 0)$ is the initial state.

4 Queueing model with linear rates

In this section we find the roots of $M_{N+1}(s)$ for the queueing model with linear rates (2.2).

Theorem 4 *For the queueing model with linear λ_n and μ_n given by (2.2) the roots of $M_{N+1}(s)$ are given by*

$$s_k=-k(a+b),\quad k=0,1,\ldots,N$$

Proof We use the following identity to prove this theorem.

Identity

If $\eta_k=\xi-(k-1)(a-b)$; $\beta_k=(N-k)a$ and $\nu_k=kb$, then

$$\begin{vmatrix} \eta_1 & \beta_1 & & & \\ \nu_1 & \eta_2 & \beta_2 & & \\ & \nu_2 & \eta_3 & \beta_3 & \\ & & & \ddots & \\ & & & & \eta_N \end{vmatrix}_{N\times N}=\prod_{i=1}^{N}\{\xi-(N-i)a+(i-1)b\}.$$

On substituting (2.2) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is, $\xi = s + Na + b$. For this ξ we have

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2, 3, 4, \ldots, N$$

$$\lambda_k = \beta_k, \quad k = 0, 1, 2, \ldots, N-1$$

$$\mu_k = \nu_k, \quad k = 1, 2, 3, \ldots, N.$$

Therefore,

$$M_{N+1}(s) = s \prod_{i=1}^{N} (\xi - (N-i)a + (i-1)b).$$

Substituting for ξ we get,

$$M_{N+1}(s) = \prod_{i=0}^{N} (s + i(a+b)).$$

Hence the roots are

$$s_k = -k(a+b), \quad k = 0, 1, 2, \ldots, N.$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{a+b}.$$

Using (3.1) and the above theorem explicit expression for $B_i(s)$ $(i = 0, 1, 2, \ldots, N)$ at the roots of $M_{N+1}(s)$ is obtained by induction as given in the following theorem.

Theorem 5 *For the queueing model with linear λ_n and μ_n given by (2.2)*

$$B_i(s_k) = \begin{cases} \prod_{j=0}^{i-1} \lambda_j, & k = 0 \\ i!\gamma_i^k, & k = 1, 2, \ldots, N. \end{cases} \quad \textit{for } i = 0, 1, \ldots, N,$$

where

$$\gamma_i^k = \sum_{j=0}^{i} (-1)^j \binom{k}{j} \binom{N-k}{i-j} b^j a^{i-j}.$$

After considerable simplification we obtain the following result using theorems 4 and 5.

Theorem 6 *For the queueing model with linear λ_n and μ_n given by (2.2) we have for $r = 0, 1, \ldots, N$,*

$$P_r(j) = \begin{pmatrix} N \\ j \end{pmatrix} \frac{\rho^j}{(1+\rho)^N} + \frac{a^{N-m-j}\rho^j (N-m)!m!}{(1+\rho)^N} \sum_{k=1}^{N} \frac{\gamma_m^k \gamma_j^k \left(\gamma_N^k\right)^{-1}}{\prod\limits_{j=0, j\neq k}^{N} (j-k)} (1-k(a+b))^r.$$

where $\rho = \frac{a}{b}$ and $m \geq 0$ is the initial condition.

As $r \to \infty$, we have from the result of theorem 4.3 the steady state probabilities,

$$p(j) = \left(\begin{smallmatrix} N \\ j \end{smallmatrix} \right) \frac{\rho^j}{(1+\rho)^N}, \quad j = 0, 1, 2, \ldots, N.$$

In the following section we find the roots of $M_{N+1}(s)$ for the queueing model with quadratic rates given by (2.3) and (2.4) and the corresponding steady state probabilities and relaxation times.

5 Queueing models with quadratic rates

In the following theorem the roots of $M_{N+1}(s)$ are found for the queueing model with quadratic rates (2.3).

Theorem 7 *For the queueing model with quadratic λ_n and μ_n given by (2.3) the roots of $M_{N+1}(s)$ are given by*

$$s_j = -2j\left[2b - c(2j-5) + 2Nc\right], \quad j = 0, 1, 2, \ldots, N.$$

Proof We use the following identity to prove this theorem.
Identity
If $\eta_k = \xi + 2(k^2 - 1)$; $\beta_k = (N-k)(b-(k-1)c)$; $\nu_k = (N+k+1)(b+(k+2)c)$; then

$$\begin{vmatrix} \eta_1 & \beta_1 & & & \\ \nu_1 & \eta_2 & \beta_2 & & \\ & \nu_2 & \eta_3 & \beta_3 & \\ & & & \ddots & \\ & & & & \eta_N \end{vmatrix}_{N\times N} = \prod_{i=0}^{N-1} \{X + 2iY + i[N-(i-3)]Z\},$$

where $X = \xi - 2(N-1)b$; $Y = 2b - 5c$ and $Z = 4c$.

On substituting (2.3) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is,

$$\xi = s + 2b(N+1) + 4Nc + 6c.$$

For this value of ξ we have

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2, 3, 4, \ldots, N$$

$$\lambda_k = \beta_k, \quad k = 0, 1, 2, \ldots, N-1$$

$$\mu_k = \nu_k, \quad k = 1, 2, 3, \ldots, N.$$

Therefore,

$$M_{N+1}(s) = s \prod_{i=0}^{N-1} (X + 2iY + i(N+3-i)Z).$$

On substituting the values of X, Y and Z we get,

$$M_{N+1}(s) = s \prod_{i=0}^{N-1} (s + 2(i+1)\left[2b - c(2i-3) + 2Nc\right]).$$

Hence the roots of $M_{N+1}(s)$ are given by

$$s_j = -2j\left[2b - c(2j-5) + 2Nc\right], \quad j = 0, 1, 2, \ldots, N.$$

□

The relaxation time for the model under consideration is given by

$$T = \frac{1}{4Nc + 4b - 6c}.$$

The transient system size probabilities can be calculated using the above roots from (2.13).

The steady state probabilities calculated using (2.12) are given by

$$p(j) = \frac{\prod_{i=j+1}^{N} (N+i+1)[b + (i+2)c] \prod_{i=0}^{j-1} (N-i)[b - (j-i)c]}{2^N \prod_{i=1}^{N} (i+1)[2b - c(2i-3) + 2Nc]}, \quad j = 0, 1, 2, \ldots, N.$$

In the following theorem the roots of $M_{N+1}(s)$ for the queueing model with quadratic rates (2.4) are found.

Theorem 8 *For the queueing model with quadratic* λ_n *and* μ_n *given by (2.4) the roots of* $M_{N+1}(s)$ *are given by*

$$s_j = -j\left[(b+c)-(N-j+1)d\right], \quad j = 0,1,2,\ldots,N.$$

Proof We use the following identity to prove this theorem.
Identity
If $\eta_k = \xi - (k-1)(b-c) + (k-1)\{N-2(k-1)\}d$; $\beta_k = (N-k)\{b+(k-1)d\}$ and $\nu_k = k\{c-(k-1)\}d$ then

$$\begin{vmatrix} \eta_1 & \beta_1 & & & \\ \nu_1 & \eta_2 & \beta_2 & & \\ & \nu_2 & \eta_3 & \beta_3 & \\ & & & \ddots & \\ & & & & \eta_N \end{vmatrix}_{N\times N} = \prod_{i=1}^{N}\{\xi-(N-i)b+(i-1)c-(i-1)(N-i)d\}\ .$$

On substituting (2.4) in (2.10) and comparing the resultant determinant with the determinant in the identity we have,

$$\eta_1 = s + \lambda_0 + \mu_1.$$

That is,

$$\xi = s + N(b-d) + c.$$

With this value of ξ we could prove

$$s + \lambda_{n-1} + \mu_n = \eta_n, \quad n = 2,3,4,\ldots,N$$

$$\lambda_k = \beta_k, \quad k = 0,1,2,\ldots,N-1$$

$$\mu_k = \nu_k, \quad k = 1,2,3,\ldots,N.$$

Therefore,

$$M_{N+1}(s) = \prod_{i=0}^{N}\left(s + i(b+c) - i(N-i+1)d\right).$$

Hence the roots of $M_{N+1}(s)$ are given by

$$s_j = -j\left[(b+c)-(N-j+1)d\right], \quad j = 0,1,2,\ldots,N.$$

$\square$

The relaxation time for the model under consideration is given by

$$T = \frac{1}{-dN+b+c}.$$

The transient system size probabilities can be calculated using the above roots from (2.13).

The steady state probabilities calculated using (2.12) are given by

$$p(j) = \frac{\prod_{i=j+1}^{N} i[c-(i-1)d] \prod_{i=0}^{j-1} (N-i)[b+(j-i)d]}{\prod_{i=1}^{N} i[b+c+(N-i+1)d]}, \quad j = 0, 1, 2, \ldots, N.$$

If $d = 0$, this model reduces to the linear model discussed in section 4.

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