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The Stability of Nash-Cournot Equilibria in Labor Managed Oligopolies

Weiye Li
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Abstract

This paper examines the asymptotical stability of equilibria in discrete dynamic labor-managed oligopolies. First the equivalence of the equilibrium problem of a large class of nonlinear games and the equilibrium problem of a class of discrete dynamic systems is verified. Stability conditions are then derived for a certain class of dynamic models, and these results are finally applied to labor-managed oligopolies. The economic interpretation of the stability conditions are also presented.

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1 Introduction

The stability of dynamic economic systems is one of the most frequently discussed problem areas in the modern economic literature. Stability conditions are usually obtained by using the stability theory of linear or nonlinear differential and difference equations.

One of the most important model classes consists of the different variants of the Cournot oligopoly model. A comprehensive summary of such models and stability results in both continuous and discrete cases is presented in Okuguchi [1], and Okuguchi and Szidarovszky [2].

In this paper the stability of labor-managed dynamic oligopolies will be examined. General stability conditions will be first introduced, and will be then applied to special models.

The paper is organized as follows. In Section 2 some general results are presented, and in Section 3, these results are applied to a special class of dynamic models, and in Section 4 the further special case of labor-managed oligopolies will be analyzed. Section 5 concludes the paper.

2 Nash-Cournot Equilibria and Fixed Points of Dynamic Systems

Oligopoly is a state of industry where a small N number of firms produce M homogeneous goods or close substitutes competitively. The multiproduct *Cournot Oligopoly* is an N-person noncooperative game defined as follows. Let $x_k^{(m)}$ denote the output of firm k of product m, then the output of the firm can be characterized by output vector $\mathbf{x_k} = (\mathbf{x_k^{(1)}}, ..., \mathbf{x_k^{(M)}})$. Let $\mathbf{s} = \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{x_k}$ denote the output vector of the industry, and assume that the price vector depends on $\mathbf{s}$: $\mathbf{p} = \mathbf{p}(\mathbf{s})$. If c_k $(k = 1, 2, ..., N)$ denotes the cost function of firm k, then its profit can be expressed as

$$\varphi_k(\mathbf{x_1},\ ...,\ \mathbf{x_N}) = \mathbf{x_k^T}\,\mathbf{p}(\mathbf{s}) - \mathbf{c_k}(\mathbf{x_k}). \tag{2.1}$$

If $X_k \subset \mathbb{R}_+^{\mathbb{M}}$ denotes the set of all feasible outputs for firm k, then the resulting game can be given in strategic form as $\Gamma = (N; X_1, ..., X_N; \varphi_1, ..., \varphi_N)$. In the single product case, $M = 1$, and we may select $X_k = [0, L_k]$, where L_k is the capacity limit of firm k.

A vector $\mathbf{x}^* = (\mathbf{x_1^*}, \mathbf{x_2^*}, ..., \mathbf{x_N^*}) \in \mathbf{X} = \mathbf{X_1} \oplus \mathbf{X_2} \oplus ... \oplus \mathbf{X_N}$ is called a Nash-Cournot equilibrium point of game Γ, if for $k = 1, 2, ..., N$,

1. $\mathbf{x_k^*} \in \mathbf{X_k}$;
2. For arbitrary $\mathbf{x_k} \in \mathbf{X_k}$,

$$\varphi_k(\mathbf{x_1^*}, ..., \mathbf{x_{k-1}^*}, \mathbf{x_k^*}, \mathbf{x_{k+1}^*}, ..., \mathbf{x_N^*}) \quad \geq \quad \varphi_k(\mathbf{x_1^*}, ..., \mathbf{x_{k-1}^*}, \mathbf{x_k}, \mathbf{x_{k+1}^*}, ..., \mathbf{x_N^*}).$$

In other words, the Nash-Cournot equilibrium is an N-tuple of strategies at which each player maximizes his own payoff with respect to his own strategy selection, given the strategy choices of all other players. If a Nash-Cournot equilibrium $\mathbf{x}^*$ exists and is an interior point of X, and all φ_k are continuously differentiable in X_k, then for $k = 1, 2, ..., N$ and $m = 1, 2, ..., M$,

$$\frac{\partial \varphi_k}{\partial x_k^{(m)}} \quad = \quad 0, \tag{2.2}$$

where the partial derivatives are taken with respect to each component.

In the continuous case, it is natural to assume that the rate of change of the k-th player's strategy selection with respect to time is positively proportional to his marginal payoff with respect to his strategy, namely,

$$\dot{\mathbf{x}}_k \quad = \quad \mathbf{C_k} \nabla_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{x}), \qquad\qquad \mathbf{k = 1,\ 2,\ ...,\ N}, \tag{2.3}$$

where $\nabla_k \varphi_k$ denotes the gradient of φ_k with respect to $\mathbf{x_k}$, and $\mathbf{C_k}$ is a diagonal constant matrix with positive diagonal elements. It is obvious that the interior Nash-Cournot equilibrium is a fixed point of the dynamical system (2.3).

In the discrete case, each player maximizes his payoff in every time period based on his knowledge on the strategies expected from the other players. Such expectations rely in general on the actual and the expected strategies taken in the last time periods.

Assume that equation (2.2) has a unique soloution $\mathbf{x_k}$ in terms of variables $\mathbf{x_1}, ..., \mathbf{x_{k-1}}, \mathbf{x_{k+1}}, ..., \mathbf{x_N}$ as

$$\mathbf{x_k} \quad = \quad \mathbf{f_k}(\mathbf{x_1},\ ...,\ \mathbf{x_{k-1}},\ \mathbf{x_{k+1}},\ ...,\ \mathbf{x_N}).$$

Besides, let $\mathbf{x_k(t)}$ be the actual strategy of player k and $\tilde{\mathbf{x}}_i^k(t)$ his expectation on the strategy of player i at time t $(i \neq k)$. Assume $\tilde{\mathbf{x}}_i^k(t+1)$ is a function of only $\mathbf{x_i(t)}$ and $\tilde{\mathbf{x}}_i^k(t)$, that is,

$$\tilde{\mathbf{x}}_i^k(t+1) \quad = \quad \mathbf{g_i^k}(\mathbf{x_i(t)},\ \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}(\mathbf{t})). \tag{2.4}$$

In addition, we assume that

$$\mathbf{g_i^k}(\mathbf{a}, \mathbf{a}) \quad = \quad \mathbf{a} \tag{2.5}$$

for any vector $\mathbf{a}$ and $k \neq i$, for which the reason is obvious.

As a special case, if $\mathbf{g_i^k}$ is a convex linear combination, i.e.,

$$\mathbf{g_i^k}(\mathbf{a}, \mathbf{b}) \quad = \quad \mathbf{D_i^k}\,\mathbf{a} + (\mathbf{I} - \mathbf{D_i^k})\,\mathbf{b}, \tag{2.6}$$

where $\mathbf{D_i^k}$ is a diagonal matrix with entries in $(0, 1]$. This scheme is called the *adaptive expectation.* Obviously, any adaptive expectation satisfies the natural assumption (2.5).

Due to the fact that each player maximizes his payoff based on the expected strategies of the others, and assuming interior optimum throughout, it is easy to see that for all k,

$$\mathbf{x_k(t)} \quad = \quad \mathbf{f_k}(\tilde{\mathbf{x}}_\mathbf{1}^\mathbf{k}(\mathbf{t}),\ ...,\ \tilde{\mathbf{x}}_\mathbf{k-1}^\mathbf{k}(\mathbf{t}),\ \tilde{\mathbf{x}}_\mathbf{k+1}^\mathbf{k}(\mathbf{t}),\ ...,\ \tilde{\mathbf{x}}_\mathbf{N}^\mathbf{k}(\mathbf{t})). \tag{2.7}$$

The dynamical system with state variables

$$\{\mathbf{x_k(t)},\ \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}(\mathbf{t}) \ \Big| \ k \neq i;\ k, i = 1, ..., N\} \tag{2.8}$$

is well determined by relations (2.4), (2.5), (2.7) and the choice of the initial expectations $\{\tilde{\mathbf{x}}_i^k(0) \,\big|\, k \neq i;\ k, i = 1, ..., N\}$. If we start the process at the Nash-Cournot equilibrium by setting the initial expectations $\tilde{\mathbf{x}}_i^k(0) = \mathbf{x_k^*}$ for all $k \neq i$, then $\mathbf{x_k(0)} = \mathbf{x_k^*}$ by (2.7) and (2.2), $\tilde{\mathbf{x}}_i^k(1) = \mathbf{x_k^*}$ by the assumption of $\mathbf{g_k}$, etc. Inductively, we have $\tilde{\mathbf{x}}_i^k(t) = \mathbf{x_k(t)} = \mathbf{x_k^*}$ for all t. Thus the N copies of the Nash-Cournot equilibrium is the fixed point of the dynamical system (2.8) when the order of variables in the system is arraged properly.

The above results can be summarized in the following theorem.

Theorem 2.1 *Assume that for all k, φ_k is continuously differentiable as an MN variable function. Then the interior Nash-Cournot equilibrium of the above N-person game corresponds to some fixed point of the dynamical system governed by (2.3) (when the system is continuous), or (2.2), (2.4), (2.5) and (2.7) (when the system is discrete).*

It is also easy to see that any fixed point is an interior equilibrium if φ_k is concave in $\mathbf{x_k}$ for all k. Therefore a corollary is derived.

Corollary 2.2 *In addition to the assumption of Theorem 2.1, assume that φ_k is concave in $\mathbf{x_k}$ for all k. Then the interior Nash-Cournot equilibrium problem of the above N-person game is equivalent to the fixed point problem of the dynamical system governed by (2.3) (when the system is continuous), or (2.2), (2.4), (2.5) and (2.7) (when the system is discrete).*

The stability or asymptotical stability of the Nash-Cournot equilibrium is of great interest in economic theory. It is well known that the eigenvalues of the Jacobian matrix of the transition function of a dynamical system at the fixed point can determine the stability and asymptotical stability in most cases. See for example, Bellman [3], and Li and Szidarovszky [4]. For instance, for a continuous system, if all eigenvalues have negative real parts, then the fixed point is asymptotically stable; for a discrete system, if all eigenvalues are inside the unit circle, then this fixed point is asymptotically stable. It is also known from the theory of differential and difference equations that if at least one eigenvalue of the Jacobian matrix at the equilibrium has positive real parts then the equilibrium of the continuous system is unstable; and if at least one eigenvalue is outside the unit circle, then the equilibrium of the discret system is unstable. These sufficient conditions are easy to be applied to the continuous system (2.3) by considering the eigenvalues of the Jacobian matrix $\frac{\partial(\mathbf{C_1}\frac{\partial\varphi_1}{\partial\mathbf{x_1}}, ..., \mathbf{C_N}\frac{\partial\varphi_N}{\partial\mathbf{x_N}})}{\partial(\mathbf{x_1}, ..., \mathbf{x_N})}$.

In the rest of this paper the discrete case will be analysed. Introduced first the notations

$$\mathbf{x} \;=\; (\mathbf{x_1},\, \mathbf{x_2},\, ...\,,\, \mathbf{x_N})^{\mathbf{T}} \in \mathbb{R}^{\mathbb{NM}},$$

$$\tilde{\mathbf{x}} \;=\; (\tilde{\mathbf{x}}_2^1,\, \tilde{\mathbf{x}}_3^1,\, ...,\, \tilde{\mathbf{x}}_N^1; \tilde{\mathbf{x}}_1^2,\, \tilde{\mathbf{x}}_3^2,\, ...,\, \tilde{\mathbf{x}}_N^2,\;\; ...;\, \tilde{\mathbf{x}}_1^N,\, \tilde{\mathbf{x}}_2^N,\, ...,\, \tilde{\mathbf{x}}_{N-1}^N)^T \in \mathbb{R}^{(\mathbb{N}-\mathbb{1})\mathbb{NM}}.$$

Accordingly, denote $\mathbf{g}$, a mapping from $(X_1 \oplus X_2 \oplus \cdots \oplus X_N) \oplus (X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$ into $(X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$, by

$$\mathbf{g}(\mathbf{x}, \tilde{\mathbf{x}}) \;=\; (\mathbf{g_i^k}(\mathbf{x_i},\, \tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}) \,\big|\, \mathbf{i} \neq \mathbf{k};\; \mathbf{k},\, \mathbf{i} = \mathbf{1}, \mathbf{2}, ..., \mathbf{N}).$$

Denote $\mathbf{f}$, a mapping from $(X_2 \oplus \cdots \oplus X_N) \oplus (X_1 \oplus X_3 \oplus \cdots \oplus X_N) \oplus \cdots \oplus (X_1 \oplus \cdots \oplus X_{N-1})$ into $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)$, by

$$\mathbf{f}(\tilde{\mathbf{x}}) \;=\; (\mathbf{f_k}(\tilde{\mathbf{x}}_\mathbf{i}^\mathbf{k}, ..., \tilde{\mathbf{x}}_\mathbf{k-1}^\mathbf{k}, \tilde{\mathbf{x}}_\mathbf{k+1}^\mathbf{k}, ..., \tilde{\mathbf{x}}_\mathbf{N}^\mathbf{k}) \,\big|\, \mathbf{k} = \mathbf{1}, \mathbf{2}, ..., \mathbf{N}).$$

Then system (2.8) is described by the following simple equalities

$$\begin{aligned} \mathbf{x}(\mathbf{t}) &= \mathbf{f}(\tilde{\mathbf{x}}(\mathbf{t})) \\ \tilde{\mathbf{x}}(t+1) &= \mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})), \end{aligned} \tag{2.9}$$

which can be rewritten as

$$\begin{aligned} \mathbf{x}(\mathbf{t}+\mathbf{1}) &= \mathbf{f}(\mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})) \\ \tilde{\mathbf{x}}(t+1) &= \mathbf{g}(\mathbf{x}(\mathbf{t}),\, \tilde{\mathbf{x}}(\mathbf{t})). \end{aligned} \tag{2.10}$$

The N-direct sum of the Nash-Cournot equilibrium up to some permutation of the components is a fixed point of system (2.9) or (2.10). The Jacobian matrix of the transition function is

$$\frac{\partial(\mathbf{x(t+1)},\, \tilde{\mathbf{x}}\mathbf{(t+1)})}{\partial(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)})} = \begin{pmatrix} D\mathbf{f}(\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}))\cdot \mathbf{D_1 g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) & D\mathbf{f}(\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}))\cdot \mathbf{D_2 g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) \\ D_1\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) & D_2\mathbf{g}(\mathbf{x(t)},\, \tilde{\mathbf{x}}\mathbf{(t)}) \end{pmatrix},$$

where D, D_1, and D_2 are differential operations with respect to $\mathbf{g}$, $\mathbf{x}$, $\tilde{\mathbf{x}}$, respectively. That is, $D\mathbf{f}$, $D_1\mathbf{g}$, and $D_2\mathbf{g}$ are the Jacobian matrices of $\mathbf{f}$ and $\mathbf{g}$ with respect to these variables. For simplicity, let us denote the Jacobian matrix of (2.10) at the fixed point as

$$\mathbf{J} = \begin{pmatrix} D\mathbf{f}\cdot \mathbf{D_1 g} & D\mathbf{f}\cdot \mathbf{D_2 g} \\ D_1\mathbf{g} & D_2\mathbf{g} \end{pmatrix}.$$

Let $(\mathbf{u},\mathbf{v})^{\mathbf{T}}$ be an eigenvector associated to an eigenvalue λ of $\mathbf{J}$. Then

$$\begin{aligned} D\mathbf{f}(\mathbf{D_1 g}\,\mathbf{u} + \mathbf{D_2 g}\,\mathbf{v}) &= \lambda \mathbf{u} \\ D_1\mathbf{g}\,\mathbf{u} + \mathbf{D_2 g}\,\mathbf{v} &= \lambda \mathbf{v}. \end{aligned}$$

Simple substitution derives either $\lambda = 0$ or $\mathbf{u} = \mathbf{Df}\,\mathbf{v}$. A zero eigenvalue does not distroy stability and asymptotical stability. If $\lambda \neq 0$, then

$$(D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2 g})\,\mathbf{v} = \lambda \mathbf{v}.$$

That is, any nonzero eigenvalue of $\mathbf{J}$ must be an eigenvalue of matrix

$$D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2 g}, \tag{2.11}$$

whose size is only $(N-1)NM \times (N-1)NM$. If this matrix has all eigenvalues inside the unit circle, then the Nash-Cournot equilibrium is asymptotically stable.

3 Stability in Single-product Oligopolies with Payoff Fuctions $\varphi_k(x_k,\, \Sigma_{i\neq k}\, x_i)$ and Adapted Expectations

The general approach outlined above requires a tedious computation in manipulating with matrix (2.11). In practice, this approach can be handled tackfully. The main idea is to reduce the number of variables in the dynamical system. This idea can be realized when the firms form expectations on the output of the rest of the industry, since the $N(M-1)$-dimensional expectation variable becomes only M-dimensional. Another case, when reduction in the dimension is possible, especially when the M goods are mutaually independent, i.e., the payoff function of each player is the sum of M functions where each of them is a payoff corresponding to a single good. Thus system (2.9) can be decomposed into M equivalent smaller dimensional subsystems.

Consider next a special single product case $\Gamma = (N; X_1, ..., X_N; \boldsymbol{\varphi_1}, ..., \boldsymbol{\varphi_N})$, where $X_k = [0, L_k]$, $\varphi_k(x_1,\, x_2,\, ...,\, x_N) = \phi(x_k, \sum_{i\neq k} x_i)$. For example, in the case of single product oligopolies without product differentiation, $\varphi_k(x_k, \sum_{i\neq k} x_i) = x_k\, p(x_k + \sum_{i\neq k} x_i) - c_k(x_k)$.

If adaptive expectations are assumed, it coincides with $g_i^k(a,b) = d_k a + (1-d_k)b$ for all $i \neq k$, and thus

$$\tilde{s}_k(t+1) \;=\; d_k s_k(t) + (1-d_k)\tilde{s}_k(t)$$

with $s_k(t) = \sum_{i\neq k} x_i(t)$ and $\tilde{s}_k(t) = \sum_{i\neq k} \tilde{x}_i(t)$. This gives us

$$x_k(t) \;=\; f_k(\tilde{s}_k(t)), \qquad \text{and} \qquad \tilde{s}_k(t+1) \;=\; g_k(s_k(t), \tilde{s}_k(t)).$$

Thus system (2.9) can be reduced to

$$\begin{aligned} \mathbf{x}(\mathbf{t}) &= \mathbf{f}(\tilde{\mathbf{s}}(\mathbf{t})) \\ \tilde{\mathbf{s}}(t+1) &= \mathbf{g}(\mathbf{s}(\mathbf{t}), \tilde{\mathbf{s}}(\mathbf{t})) \end{aligned} \tag{3.1}$$

In this case, matrix (2.11) becomes

$$\begin{aligned} & D_1\mathbf{g}\,\mathbf{Df} + \mathbf{D_2g} \\ = & \begin{pmatrix} 0 & d_1 & \cdots & d_1 \\ d_2 & 0 & \cdots & d_2 \\ \vdots & \vdots & & \vdots \\ d_N & d_N & \cdots & d_N \end{pmatrix} \begin{pmatrix} Df_1 & & & \\ & Df_2 & & \\ & & \ddots & \\ & & & Df_N \end{pmatrix} + \begin{pmatrix} 1-d_1 & & & \\ & 1-d_2 & & \\ & & \ddots & \\ & & & 1-d_N \end{pmatrix} \\ = & \Lambda + \mathbf{a}\,\mathbf{b^T}, \end{aligned}$$

with

$$\Lambda \;=\; \begin{pmatrix} 1-d_1-d_1Df_1 & & & \\ & 1-d_2-d_2Df_2 & & \\ & & \ddots & \\ & & & 1-d_N-d_NDf_N \end{pmatrix},$$

$$\mathbf{a} \;=\; \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} \;=\; \begin{pmatrix} Df_1 \\ Df_2 \\ \vdots \\ Df_N \end{pmatrix}.$$

The characteristic polynomial of this matrix is given as

$$\begin{aligned} \phi(\lambda) &= \det(\Lambda - \lambda\mathbf{I} + \mathbf{ab}^T) \\ &= \det(\Lambda - \lambda\mathbf{I}) \cdot \det(\mathbf{I} + (\Lambda - \lambda\mathbf{I})^{-1}\mathbf{ab}^T)\,. \end{aligned}$$

It is well known that if $\mathbf{I}$ is the $n \times n$ identity matrix, and $\mathbf{u}$ and $\mathbf{v}$ are n-dimensional real (or complex) vectors, then

$$\det(\mathbf{I} + \mathbf{uv}^T) \;=\; 1 + \mathbf{v}^T\mathbf{u},$$

which can be proved easily by using finite induction with respect to n. Therefore

$$\begin{aligned}\phi(\lambda) &= \det(\Lambda - \lambda\mathbf{I}) \cdot [1 + \mathbf{b}^T(\Lambda - \lambda\mathbf{I})^{-1}\mathbf{a}] \\ &= \prod_{k=1}^{N}(1 - d_k - d_k\, Df_k - \lambda)\left(1 + \sum_{k=1}^{N}\frac{d_k\, Df_k}{1 - d_k - d_k\, Df_k - \lambda}\right).\end{aligned} \tag{3.2}$$

Introduce the notation $\gamma_k = 1 - d_k - d_k\, Df_k$, $\xi_k = d_k\, Df_k$, and denote by $\eta_1 < \eta_2 < \cdots < \eta_s$ the distinct γ_k values with multiplicities $r_1, r_2, \ldots, r_s$. Let $I_j = \{i \,\big|\, \gamma_i = \eta_j\}$, and $\theta_j = \sum_{i \in I_j} \xi_i$ for $j = 1, 2, \ldots, s$. Then solving for the roots of (3.2) becomes finding the solution to equation

$$\prod_{j=1}^{s}(\lambda - \eta_j)^{r_j}\left(\sum_{j=1}^{s}\frac{\theta_j}{\lambda - \eta_j} - 1\right) = 0. \tag{3.3}$$

The left hand side is an Nth-degree polynomial with roots η_j with multiplicities $r_j - 1$ when $\theta_j \neq 0$, or with multiplicities r_j when $\theta_j = 0$. All other roots of (3.3) are the roots of function

$$G(\lambda) = \sum_{j=1}^{s}\frac{\theta_j}{\lambda - \eta_j} - 1. \tag{3.4}$$

Finding the roots of this function is equivalent to finding the solution of a polynomial equation of degree which equals the number of nonzero θ_j. It is easy to see that if all roots of $G(\cdot)$ are real, then (3.3) has N real roots.

The following result gives a characterization of the roots of $G(\cdot)$.

Lemma 3.1 *Assume that all $\theta_j \neq 0$, and the number of sign changes of sequence $\{\theta_1, \theta_2, \ldots, \theta_s\}$ is at most 1 in such a way that when the sign change occurs, it is from "$-$" to "$+$". Then (3.4) has exactly s real roots. If there is one sign change, then $s-2$ roots are in (η_1, η_s), one is in $(-\infty, \eta_1)$ and one is in (η_s, ∞). If there is no sign change, then $s-1$ roots are in (η_1, η_s), and the last root is in $(-\infty, \eta_1)$ when all $\theta_j < 0$, or in (η_s, ∞) when all $\theta_j > 0$.*

Proof: Under the assumption of the lemma, $G(\cdot)$ has the following properties.

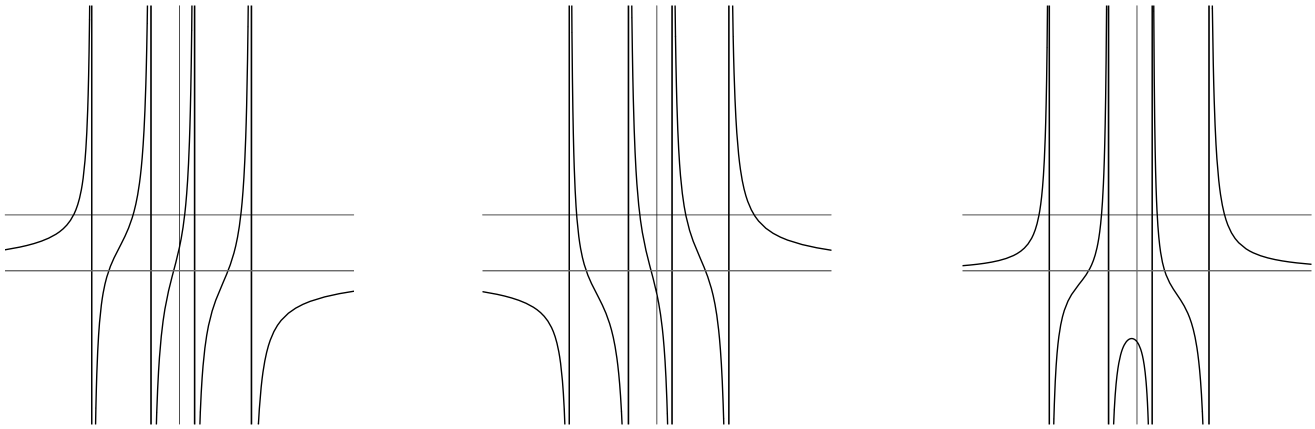
(a) $G(-\infty) = G(\infty) = -1$.

(b) If all $\theta_j < 0$, then $G(\lambda) > -1$ for any $\lambda < \eta_1$; $G(\lambda) < -1$ for any $\lambda > \eta_s$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = +\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = -\infty$.

(c) If all $\theta_j > 0$, then $G(\lambda) < -1$ for any $\lambda < \eta_1$; $G(\lambda) > -1$ for any $\lambda > \eta_s$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = -\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = +\infty$.

(d) If the only sign change is at l and $\theta_l < 0$ and $\theta_{l+1} > 0$, then $G(\lambda) < -1$ for any $\lambda \in (\eta_l, \eta_{l+1})$. Furthermore, $\lim_{\lambda \to \eta_j - 0} G(\lambda) = +\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = -\infty$ for $j \leq l$; $\lim_{\lambda \to \eta_j - 0} G(\lambda) = -\infty$, $\lim_{\lambda \to \eta_j + 0} G(\lambda) = +\infty$ for $j > l$.

Using the Intermediate Value Theorem and the above properties of $G(\cdot)$, it is easy to see that the graph of $G(\cdot)$ has only three patterns which correspond to the three cases as given before, and the graph intersects the horizontal axis exactly s times in all cases.



All $\theta_j < 0$ All $\theta_j > 0$ One sign change

□

From the above discussion, we know that the characteristic equation of matrix (2.11) has at most $N-2$ roots in the closed interval $[\eta_1, \eta_s]$. To ensure that all the eigenvalues have absolute values less than 1, we need that all $|\eta_j| < 1$ plus either $G(-1) < 0$ or $G(1) < 0$, or both $G(-1) < 0$ and $G(1) < 0$, fully depending whether there are either all $\theta_j \leq 0$ or all $\theta_j \geq 0$, or there is one sign change occuring in the way as stated in Lemma 3.1.

Theorem 3.2 *Assume that system (3.1) has an interior Nash-Cournot equilibrium and all conditions of Lemma 3.1 are met. Then any one of the following three cases ensures that the Nash-Cournot equilibrium is asymptotically stable:*

1. All $|\eta_j| < 1$, all $\theta_j \leq 0$, and $G(-1) < 0$.

2. All $|\eta_j| < 1$, all $\theta_j \geq 0$, and $G(1) < 0$.

3. All $|\eta_j| < 1$, $\theta_j \leq 0$ for $j < l$, $\theta_l < 0$, $\theta_{l+1} > 0$, $\theta_j \geq 0$ for $j > l+1$ and $G(-1) < 0$, $G(1) < 0$.

As a special case, assume that all $Df_k \in (-1, 0]$. Then all $\gamma_k = 1 - m_k - m_k\, Df_k \in [0, 1)$, and all $\xi_k = m_k\, Df_k \leq 0$. So all $|\eta_j| < 1$, and all $\theta_j = \sum_{i \in I_j} \xi_i \leq 0$. Therefore, we have

Corollary 3.3 *Assume that system (3.1) has an interior Nash-Cournot equilibrium. If for all k, $-1 < Df_k \leq 0$, and $G(-1) < 0$, then the equilibrium is asymptotically stable.*

Similarly to Theorem 3.2, one can prove the following necessary stability conditions.

Theorem 3.4 *Assume that system (3.1) has an interior stable Nash-Cournot equilibrium, all conditions of Lemma 3.1 are met, and $s \geq 2$.*

1. If all $\theta_j \leq 0$, then $|\eta_j| < 1$ for $j = 1, 2, ..., s-1$. Furthermore, either $\eta_s \leq 1$ and $G(-1) \leq 0$ or $\eta_s > 1$ and $G(-1) \leq 0$ and $G(1) \geq 0$;

2. If all $\theta_j \geq 0$, then $|\eta_j| < 1$ for $j = 2, ..., s$. Furthermore, either $\eta_1 \geq -1$ and $G(1) \leq 0$ or $\eta_1 < -1$ and $G(-1) \geq 0$ and $G(1) \leq 0$;

3. If $\theta_j \geq 0$ for $j < l$, $\theta_l < 0$, $\theta_{l+1} > 0$, $\theta_j \geq 0$ for $j > l+1$, then $|\eta_j| < 1$ for $j = 1, 2, ..., s$. Furthermore,$G(-1) \leq 0$ and $G(1) \leq 0$.

4 Stability in Labor Managed Oligopolies

As a particular example, we will discuss the asymptotical stability of the Nash-Cournot equilibrium of the single product case of labor managed oligopolies. Assume now that N firms produce a single good and the payoff function of each firm is the surplus per labor of the firm. Let p be the price (or inverse demand) function which is assumed to be a function of the total production of the industry. Let w be the competitive wage rate, and c_k the fixed cost of firm k. Let h_k be the number of labors in firm k as a function of the production level of firm k. Then the payoff function of firm k is given as

$$\varphi_k(x_1,\ \ldots,\ x_N) = \frac{x_k\, p(s) - w\, h_k(x_k) - c_k}{h_k(x_k)} \tag{4.1}$$

with $s = \sum_{i=1}^N x_i$. Assuming interior equilibrium, we have

$$\frac{\partial \varphi_k}{\partial x_k} = \frac{(p(s) + x_k\, p'(s))\, h_k(x_k) - (x_k\, p(s) - c_k)\, h_k'(x_k)}{h_k^2(x_k)} = 0. \tag{4.2}$$

The existence of the Nash-Cournot equilibrium has been proved under realistic conditions in Okuguchi [5]. From (4.2), we get

$$(p(s) + x_k\, p'(s))\, h_k(x_k) - (x_k\, p(s) - c_k)\, h_k'(x_k) = 0. \tag{4.3}$$

Assume that for each $s_k = \sum_{i \neq k} x_i$, there is a unique solution for x_k, $x_k = f_k(s_k)$. Using the chain rule to differentiate both sides of equality (4.3) with respect to s_k, we get

$$\begin{aligned}
&[p'(s)\,(1 + Df_k) + Df_k\, p'(s) + x_k\, p''(s)\,(1 + Df_k)]\, h_k(x_k) + [p(s) + x_k\, p'(s)]\, h_k'(x_k)\, Df_k \\
&\qquad - [Df_k\, p(s) + x_k p'(s)(1 + Df_k)]\, h_k'(x_k) - [x_k\, p(s) - c_k]\, h_k''\, Df_k \;=\; 0.
\end{aligned}$$

Solve for Df_k to get

$$\begin{aligned}
Df_k &= \frac{x_k\, p'(s)\, h_k'(x_k) - [p'(s) + x_k\, p''(s)]\, h_k(x_k)}{[2p'(s) + x_k\, p''(s)]\, h_k(x_k) - [x_k\, p(s) - c_k]\, h_k''(x_k)} \\
&= \frac{p'(s)\, [x_k\, h_k(x_k)]' - [x_k\, p(x_k)]''\, h_k(x_k)}{[x_k\, p(x_k)]''\, h_k(x_k) - [x_k\, p(s) - c_k]\, h_k''(x_k)}.
\end{aligned} \tag{4.4}$$

We can study the magnitudes of Df_k to draw conclusions on the stability or asymptotical stability of the labor-managed oligopoly by using the results in Section 3. As a special case, consider the linear case, when $p(s) = As + b$ with $A < 0$, $b > 0$, and $h_k(x_k) = a_k x_k + b_k$, with $a_k > 0$, $b_k \geq 0$ for all k. From (4.4), $Df_k = -\frac{b_k}{2h_k} \in (-1, 0]$. Thus Corollary 3.3 implies the following result:

Corollary 4.1 *Assume that a linear labor managed oligopoly has an interior Nash-Cournot equilibrium, and assume that* $G(-1) = \sum_{k=1}^N \frac{\frac{d_k b_k}{2h_k}}{2 - d_k\left(1 - \frac{b_k}{2h_k}\right)} - 1 < 0$ *at the equilibrium. Then the equilibrium is asymptotically stable. If* $G(-1) > 0$*, then the equilibrium is unstable.*

5 Economic Interpretation

Notice that the stability condition can be rewritten as

$$\sum_{k=1}^{N} \frac{\frac{d_k b_k}{2h_k}}{2 - d_k\left(1 - \frac{b_k}{2h_k}\right)} \quad < \quad 1. \tag{5.1}$$

Here $d_k \in (0,1]$, $h_k > 0$, $b_k \geq 0$. This condition holds if all the numbers $\frac{d_k b_k}{2h_k}$ are sufficiently small. It happens if for all k, either d_k or b_k is small enough, or h_k is sufficiently large. That is, either the speed of adjustment is small requiring slow speed in following the actual data in the adaptive expectation, or the labor requirement for producing zero output is small, or the labor requirement for producing equilibrium output is large enough. As a further special case assume symmetry, that is, assume that $d_k \equiv d$, $b_k \equiv b$, and $h_k \equiv h$. Then (5.1) reduces to the following:

$$\frac{\frac{dbN}{2h}}{2 - d\left(1 - \frac{b}{2h}\right)} \quad < \quad 1,$$

which can be further simplified as follows:

$$\frac{db}{2h}(N-1) + d \quad < \quad 2. \tag{5.2}$$

If b and h are given, then (5.2) is equivalent to the relation

$$d \quad < \quad \frac{2}{1 + \frac{b(N-1)}{2h}}\ . \tag{5.3}$$

If d and h are fixed, then (5.2) can be rewritten as

$$b \quad < \quad \frac{2-d}{\frac{d(N-1)}{2h}}\ , \tag{5.4}$$

and if b and d are given, then

$$h \quad > \quad \frac{db(N-1)}{2(2-d)}\ , \tag{5.5}$$

given actual upper bounds for the different variables. In the literature of labor managed oligopolies several authors assume that $h_k(0) = 0$ for all k. In this case $b_k = 0$, and therefore condition (5.1) is always satisfied showing that the interior equilibrium is always asymptotically stable.

6 Conclusions

In this paper dynamic labor managed oligopolies were examined with discrete time scales. In addition to deriving general stability conditions for the nonlinear case, the special case of linear price and labor functions was investigated in details. We have shown that the interior equilibrium is asymptotically stable if for all firms, either the speed of adjustment, or the labor requirement for zero output is sufficiently small, or the labor needed to produce the equilibrium output is large enough. Particular bounds have been derived for the relevant parameters.

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REDUCTION OF THE MODIFIED POINCARÉ DIFFERENTIAL EQUATION TO BIRKHOFF MATRIX FORM

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In this paper is given the reduction of the modified Poincaré linear differential equation with one n-tiple regular singularity to the Birkhoff canonical matrix form.

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1. Introduction

In the previous paper [1] is shown that the general Poincaré linear differential equation reduces to the following Cauchy matrix form $(xI - D)\frac{dY}{dx} = QY$, where I is $n \times n$ unit matrix, Q is $n \times n$ matrix of transformation, $D = diag(d_1, d_2, \cdots, d_n)$ and d_i, $(1 \le i \le n)$ a distinct regular singularities. If we substitute $d_i = 0$, $(1 \le i \le n)$ into the last equality, then very easy we find that the modified Poincaré linear differential equation with one n-tiple regular singularity reduces to the following Birkhoff matrix form $xI\frac{dY}{dx} = QY$, which means that Birkhoff canonical matrix system is a special case of the Cauchy matrix system. Just, this type of the reduction will be an object of investigation in the present paper.

2. Main Results

Now we will prove the following result.

Theorem. *The modified Poincaré differential equation*

$$x^n y^{(n)} = \sum_{i=1}^{n}\Big(\sum_{j=0}^{ri} a_{ij}x^j\Big)x^{n-i}y^{(n-i)}, \tag{1}$$

reduces to the Birkhoff matrix form

$$xI\frac{dU}{dx} = Q(x)U. \tag{2}$$

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by the linear transformation with polynomials in x^{-1} as its coefficients

$$Q(x)=\begin{bmatrix} q_{11}(x) & x^r & 0 & \cdots & 0 & 0 \\ q_{21}(x) & q_{22}(x) & x^r & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ q_{n-1,1}(x) & q_{n-1,2}(x) & q_{n-1,3}(x) & \cdots & q_{n-1,n-1}(x) & x^r \\ q_{n1}(x) & q_{n2}(x) & q_{n3}(x) & \cdots & q_{n,n-1}(x) & q_{nn}(x) \end{bmatrix}, \tag{3}$$

where

$$\begin{gathered} q_{ii}(x)=\rho_i-(i-1)r+q_{ii}^1x+\cdots+q_{ii}^{r-1}x^{r-1}, \quad (i\neq n) \\ q_{ij}(x)=q_{ij}^0+q_{ij}^1x+\cdots+q_{ij}^{r-1}x^{r-1}, \quad (i>j;i\neq n) \\ \\ q_{nj}(x)=q_{nj}^0+q_{nj}^1x+\cdots+q_{nj}^{r-1}x^{r-1}+a_{n+1-j,r(n+1-j)}x^r, \\ q_{nn}(x)=\rho_n-(n-1)r+q_{nn}^1x+\cdots+q_{nn}^{r-1}x^{r-1}+a_{1r}x^r. \end{gathered} \tag{4}$$

Proof. For the equation (1) the characteristic constants ρ_i, $(1\leq i\leq n)$ of the regular singularity at the coordinate origin $x=0$, are given as roots of the equation

$$I(\rho)\equiv[\rho]_n-\sum_{i=1}^n a_{i0}[\rho]_{n-i}=0, \quad [\rho]_k=\rho(\rho-1)\cdots(\rho-k+1), \quad [\rho]_0=1$$

such that $\rho_i\neq\rho_j(mod1)$, $(i\neq j;1\leq i,j\leq n)$ and the principal characteristic constants λ_i, $(1\leq i\leq n)$ of the irregular singularity of rank 1 at infinity $x=\infty$, are given as roots of the equation

$$J(\lambda)\equiv\lambda^n-\sum_{i=1}^n a_{i,ri}\lambda^{n-i}=0.$$

¿From the equality

$$y_p(x)=x^{-(r-1)p}y^{(p)}, \tag{5}$$

after differentiation yields to

$$xy_p'+(r-1)py_p=x^ry_{p+1}, \quad (0\leq p\leq n-1). \tag{6}$$

If we multiply the equation (1) by x^{-rn}, then it obtains the form

$$y_n=\sum_{i=1}^n F_i(x)y_{n-i}, \tag{7}$$

where

$$F_i(x)=\sum_{j=0}^{ri}a_{ij}x^{j-ri}, \quad (1\leq i\leq n). \tag{8}$$

Now according to (6) and (7), we obtain

$$DY = D \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} s_1 & x^r & 0 & \cdots & 0 & 0 \\ 0 & s_2 & x^r & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & s_{n-1} & x^r \\ x^r F_n(x) & x^r F_{n-1}(x) & x^r F_{n-2}(x) & \cdots & x^r F_2(x) & s_n + x^r F_1(x) \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix} =$$

$$= L(x)Y, \qquad (D \equiv x\frac{d}{dx}; \quad s_i = -(i-1)(r-1); \quad 1 \le i \le n). \tag{9}$$

By the substitution of Turrittin [2],

$$U = C(x)Y, \qquad C(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{21}(x) & 1 & 0 & \cdots & 0 & 0 \\ c_{31}(x) & c_{32}(x) & 1 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{n1}(x) & c_{n2}(x) & c_{n3}(x) & \cdots & c_{n,n-1}(x) & 1 \end{bmatrix} \tag{10}$$

we obtain the equality

$$DU = [DC(x) + C(x)L(x)]C^{-1}(x)U = Q(x)U, \tag{11}$$

and hence it follows that

$$DC(x) + C(x)L(x) = Q(x)C(x). \tag{12}$$

From (12) can be determined only $c_{ij}(x)$, $(i > j)$ such that all elements of the matrix $Q(x)$ are polynomials of the required form (4).

If we substitute

$$C(x)L(x) = \Big[p_{ij}(x)\Big]_{n\times n} \quad \text{and} \quad Q(x)C(x) = \Big[t_{ij}(x)\Big]_{n\times n},$$

then we obtain

$$p_{ij}(x) = x^r c_{i,j-1}(x) + s_j c_{ij}, \quad (1 \le i \le n-1) \tag{13}$$

$$p_{nj}(x) = x^r c_{n,j-1}(x) + s_j c_{nj}(x) + x^r F_{n-j+1}(x)$$

and

$$t_{ij}(x) = q_{ij}(x) + \sum_{\nu=j+1}^{i} q_{i\nu}(x) c_{\nu j}(x) + x^r c_{i+1,j}(x), \tag{14}$$

where it is

$$c_{ii}(x) = 1; \quad c_{ij}(x) = 0, \quad (j > i); \quad c_{i0}(x) = 0.$$

¿From (12) it also follows that

$$Dc_{ij}(x) + p_{ij}(x) = t_{ij}(x), \quad (j < i + 2). \tag{15}$$

These relations are identically satisfied for the elements over the diagonal, because it is $p_{i,i+1}(x) = x^r$ and $t_{i,i+1} = x^r$.

For the diagonal elements

$$p_{ii}(x) = t_{ii}(x),$$

it follows that

$$x^r c_{i,i-1}(x) + s_i = q_{ii}(x) + x^r c_{i+1,i}(x), \quad (1 \leq i \leq n-1) \tag{16}$$

$$x^r c_{n,n-1}(x) + s_n + x^r F_1(x) = q_{nn}(x), \tag{17}$$

and for the j-th subdiagonal elements we obtain

$$Dc_{i,i-j}(x) + x^r c_{i,i-j-1}(x) + s_{i-j} c_{i,i-j}(x) =$$

$$= q_{i,i-j}(x) + \sum_{\nu=0}^{j-1} q_{i,i-\nu}(x) c_{i-\nu,i-j}(x) + x^r c_{i+1,i-j}(x), \quad (2 \leq i \leq n-1) \tag{18}$$

$$Dc_{n,n-j}(x) + x^r c_{n,n-j-1}(x) + s_{n-j} c_{n,n-j}(x) + x^r F_{j+1}(x) =$$

$$= q_{n,n-j}(x) + \sum_{\nu=0}^{j-1} q_{n,n-\nu}(x) c_{n-\nu,n-j}(x), \quad (1 \leq j \leq n-1). \tag{19}$$

¿From the equalities (16) - (19) can be considered the subdiagonal elements of $C(x)$ which can be determined as polynomials of the same degree. Now we will substitute

$$c_{i,i-j}(x) = x^{-r} c^1_{i,i-j}(x) + x^{-2r} c^2_{i,i-j}(x) + \cdots + x^{-jr} c^j_{i,i-j}(x) \tag{20}$$

$$(2 \leq i \leq n; \quad 1 \leq j \leq i-1)$$

where $c^m_{i,i-j}(x)$, $(1 \leq m \leq j)$ are polynomials of x of degree in the best case $r-1$. The coefficients $F_i(x)$, $(1 \leq i \leq n)$ can be represented in the form

$$F_i(x) = a_{i,ri} + x^{-r} F^1_i(x) + x^{-2r} F^2_i(x) + \cdots + x^{-ir} F^i_i(x), \quad (1 \leq i \leq n) \tag{21}$$

where $F^m_i(x)$, $(1 \leq m \leq i)$ are polynomials of degree $r-1$, i.e.

$$F^m_i(x) = F^m_i(0) + F^m_i(1)x + \cdots + F^m_i(r-1)x^{r-1}. \tag{22}$$

Particularly, it is $F_i^i(0) = a_{i0}$, $(1 \le i \le n)$. Now we will introduce the following remarks

$$D\Big[x^{-mr}c^m(x)\Big] = D\Big(x^{-mr}\sum_{j=0}^{r-1}\mu_j x^j\Big) =$$

$$= x^{-mr}\Big[\sum_{j=0}^{r-1}\mu_j(-mr+j)x^j\Big] \equiv x^{-mr}D_{mr}c^m(x),$$

$$q(x)c(x) = \Big(\sum_{j=0}^{r} t_j x^j\Big)\Big(\sum_{j=0}^{r-1}\mu_j x^j\Big) =$$

$$= x^r\Big[(t_r\mu_0 + t_{r-1}\mu_1 + \cdots + t_1\mu_{r-1}) + (t_r\mu_1 + t_{r-1}\mu_2 + \cdots + t_2\mu_{r-1})x + \cdots + (t_r\mu_{r-1})x^{r-1}\Big] +$$

$$+\Big[(t_0\mu_0) + (t_0\mu_1 + t_1\mu_0)x + \cdots + (t_0\mu_{r-1} + t_1\mu_{r-2} + \cdots + t_{r-1}\mu_0)x^{r-1}\Big] \equiv$$

$$\equiv x^r[q(x)c(x)]^0 + [q(x)c(x)]^1.$$

Substituting (20) and (21) into (18) and (19), we obtain

$$D_{mr}c^m_{i,i-j} + s_{i-j}c^m_{i,i-j} + c^{m+1}_{i,i-j-1} =$$

$$= \sum_{\nu=0}^{r-1}\Big\{\Big[q_{i,i-\nu}c^m_{i-\nu,i-j}\Big]^1 + \Big[q_{i,i-\nu}c^{m+1}_{i-\nu,i-j}\Big]^0\Big\} + c^{m+1}_{i+1,i-j}, \tag{23}$$

$$q_{i,i-j} = c^1_{i,i-j-1} - \sum_{\nu=0}^{j-1}\Big[q_{i,i-\nu}c^1_{i-\nu,i-j}\Big]^0 - c^1_{i+1,i-j}, \tag{24}$$

$$D_{mr}c^m_{n,n-j} + s_{n-j}c^m_{n,n-j} + c^{m-1}_{n,n-j-1} + F^{m+1}_{j+1} =$$

$$= \sum_{\nu=0}^{r-1}\Big\{\Big[q_{n,n-\nu}c^m_{n-\nu,n-j}\Big]^1 + \Big[q_{n,n-\nu}c^{m+1}_{n-\nu,n-j}\Big]^0\Big\}, \tag{25}$$

$$q_{n,n-j} = c^1_{n,n-j-1} + F^1_{j+1} + a_{j+1,r(j+1)}x^r - \sum_{\nu=0}^{j-1}\Big[q_{n,n-\nu}c^1_{n-\nu,n-j}\Big]^0, \quad (1 \le m \le j) \tag{26}$$

where

$$c^m_{i-\nu,i-j} = 0, \qquad (m > j-\nu).$$

¿From (26) it just follows that

$$q_{n,n-j} = a_{j+1,r(j+1)}x^r + \cdots, \qquad (0 \le j \le n-1). \tag{27}$$

Now we are able to determine all coefficients of the polynomials $c_{ij}^m(x)$ and $q_{ij}(x)$. So, suppose that $\xi_0^1, \xi_0^2, \cdots, \xi_0^n$ are constants, such that holds the following equality

$$[\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu). \tag{28}$$

Now we choose an unknown variable ζ which satisfies the following equalities

$$\begin{gathered} \xi_0^1 + [\zeta - (n-1)] = \mu_0^1 \\ \xi_0^j + [\zeta - (n-j)]\mu_0^{j-1} = \mu_0^j, \quad (2 \le j \le n-1) \\ \zeta\mu_0^{n-1} + \xi_0^n = 0. \end{gathered} \tag{29}$$

Let ζ be one of the values of ρ_ν, for example let $\rho = \rho_n$, such that it holds

$$\begin{vmatrix} \mu_0^1 + \rho - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho \end{vmatrix} =$$

$$= [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n}(\rho - \rho_\nu). \tag{30}$$

It is easy to verify the validity of (28). Since

$$\begin{vmatrix} \mu_0^1 + y_1 & -1 & 0 & \cdots & 0 \\ \mu_0^2 & y_2 & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & y_{n-1} \end{vmatrix} =$$

$$= y_1y_2\cdots y_{n-1} + \mu_0^1 y_2y_3\cdots y_{n-1} + \mu_0^2 y_3y_4\cdots y_{n-1} + \cdots + \mu_0^{n-1},$$

the determinant can be denoted by $f(\rho)$,

$$f(\rho) = [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1}$$

and

$$(\rho - \rho_n)f(\rho) = [\rho]_n - (\rho_n - n + 1)[\rho]_{n-1} + \mu_0^1[\rho]_{n-1} - (\rho_n - n + 2)\mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1}[\rho]_1 - \rho_n\mu_0^{n-1} =$$

$$= [\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho-\rho_\nu) = (\rho-\rho_n)\prod_{\nu=1}^{n-1}(\rho-\rho_\nu).$$

Now we will start by determination of $c^j_{n,n-j}(x)$, $(1 \le j \le n-1)$ and $q_{nn}(x)$. From (17) and (25) we obtain

$$c^1_{n,n-1} + s_n + F^1_1 = q_{nn} - a_{1r}x^r,$$

$$D_{jr}c^j_{n,n-j} + s_{n-j}c^j_{n,n-j} + c^{j+1}_{n,n-j-1} + F^{j+1}_{j+1} = [q_{nn}c^j_{n,n-j}]^1, \qquad (1 \le j \le n-1). \tag{31}$$

If we substitute

$$c^j_{n,n-j} = (\mu^j_0 + \mu^j_1 x + \cdots + \mu^j_{r-1}x^{r-1}), \quad (1 \le j \le n-1)$$

$$q_{nn} - a_{1r}x^r = t_0 + t_1x + \cdots + t_{r-1}x^{r-1},$$

then from (31) it follows that

$$\mu^1_0 = t_0 - s_n - a_{10} = t_0 + (n-1)r - (n-1) - a_{10}, \tag{32}$$

$$\mu^1_m = t_m - F^1_1(m), \quad (1 \le m \le r-1) \tag{33}$$

$$\begin{aligned}\mu^{j+1}_0 &= (t_0 + jr - s_{n-j})\mu^j_0 - a_{j+1,0} = \\ &= [t_0 + (n-1)r - (n-r-1)]\mu^j_0 - a_{j+1,0},\end{aligned} \tag{34}$$

$$\begin{aligned}\mu^{j+1}_m &= [t_0 + (n-1)r - m - (n-j-1)]\mu^j_m + t_m\mu^j_0 + \\ &+ (t_1\mu^j_{m-1} + \cdots + t_{m-1}\mu^j_1) - F^{j+1}_{j+1}(m) \\ &(1 \le j \le n-1; 1 \le m \le r-1),\end{aligned} \tag{35}$$

where $\mu^n_m = 0$. By application of (28),(29) and (30) into (32) and (34), then it just follows that $t_0 + (n-1)r$ is one of the roots of ρ_i, $(1 \le i \le n)$ of the equation

$$[\rho]_n - \sum_{i=1}^{n} a_{i0}[\rho]_{n-i} = 0.$$

By substitution

$$t_0 + (n-1)r = \rho_n, \tag{36}$$

can be determined μ^j_0, $(1 \le j \le n-1)$. Using (33) and (35) we will determine μ^j_m, $(1 \le j \le n-1)$ and successively t_m. According to the equation (30), the determinant

$$\begin{vmatrix} \mu^1_0 + \rho_n - m - (n-2) & -1 & 0 & \cdots & 0 \\ \mu^2_0 & \rho_n - m - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu^{n-2}_0 & 0 & 0 & \cdots & -1 \\ \mu^{n-1}_0 & 0 & 0 & \cdots & \rho_n - m \end{vmatrix} =$$

$$= [\rho_n - m]_{n-1} + \mu_0^1[\rho_n - m]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho_n - m - \rho_\nu)$$

is again different from zero, which means that only the coefficients $c^j_{n,n-j}(x)$ and $q_{nn}(x)$ can be determined.

Further successively will be determined the coefficients $c^j_{i,i-j}(x)$, $(1 \le j \le i-1)$ and $q_{ii}(x)$. ¿From (16) and (23), it follows that

$$c^1_{i,i-1} + s_i = q_{ii} + c^1_{i+1,i}$$

$$D_{jr}c^j_{i,i-j} + s_{i-j}c^j_{i,i-j} + c^{j+1}_{i,i-j-1} = [q_{ii}c^j_{i,i-j}]^1 + c^{j+1}_{i+1,i-j}, \quad (1 \le j \le i-1). \tag{37}$$

Further we apply the mathematical induction. Let

$$c^{j+1}_{i+1,i-j}(x) = \xi_0^{j+1} + \xi_1^{j+1}x + \cdots + \xi_{r-1}^{j+1}x^{r-1}, \qquad (0 \le j \le i-1)$$

be polynomials and let the constants ξ_0^j, $(1 \le j \le i)$ satisfy the equation

$$[\rho]_i + \xi_0^1[\rho]_{i-1} + \cdots + \xi_0^i = \prod_{\nu=1}^{i}(\rho - \rho_\nu). \tag{38}$$

By the changes

$$c^j_{i,i-j} = (\mu_0^j + \mu_1^j x + \cdots + \mu_{r-1}^j x^{r-1}), \qquad (1 \le j \le i-1)$$

$$q_{ii} = (t_0 + t_1 x + \cdots + t_{r-1}x^{r-1}),$$

we obtain

$$\mu_0^1 = t_0 + (i-1)r - (i-1) + \xi_0^1, \tag{39}$$

$$\mu_m^1 = t_m + \xi_m^1, \quad (1 \le m \le r-1) \tag{40}$$

$$\mu_0^{j+1} = [t_0 + (i-1)r - (i-j-1)]\mu_0^j + \xi_0^{j+1}, \tag{41}$$

$$\mu_m^{j+1} = [t_0 + (i-1)r - m - (i-j-1)]\mu_m^j + t_m\mu_0^j + (t_1\mu_{m-1}^j + \cdots + t_{m-1}\mu_1^j) + \xi_m^{j+1} \tag{42}$$

$$(1 \le j \le i-1; \quad 1 \le m \le r-1)$$

where $\mu_m^i = 0$. Applying the equation (28)-(30) to the equations (39)-(40), it follows

$$t_0 + (i-1)r = \rho_i \tag{43}$$

and

$$[\rho]_{i-1} + \mu_0^1[\rho]_{i-2} + \cdots + \mu_0^{i-1} = \prod_{\nu=1}^{i-1}(\rho - \rho_\nu). \tag{44}$$

Again, assuming that (3) and (4) hold, can be determined $c^j_{i,i-j}(x)$, $(1 \leq j \leq i-1)$ and $q_{ii}(x)$. Thus by mathematical induction we proved that for each i, $(1 \leq i \leq n)$, can be determined $c^j_{i,i-j}(x)$, $(1 \leq j \leq i-1)$ and $q_{ii}(x)$. By using of (36), (38) (43) and (44), it obtains also that $q_{ii}(x)$ has the form

$$q_{ii}(x) = \rho_i - (i-1)r + q^1_{ii}x + \cdots + q^{r-1}_{ii}x^{r-1} + q^r_{ii}x^r, \quad (1 \leq i \leq n)$$

where $q^r_{ii} = 0$, $(i \neq n)$ and $q^r_{nn} = a_{1r}$.

Finally, we will prove that the sets of polynomials

$$\{q_{n,n-j}(x), c^m_{n-j-m}(x); \quad 1 \leq m \leq n-j-1\}$$

and

$$\{q_{i,i-j}(x), c^m_{i,i-j-m}(x); \quad 1 \leq m \leq i-j-1\}, \quad (n-1 \geq i \geq 1)$$

can be determined successively for values n and $(1 \leq j \leq n-1)$. It will be proven by induction. Let the sets $\{q_{n,n-\nu}, c^m_{n-\nu-m}\}$ and $\{q_{i,i-\nu}, c^m_{i-\nu-m}\}$, $(1 \leq i \leq n-1)$ are known for $0 \leq \nu \leq j-1$. Then from (25) we obtain

$$D_{mr}c^m_{n,n-j-m} + s_{n-j-m}c^m_{n,n-j-m} + c^{m+1}_{n,n-j-m-1} =$$

$$= [q_{nn}c^m_{n,n-j-m}]^1 + [q_{n,n-j}c^m_{n-j,n-j-m}]^1 +$$

$$+ \sum_{\nu=1}^{j-1}[q_{n,n-\nu}c^m_{n-\nu,n-j-m}]^1 + \sum_{\nu=1}^{j-1}[q_{n,n-\nu}c^{m+1}_{n-\nu,n-j-m}]^0 - F^{m+1}_{j+m+1}, \qquad (1 \leq m \leq n-j-1). \quad (45)$$

By the change

$$c^j_{n-j,n-j-m} = \xi^m_0 + \xi^m_1 x + \cdots + \xi^m_{r-1}x^{r-1}, \qquad (1 \leq m \leq n-j-1)$$

and using (38) and (44), we obtain

$$[\rho]_{n-j-1} + \xi^1_0[\rho]_{n-j-2} + \cdots + \xi^{n-j-1}_0 = \prod_{i=1}^{n-j-1}(\rho - \rho_i). \quad (46)$$

If we substitute

$$q_{n,n-j} = t_0 + t_1 x + \cdots + t_{r-1}x^{r-1},$$

$$c^m_{n,n-j-m} = \mu^m_0 + \mu^m_1 x + \cdots + \mu^m_{r-1}x^{r-1}, \qquad (1 \leq m \leq n-j-1)$$

then from (26) follow the equalities

$$t_r = a_{j+1,r(j+1)}, \quad (47)$$

$$t_\nu = \mu^1_\nu + \cdots, \qquad (0 \leq \nu \leq r-1) \quad (48)$$

and from (45) we obtain

$$(-mr + \nu + s_{n-j-m})\mu^m_\nu + \mu^{m+1}_\nu =$$

$$= (p_0\mu^m_\nu + \cdots + p_\nu\mu^m_0) + (t_0\xi^m_\nu + \cdots + t_\nu\xi^m_0) + \cdots \quad (49)$$

$$(1 \leq m \leq n-j-1; \quad 0 \leq \nu \leq r-1)$$

where we can substitute

$$q_{nn} = p_0 + p_1 x + \cdots + p_{r-1} x^{r-1} + p_r x^r, \qquad (p_0 = \rho_n - (n-1)r).$$

The equation (49) can be represented in the following form

$$\mu_\nu^{m+1} = [\rho_n - jr - \nu - (n-j-m-1)]\mu_\nu^m + t_\nu \xi_0^m +$$
$$+(p_1 \mu_{\nu-1}^m + \cdots + p_\nu \mu_0^m) + (t_0 \xi_\nu^m + \cdots + t_{\nu-1}\xi_1^m) + \cdots \tag{50}$$

and hence we obtain the determinant

$$\begin{vmatrix} \xi_0^1 + \rho_n - jr - \nu - (n-j-2) & -1 & 0 & \cdots & 0 \\ \xi_0^2 & \rho_n - jr - \nu - (n-j-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \xi_0^{n-j-2} & 0 & 0 & \cdots & -1 \\ \xi_0^{n-j-1} & 0 & 0 & \cdots & \rho_n - jr - \nu \end{vmatrix} =$$

$$= [\rho_n - jr - \nu]_{n-j-1} + \xi_0^1 [\rho_n - jr - \nu]_{n-j-2} + \cdots + \xi_0^{n-j-1} =$$

$$= \prod_{i=1}^{n-j-1} (\rho_n - rj - \nu - \rho_i) \neq 0.$$

Thus, the set $\{q_{n,n-j}, c_{n,n-j-m}^m\}$ is determined uniquely. In order to prove that the sets $\{q_{i,i-\nu},\ c_{i,i-\nu-m}^m\}$, $(1 \leq i \leq n-1)$ can uniquely be determined, we apply again mathematical induction. The evaluation of their proof is not need here, since such sets of polynomials are just the same as the previous. ∎

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Space-time distributions

Mihaela D. Iftime

Abstract

The space-time foliation Σ compatible with the gravitational field g on a 4-manifold M determines a fibration π of M, $\pi : M \longrightarrow \mathcal{N}$ is surjective submersion over the 1-dimensional leaves space $\mathcal{N}$. M is then written as a disjoint union of the leaves of Σ, which are 3-dimensional spacelike surfaces on M.

The decomposition, $TM = \Sigma \oplus T^0M$, also implies that we can define a lift of the curves on $\mathcal{N}$ to curves (non-spacelike) on M.

The stable causality condition holds on M [2] means that Σ is a causal space-time distribution, generated by an exact timelike 1-form $\omega^0 = \mathrm{d}t$ where t is some real function on M. In this case M is written as a disjoint union of a family of spacelike 3-surfaces of constant t, which cover $D^+(S)$ of a initial 3-surface S of M.

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1 Introduction

Note: We will denote by small Latin indices $a, b, \ldots$ numbers ranging from 0 to 3 and the small Greek indices $\alpha, \beta, \ldots$ numbers ranging from 1 to 3.

The space-time structure on a 4-dimentional manifold M is a uniquely defined by a triplet (g^R, g, Σ), where g and g^R represent a Riemannian, respectively a Lorentzian metric on M and $\Sigma \subset \tau_M$, the space-time distribution on M compatible with the gravitational field g, is a 3-dimentional spatial tangent subbundle on M orthocomplement (with respect both g and g^R) of the 1-dimentional timelike subbundle T^0M generated by the timelike globally defined vector field $h^0 \in \mathcal{X}(M)$, $g^R(h^0\,, h^0) = -g(h^0\,, h^0) = 1$ which indicates the time orientation (locally) at every point of M.

If g^R is a Riemannian metric on M (which exists on every paracompact space) and a time-orientation defined by $h^0 \in \mathcal{X}(M)$ then we have a Lorentz metric on M determined by the expression :

$$(1) \quad g(X,\, Y) = g^R(X,\, Y) - 2g^R(X,\, h^0)g^R(Y,\, h^0) \quad \forall X,\, Y \in \mathcal{X}(M)$$

Locally, relating to an orthonormal frame $\{e_a\} = \{h^\alpha,\, h^0\}$ we find:

$$(2) \quad g_{ab} = g^R_{ab} - 2\omega^0_a\omega^0_b,$$

where $\omega^0 = g^R(h^0,\, \cdot)$ is the unitary timelike 1-form.

The Pfaff equation $\omega^0 = 0$ generates the 3-dimensional spatial spaces Σ_p in each fibre T_pM of the tangent bundle τ_M and these subspaces are orthogonal to the timelike field h^0 in $p \in M$.

The tensor

$$(3) \quad h_{ab} = g_{ab} + \omega^0_a\omega^0_b$$

defines a Riemannian metric on the tangent subbundle Σ.

We have the following decomposition of tangent bundle:

$$(4) \quad TM = \Sigma \oplus T^0M$$

So, given a Riemannian metric g^R_{ab} on M, each 3-dimensional distribution Σ on M generated by a nonzero 1-form σ on M represents a space-time distribution compatible with a gravitational field g defined by the expression (2), where we take $\omega^0 = -\frac{\sigma}{g^R(\sigma,\sigma)}$.

Inverse, given a gravitational field g on M we find a spacetime distribution Σ compatible with g and a Riemannian metric g^R which satisfy the expression (2).

Two such different triplets $(g_i,\, g_i^R,\, \Sigma_i),\ \ i = 1,\, 2$ will describe the same space-time structure on M if the space-time distributions coincide, $\Sigma_1 = \Sigma_2$.

The geometric equivalence principle — which states that we may introduce an orthonormal tetrad field $\{h^\alpha,\, h^0\}$ at each point of M — is also sufficient for defining the space-time structure on a 4-manifold M.

The metric g_{ab} can be transformed to

(5) $\quad g_{ab} = \eta_{ab} = diag(1,\, 1,\, 1,\, -1)$

at any point of M.

Also for any point $p \in M$ and its normal convex neighbourhood $U_p \subset M,\ U_p = \exp_p W_0$, where $W_0 \subset T_pM$ is an open neighbourhood of the origin of T_pM, we can have a causal structure; all events q in U_p are divided into three groups according whether the causal relation to p is timelike, null or spacelike and these relations are independent of the choice of coordinate system. But we don't know how to split an arbitrary space-time in space and time globally.

An integrable distribution

$$\Sigma = \bigcup_{p \in M} \Sigma_p$$

is a spacetime foliation of spatial hypersurfaces S of M; $\Sigma_p = T_pS,\ \forall p \in S$.

In this case, at every point $p \in M$ we can define a local coordinate neighbourhood $U_i \subset M$ with local adapted coordinates (x^a) so that the spacelike vector fields $\{\frac{\partial}{\partial x^\alpha}\}$ generate Σ locally.

The Pfaff equation $x^0 =$ const. locally defines the spatial hypersurfaces S of M (i.e. the maximal integral manifolds or the leaves of Σ).

We have,

$$\mathcal{U} = \{U_i,\ t_i : \tau_M^{-1}(U_i) \longrightarrow U_i \times \mathrm{R}^4\}$$

a bundle atlas on TM which is compatible with the subbundle $\Sigma \subset \tau_M$, so that

$$\mathcal{U}^\Sigma = \{U_i,\ t_i^{-1} : \tau_M^{-1}(U_i) \cap \Sigma \longrightarrow U_i \times \mathrm{R}^3\}$$

is a bundle atlas on Σ, $t_i(\tau_M^{-1}(U_i) \cap \Sigma) = U_i \times \mathrm{R}^3$.

We can have directly: for any adapted local coordinate system $(U_i,\, (x^a))$, then $(x^a,\, \dot{x}^a)$ are local bundle coordinates on TM and $\{\frac{\partial}{\partial x^a}\}$ are the dual sections. Then, from the local trivialisation of τ_M

(6) $\quad t_i = (\tau_{M_{|\tau_M^{-1}(U_i)}},\, \dot{x}^a) : \tau_M^{-1}(U_i) \longrightarrow U_i \times \mathrm{R}^4$

we obtain a local trivialisation on subbundle Σ

(7) $\quad t_i^\Sigma = (\tau_{M_{|\tau_M^{-1}(U_i)\cap\Sigma}},\, \dot{x}^\alpha) : \tau_M^{-1}(U_i) \cap \Sigma \longrightarrow U_i \times \mathrm{R}^3.$

2 Family of spatial hypersurfaces of spacetime

A space-time foliation Σ on M generates a fibration π as follows: we define an equivalence relation on M, $x, y \in M$, $x \sim y$ if x, y are in the same leaf of the foliation.

We denote by $\mathcal{N} = M_{|\sim}$ the space of leaves which has the natural quotient structure of 1-dimensional differentiable manifold, so that $\pi : M \longrightarrow \mathcal{N}$ is a surjective submersion. The leaves of the foliation Σ coincide with the fibres of π, $\pi^{-1}(\nu) = S_\nu$ and $\Sigma = \mathrm{Ker}\pi_*$. We also may suppose that the fibres of π are simply connected even if it is not a necessary condition.

We can define an vertical lift of the tangent vectors on $\mathcal{N}$ to tangent vectors on M which belong to Σ.

For $\bar{X} \in T_\nu\mathcal{N}$ its vertical lift is $X^v \in T_pM$, $X^{va} = h^a_b(p)X^b$, $\forall X \in T_pM$, where $\pi_*X = \bar{X}$.

Here

$$(8) \quad h = (h^a_b) : TM \longrightarrow TM$$
$$h^a_b = g^{ac}h_{cb}$$

is an endomorphism on τ_M, a projection operator, i.e. $h^a_b h^b_c = h^a_c$, which projects any vector $X \in T_pM$ into its part lying in the subspace Σ_p,

$$(9) \quad X^a = h^a_b X^b - h^{0a}\omega^0_b X^b$$

and any $X \in \mathcal{X}(\mathcal{N})$ then $X^v \in \mathcal{V}(\Sigma)$ defined by $X^v = h(X)$ for any $X \in \mathcal{X}(M)$ so that $\pi_*X = \bar{X}$.

Here $\mathcal{V}(\Sigma) = \{X \in \mathcal{X}_{loc}(M) \ / X_p \in \Sigma_p, \forall p\}$.

The Lorentzian (metric) connection ∇ on M induces by restriction a connection ∇^S on each leaf of Σ ,

$$(10) \quad \nabla_{X^v}Y^v = \nabla^S_{X^v}h(Y) = h(\nabla_{X^v}Y) \in \Sigma.$$

We used

$$(11) \quad 0 = (\nabla_{X^v}h)Y = \nabla_{X^v}h(Y) - h(\nabla_{X^v}Y)$$

and

$$(12) \quad h_{ab;c} = (g_{ab} + \omega^0_a\omega^0_b)_{;c} = 0$$

It is worth notice that the space-time is a disjoint union of the leaves of the foliation Σ, i.e.

$$M = \bigcup_{\nu \in \mathcal{N}} S_\nu$$

where $S_\nu = \pi^{-1}(\nu)$ represents a family of spatial hypersurfaces of M. From decomposition (4) we can define a transport of the those fibres S_ν along the curves $\gamma(\tau)$ on $\mathcal{N}$. That is to say, a curve $\gamma(\tau) : [0, 1] \longrightarrow \mathcal{N}$, $\gamma(0) = \nu_0$, $\gamma(1) = \nu_1$ could be lifted on M from each point of the fibre $x_i \in \pi^{-1}(\nu_0) = S$ to a nonspatial future directed curve on M of end-point $y_i \in \pi^{-1}(\nu_1) = S_{\nu_1}$.

This way we obtain an 1:1 correspondence between S and S_{ν_1}. Then we have a family of maps (homeomorphisms)

(13) $\alpha_\gamma : S_{\gamma(0)} \longmapsto S_{\gamma(1)}, \quad \alpha_\gamma : x_i \longrightarrow y_i$

satisfying natural conditions: $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$, $\alpha_{\gamma^{-1}} = (\alpha_\gamma)^{-1}$, $\alpha_\gamma = \mathrm{id}$ for $\gamma = \text{const.}$, α_γ is independent of the choice of the parameter τ on γ.

On the other hand $\mathcal{N}$ is 1-dimensional (connected) manifold. This implies that $\mathcal{N}$ must be diffeomorphic with an interval $[0, 1]$ or with circle S^1.

Supposing $\mathcal{N}$ has a fixed orientation given by the unique unitary $\bar{X} \in \mathcal{X}(\mathcal{N})$, then $\mathcal{N} = \gamma_{\bar{X}}([0, 1])$, where $\gamma_{\bar{X}} : [0, 1] \longrightarrow \mathcal{N}$ is the integral curve of $\bar{X}$. We will have the two cases: when $\gamma_{\bar{X}}$ is an injective integral curve, $\mathcal{N} = [0, 1]$ and when $\gamma_{\bar{X}}$ admits cross-points (i. e. $\gamma_{\bar{X}}$ may intersect itself), $\mathcal{N} = S^1$.

Lifting $\gamma_{\bar{X}}$ from each point $x_i \in \pi^{-1}(\nu_0) = S$ we obtain a family of timelike future directed curves $\tilde{\gamma}_{\bar{X}} : [0, 1] \longrightarrow M$, $\dot{\tilde{\gamma}}_{\bar{X}}(\tau) \in T_p^0 M$, $\pi(p) = \nu$ which start on S and intersect each leaf S_ν orthogonally.

The stable causality condition holds on M [1] means that the space-time foliation Σ is causal, being generated by a globally defined timelike 1-form $\omega^0 \in \wedge^1 M$ which is exact, i.e. $\omega^0 = \mathrm{d}t$ where t is some real function on M.

In this case, the leaves of Σ are spatial hypersurfaces of M of constant t, and

$$M = \bigcup_{\nu \in \mathcal{N}} S_\nu$$

is the disjoint union of a family of spatial hypersurfaces S_ν of constant t on M.

If we suppose that the initial surface S is a partial Cauchy surface of M we obtain M as a disjoint union of such homeomorphic partial Cauchy surfaces S_ν, which in addition satisfy the following conditions: for $\nu_i < \nu_j$ then S_{ν_j} lies to the future of S_{ν_i} and they cover $D^+(S)$.

[1] This condition of stability is equivalent to the existence of a function f on M, whose ∇f is everywhere timelike.

In the case when $\mathcal{N}$ is diffeomorphic with S^1, provided that S is a partial Cauchy surface on M, a curve $\gamma_{\bar{X}} : [0, 1] \longrightarrow \mathcal{N}$, $\gamma_{\bar{X}}(0) = \gamma_{\bar{X}}(1) \in S^1$ is lifted from a point x_i of S to a closed timelike curve $\tilde{\gamma}$ on M with $\tilde{\gamma}(0) = x_i = \tilde{\gamma}(1)$.

3 Some discussion

Physically, it seems more convenient to choose the surfaces S_ν so that they intersect $\mathcal{I}^+$ [2]. This means that every S_ν tend asymptotically to null surfaces.

Also by assumption of strong asymptotic predictability (i.e. there exists a partial Cauchy surface S of M such that (a) $\mathcal{I}^+$ lies in the boundary of $D^+(S)$ and (b) $J^+(S) \cap \partial J^-(\mathcal{I}^+)$ lies in $D^+(S)$) for some ν, the surfaces S_{ν_i} for $\nu_i > \nu$ will intersect the event horizon $\partial J^-(\mathcal{I}^+)$. Then a connected component of the nonempty set $S_{\nu_i} \backslash J^-(\mathcal{I}^+)$ will represent a black hole on the surface S_{ν_i}.

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Sufficient conditions for semilocal convergence of a fourth order multipoint iterative method for solving equations in Banach Spaces

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Abstract

We present sufficient conditions on the initial pivot for local convergence of a fourth order multipoint iterative method for solving nonlinear systems of equations. We do this by means of a system of majorizing real sequences, each one controlling either the norm of the operators involved or the error estimation. We use a technique similar to the one introduced in [2] from the reference list.

Keywords: Jarratt Method, multipoint method, nonlinear equations in Banach spaces, fourth-order method, recurrence relations, a priori error bounds.
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1 Introduction

Consider the equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator from a Banach space X in a Banach space Y. We consider the following two-step iteration

$$\begin{aligned}
y_n &= x_n - [F'(x_n)]^{-1}F(x_n) = x_n - \Gamma_n F(x_n), \\
z_n &= x_n + \frac{2}{3}(y_n - x_n) = x_n - \frac{2}{3}\Gamma_n F(x_n) \\
H(x_n, y_n) &= \Gamma_n \left[F'(z_n) - F'(x_n)\right], \\
x_{n+1} &= y_n - \frac{3}{4}\left[I + \frac{3}{2}H(x_n, y_n)\right]^{-1} H(x_n, y_n)(y_n - x_n)
\end{aligned} \tag{2}$$

for generating approximate solutions to (1).

We study the convergence of the above two-step iteration to a solution of (1). This method was introduced by Jarratt [8], [9] following the basic idea of increasing the order of convergence and avoiding the computation of the second Fréchet derivative in Newton's method, so that a two-point scheme is applied. (This idea was first found and developed by Ostrowski [11] and Traub [12] for functions of one variable).

Observe that (2) can be viewed as an extension of the classical Newton method. This method is computationally competitive for scalar equations and for systems of very few equations, which was Jarratt's original purpose (see[8],[9],[12]).

For this study, we use the technique considered in [2],[3] for one-point iterative processes that consists of constructing a system of real sequences, such that the convergence of the iterative process in Banach spaces follows from the analysis of these real sequences. The use of real sequences simplifies the study of iterations in Banach spaces by means of the majorizing principle [10], [1].

In Section 2, we construct a system of real sequences which are analysed and some properties are given. Next, in Section 3, we establish a convergence theorem and also give an explicit expression for the error bound of the method (2). Finally, a well-known integral equation is considered and we reach better error bounds than the ones obtained by other authors.

2 General Recurrence Relations

Let F be a nonlinear three times Fréchet-differentiable operator defined on some open convex subset Ω of a Banach space X with values in another Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Let us assume that

(c$_1$) $\|\Gamma_0\| \le \alpha$,

(c$_2$) $\|\Gamma_0 F(x_0)\| \le \beta$,

(c$_3$) $\|F''(x)\| \le M, \quad x \in \Omega$,

(c$_4$) $\|F'''(x)\| \le N, \quad x \in \Omega$,

(c$_5$) $\|F'''(x) - F'''(y)\| \le K\|x - y\|, \quad x, y \in \Omega, \quad K \ge 0$.

Let us denote $\theta = M\alpha\beta, \omega = N\alpha\beta^2$ and $\delta = K\alpha\beta^3$. We define the sequences

$$a_0 = 1 = c_0, \quad b_0 = \frac{2}{3}\theta, \quad d_0 = \frac{2-\theta}{2(1-\theta)}$$

and

$$a_{n+1} = \frac{a_n}{1 - \theta a_n d_n}, \quad b_{n+1} = \frac{2}{3}\theta a_{n+1} c_{n+1},$$

$$c_{n+1} = \frac{2^3(2-3b_n)^4}{(4-3b_n)^4}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3 + \frac{17}{108}\delta + \frac{\omega a_n}{3(2-3b_n)}\theta\right] a_{n+1}\, d_n^4,$$

$$d_{n+1} = \frac{4-3b_{n+1}}{4-6b_{n+1}} c_{n+1},$$

for all $n \ge 0$.

We then prove the following statements:

[I$_n$] $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le a_n\alpha$,

[II$_n$] $\|H(x_n, y_n)\| \le b_n$,

[III$_n$] $\|\Gamma_n F(x_n)\| \le c_n\beta$,

[IV$_n$] $\|x_{n+1} - x_n\| \le d_n\beta$, for all $n \geq 0$. We assume for all $n \geq 0$ that

$$x_n, y_n \in \Omega, \quad \theta a_n d_n < 1 \quad \text{and} \quad \theta < \frac{1}{2},$$

and use mathematical induction on n. It is easy to check the case when $n = 0$ by initial hypothesis. We assume that $[\mathbf{I}_n]$–$[\mathbf{IV}_n]$ are true for a fixed $n \geq 1$.

$[\mathbf{I}_{n+1}]$: Under the above hypotheses, we have:

$$\|I - \Gamma_n F'(x_{n+1})\| \leq \|\Gamma_n\| \|F'(x_n) - F'(x_{n+1})\| \leq a_n \alpha M \|x_{n+1} - x_n\| \leq \theta a_n d_n < 1$$

and, by the Banach lemma [10], Γ_{n+1} exists and

$$\|\Gamma_{n+1}\| \leq \frac{\|\Gamma_n\|}{1 - \|I - \Gamma_n F'(x_{n+1})\|} \leq \frac{a_n \alpha}{1 - \theta a_n d_n} = a_{n+1}\alpha.$$

$[\mathbf{II}_{n+1}]$:

$$\|H(x_{n+1}, y_{n+1})\| \leq \|\Gamma_{n+1}\| \|F'(x_{n+1} + \frac{2}{3}(y_{n+1} - x_{n+1})) - F'(x_{n+1})\|$$

$$\leq \frac{2\theta}{3} a_{n+1} c_{n+1} = b_{n+1}.$$

$[\mathbf{III}_{n+1}]$: By means of Taylor's theorem, we have

$$F(x_{n+1}) = \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2$$

$$+ \int_0^1 [F''(x_n + t(y_n - x_n))(1-t) - \frac{1}{2}F''(x_n + \frac{2}{3}t(y_n - x_n))]dt(y_n - x_n)^2$$

$$+ \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{2}{3}t(y_n - x_n)) - F''(x_n + t(y_n - x_n))\right] dt(y_n - x_n)$$

$$* \left[I + \frac{3}{2}H(x_n, y_n)\right]^{-1} \Gamma_n \int_0^1 F''(x_n + \frac{2}{3}t(y_n - x_n))dt(y_n - x_n)^2$$

and

$$\left\| \int_0^1 \left[F''(x_n + t(y_n - x_n))(1-t) - \frac{1}{2}F''(x_n + \frac{2}{3}t(y_n - x_n))\right] dt \right\|$$

$$\leq \left\| \int_0^1 \int_0^1 [F'''(x_n + st(y_n - x_n)) - F'''(x_n)] \, ds \, t \, (1-t) \, dt(y_n - x_n) \right\|$$

$$+\frac{1}{3}\left\|\int_0^1\int_0^1[F'''(x_n+\frac{2}{3}st(y_n-x_n))-F'''(x_n)]ds\,t\,dt(y_n-x_n)\right\|$$
$$\le\frac{17}{216}K\|y_n-x_n\|^2.$$

Hence

$$\|\Gamma_{n+1}F(x_{n+1})\|\le\|\Gamma_{n+1}\|\|F(x_{n+1})\|$$
$$\le\frac{1}{2}a_{n+1}\alpha\left[M\|x_{n+1}-y_n\|^2+\left(\frac{17}{108}K+\frac{MN}{3(2-3b_n)}\|\Gamma_n\|\right)\|\Gamma_nF(x_n)\|^4\right]$$
$$\le\frac{a_{n+1}}{2}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3+\frac{17}{108}\delta+\frac{a_n\omega}{3(2-3b_n)}\theta\right]c_n^4\beta$$
$$=\frac{8(2-3b_n)^4}{(4-3b_n)^4}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3+\frac{17}{108}\delta+\frac{a_n\omega}{3(2-3b_n)}\theta\right]a_{n+1}d_n^4\beta=c_{n+1}\beta.\quad(3)$$

[IV$_{n+1}$]:

$$\|x_{n+2}-x_{n+1}\|\le\|\frac{3}{4}\left[I+\frac{3}{2}H(x_{n+1},y_{n+1})\right]^{-1}H(x_{n+1},y_{n+1})\|\|y_{n+1}-x_{n+1}\|$$
$$\le\frac{4-3b_{n+1}}{4-6b_{n+1}}c_{n+1}\beta=d_{n+1}\beta.$$

This completes the induction and shows that the previous statements **[I$_n$]**–**[IV$_n$]** hold for all $n\in\mathbb{N}$.

3 Semilocal convergence analysis

We now prove the convergence of sequence (2) from an analysis of the sequences $\{a_n\},\{b_n\},\{c_n\}$ and $\{d_n\}$. We first need the following technical lemma, whose proof is trivial.

Lemma 3.1 *Let us define the following real functions:*

$$f(x)=\frac{27x^4(2-3x)^2}{[9x^2-24x+8]^2},$$

$$g(x,y)=f(x)\left[\frac{1}{(2-3x)^2}+\frac{17}{108}\frac{A}{y^2}+\frac{1}{3}\frac{B}{y(2-3x)}\right],\quad(4)$$

where $A, B \in \mathbb{R}_+$, *and*

$$q(x) = \frac{(2-3x)^2}{[9x^2-24x+8]^2},$$

$$p(x) = \frac{(3x-1)(27x^3-144x^2+192x-64)}{9x^3(2-3x)^2}. \tag{5}$$

Then

(i) $f : [0, \frac{1}{3}] \to \mathbb{R}$ *is an increasing function and* $f(0) = 0$,

(ii) f' *is increasing and* $f'(0) = 0$,

(iii) $g(x, y) < g(x, 1)$ *for* $y \geq 1$ *and* $x \in \left[0, \frac{1}{3}\right)$,

(iv) *the function* $g_0(x) = g(x, 1)$ *is increasing for* $x \in \left[0, \frac{1}{3}\right)$,

(v) *the function* $p(x)$ *is decreasing for* $x \in \left[0, \frac{1}{3}\right)$,

(vi) *the function* $q(x)$ *is increasing for* $x \in \left[0, \frac{1}{3}\right)$.

With these notations, we can write

$$b_{n+1} = g(b_n, a_n), \quad n \geq 0, \tag{6}$$

where g is defined in (4). Next, we give some properties for the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$.

Theorem 3.2 *Let* $\theta \in (0, 1/2)$, *and*

$$0 \leq \delta < \frac{27(2\theta-1)(\theta^3-8\theta^2+16\theta-8)}{17(1-\theta)^2},$$

$$0 \leq \omega < \frac{3(2\theta-1)(\theta^3-8\theta^2+16\theta-8)}{4\theta(1-\theta)^2} - \frac{17\delta}{36\theta}. \tag{7}$$

Then

(i) *the sequence* $\{b_n\}$ *is decreasing,*

(ii) $\theta a_n d_n < 1$ *for all* $n \geq 0$*,*

(iii) $a_n \geq 1$ *for all* $n \geq 0$*,*

(iv) *the sequence* $\{a_n\}$ *is increasing.*

Proof. We prove for all $n \geq 0$:

$(\mathbf{i}_n)$ $b_n \geq b_{n+1}$,

$(\mathbf{ii}_n)$ $\theta a_n d_n < 1$,

$(\mathbf{iii}_n)$ $a_n \geq 1$,

$(\mathbf{iv}_n)$ $a_n \leq a_{n+1}$.

We use induction on n. For $n = 0$, taking into account (4), $(\mathbf{i}_0)$ is equivalent to

$$\frac{17}{36}\frac{\delta}{\theta^3} + \frac{\omega}{\theta^2} < \frac{(3b_0 - 1)(27b_0^3 - 144b_0^2 + 192b_0 - 64)}{9b_0^3(2 - 3b_0)^2} = p(b_0).$$

This follows from (6). As $\theta < 1/2$, $a_0 = 1$ and $d_0 < 2$, $(\mathbf{ii}_0)$ is trivial and $(\mathbf{iii}_0)$ and $(\mathbf{iv}_0)$ follow immediately. We now suppose that $(\mathbf{i}_k)$–$(\mathbf{iv}_k)$ hold for $k = 1, \ldots, n-1$. By (5), lemma 3.1 **(v)**, $(\mathbf{i}_{n-1})$ and $(\mathbf{iv}_{n-1})$ we obtain $(\mathbf{i}_n)$.

$(\mathbf{ii}_n)$ is immediate from

$$\theta a_n d_n = \frac{3b_n(4 - 3b_n)}{4(2 - 3b_n)}$$

and $b_n \leq 1/3$. Finally, $(\mathbf{iii}_n)$ and $(\mathbf{iv}_n)$ follow by recurrence and the induction is concluded. ■

Theorem 3.3 *Under the assumptions of theorem 3.2, if we denote* $\gamma = b_1/b_0$*, then* $\gamma < 1$*,*

$$b_n \leq \frac{\gamma^{\frac{4^{n+1}}{3}}}{\gamma^{1/3}} b_0, \quad n \geq 1. \tag{9}$$

In addition, $\lim_n b_n = 0$ *and* $\sum_{n=0}^{\infty} b_n < +\infty$*.*

Proof. By conditions (7) and (8), $b_1 < b_0$ and $\gamma = b_1/b_0 < 1$.

In addition, as $b_1 = \gamma b_0 < b_0$, and by Lemma 3.1 (v) and (vi) it follows that $b_2 = g(b_1, a_1) \leq \gamma^4 b_1$, since $a_1 \geq a_0$. Next, by recurrence, we infer that $b_n \leq \gamma^4 b_{n-1}$ and consequently (9) is satisfied for all $n \geq 1$. It is then clear that $\lim_n b_n = 0$. Finally, $\sum_{n=0}^{\infty} b_n < +\infty$ as a consequence of $\sum_{n=0}^{\infty} b_n \leq \frac{b_0}{\gamma^{\frac{1}{3}}} \sum_{n=0}^{\infty} \gamma^{\frac{4^{n+1}}{3}}$. ∎

Theorem 3.4 *Under the assumptions of theorem 3.2, there exists an $L \in \mathbb{R}$, $L \geq 1$, such that $a_n \leq L$ for all $n \geq 0$.*

Proof. We can write

$$a_{n+1} = \prod_{i=0}^{n} \left[1 + \frac{3b_i(4 - 3b_i)}{9b_i^2 - 24b_i + 8} \right]$$

from the definition of the sequence $\{a_n\}$. Taking into account that

$$u(x) = \frac{(4 - 3x)}{9x^2 - 24x + 8}$$

is an increasing function for $x \in [0, \frac{1}{3})$, we obtain that

$$3b_i u(0) \leq 3b_i u(b_i) \leq 9b_i$$

for all $i \geq 0$. Then, as a consequence of the fact that $\sum_{n=0}^{\infty} b_n < +\infty$, it follows that

$$\prod_{n=0}^{\infty} (1 + 3b_n u(b_n)) < +\infty$$

and the proof is concluded. ∎

Theorem 3.5 *Under the assumptions of theorem 3.2, we have*

$$d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)} \frac{\gamma^{\frac{4^{n+1}}{3}}}{\gamma^{1/3}} b_0$$

for all $n \geq 0$ and $\gamma = b_1/b_0$.

Proof. Note that
$$d_n = \frac{3b_n}{4\theta a_n}\frac{4-3b_n}{2-3b_n}.$$
As the function $r(x) = (4-3x)/(2-3x)$ is increasing for all x, we get $2 = r(0) \leq r(b_n) \leq r(b_0) = (2-\theta)/(1-\theta)$. Hence
$$d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)}b_n$$
since $a_n \geq 1$ for all $n \geq 0$.

The proof is completed by theorem 3.3. ∎

Remark. Note that $\sum_{n=0}^{\infty} d_n < +\infty$ follows from the previous theorem and from taking into account that
$$\frac{6}{4\theta L}b_n \leq d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)}b_n$$
since $1 \leq a_n \leq L$ for all $n \geq 0$.

We are now ready to prove the main convergence theorem. Let us denote $\overline{B(x,r)} = \{y \in X; \|y-x\| \leq r\}$ and $B(x,r) = \{y \in X; \|y-x\| < r\}$.

Theorem 3.6 *Let F be a nonlinear operator which is three times Fréchet differentiable in an open convex domain Ω of a Banach space X with values in another Banach space Y. Let us suppose that $\Gamma_0 = F(x_0)^{-1}$ exists at some $x_0 \in \Omega$. Assume $(\mathbf{c_1}) - (\mathbf{c_5})$ and $\overline{B(x_0, R\beta)} \subset \Omega$ where $R = \lim_n \sum_{i=0}^{n} d_i$. Let us denote $\theta = M\alpha\beta, \omega = N\alpha\beta^2$ and $\delta = K\alpha\beta^3$. If $\theta \in (0, 1/2)$, and the conditions in (7) are verified, then the sequence $\{x_n\}$ given by (2) is well-defined, $x_n, y_n \in B(x_0, R\beta)$ for all $n \geq 0$, and converges to a solution $x^* \in \overline{B(x_0, R\beta)}$ of equation (1). Moreover, the solution x^* is unique in $B(x_0, \frac{2}{\alpha M} - R\beta)$. Furthermore, we also have the following error estimates:*
$$\|x^* - x_n\| \leq \sum_{i\geq n} d_i\beta \leq \beta\frac{(2-\theta)}{2(1-\theta)}\frac{1}{\gamma^{1/3}}\sum_{i\geq n}\gamma^{4^{i+1}/3} \tag{10}$$
where $\gamma = b_1/b_0$.

Proof. Under the assumptions mentioned above, it follows immediately from the previous theorems that sequence (2) is convergent. We also have

$$\|x_n - x_0\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_1 - x_0\| \leq \sum_{i=0}^{n-1} d_i\beta < R\beta.$$

Following a similar method, we obtain that

$$\|y_n - x_0\| < R\beta$$

and since $z_n = x_n + \frac{2}{3}(y_n - x_n)$, then $x_n, y_n, z_n \in B(x_0, R\beta) \subset \Omega_0$.

If x^* is the limit of the sequence (2), taking into account (3) and the continuity of F, we get that $F(x^*) = 0$.

In addition, we have $\|x_{n+1} - x_n\| \leq d_n\beta$ and consequently, for $p \geq 0$,

$$\begin{aligned}\|x_p - x_0\| &\leq \|x_p - x_{p-1}\| + \|x_{p-1} - x_{p-2}\| + \cdots + \|x_1 - x_0\| \\ &\leq (d_{p-1} + d_{p-2} + \cdots d_0)\beta.\end{aligned}$$

By letting $p \to \infty$, we obtain $\|x^* - x_0\| \leq R\beta$ and the error estimates given in (10).

Finally, to show the uniqueness, let us assume that $y^* \in B(x_0, \frac{2}{M\alpha} - R\beta)$ is another solution of $F(x) = 0$. According to [7]

$$0 = \Gamma_0[F(y^*) - F(x^*)] = \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*).$$

Then we only have to prove that the operator $\left[\int_0^1 F'(x^* + t(y^* - x^*))\, dt\right]^{-1}$ exists and therefore $y^* = x^*$. ■

Example. Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, with the max-norm and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - 1 - \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt,$$

with $s \in [0,1]$, $x \in C[0,1]$ and $0 < \lambda \leq 2$. Integral equations of this kind (called Chandrasekhar equations) arise in elasticity or neutron transport problems [4].

For $\lambda = 1/4$, it is easy to prove that

$$[F'(x)(y)](s) = y(s) - \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}y(t)\,dt - \frac{1}{4}y(s)\int_0^1 \frac{s}{s+t}x(t)\,dt,$$

$$[F''(x)(yz)](s) = -\frac{1}{4}\left[z(s)\int_0^1 \frac{s}{s+t}.y(t)\,dt + y(s)\int_0^1 \frac{s}{s+t}.z(t)\,dt\right].$$

and

$$[F'''(x)(yzu)](s) = 0$$

and starting at $x_0 = x_0(s) = s$, we have

$$F(x_0)(s) = -\frac{s}{4}\log\frac{s+1}{s},$$

$$\|F(x_0)(s)\| = \frac{\log 2}{4}.$$

The existence of Γ_0 will now be proved and we will calculate the parameters appearing in Theorem 3.6

$$\|[I - F'(x_0)]\| = \max_{s\in[0,1]} |y(s) - F'(x_0)y(s)| \le \frac{\log 2}{2} = 0.3465\ldots < 1.$$

The Banach lemma gives the existence of Γ_0 and we obtain:

$$\|\Gamma_0\| \le \frac{1}{1-(\log 2/2)} = \alpha = 1.5303\ldots$$

$$\|\Gamma_0 F(x_0)\| \le \frac{(\log 2/4)}{(\log 2/2)} = \beta = 0.2651\ldots,$$

$$\|F''(x)\| \le 0.3465\ldots = M, \qquad N = K = 0.$$

So $\theta = M\alpha\beta = 0.1406312\ldots$ and $\omega = \delta = 0$, and the conditions of Theorem 3.6 are fulfilled.

To compare the super-Halley and the Jarratt methods, we give an upper bound C to the number $10^{16}\|x^* - x_2\|$, where x_2 is the second iterate obtained by using these methods. We have that C is 31 for the super-Halley method and 0.143799 for the Jarratt method. Notice that the above operator is quadratic and the super-Halley method has order four for this kind of operators [5], [6].

Conclusion: The Jarratt and super-Halley methods both have order four as we can see. However, the error bound that we have for the Jarratt method is much better than the one obtained for the super-Halley method.

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Weighted Sobolev Descent for Singular First Order Partial Differential Equations

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Abstract

The author's work on solving singular ordinary differential equations via steepest descent based on weighted Sobolev gradients is extended to first order partial differential equations with linear singularities. Results are presented which demonstrate the improvements obtained by computing based on weighted Sobolev gradients.

Keywords: Sobolev descent, weighted Sobolev space, PDE, finite difference, numerical, pre-conditioning, least squares.
AMS subject classifications: 65N06, 65N22

1 Introduction

A general discussion of optimization techniques is given in [9] and [15]. Applying steepest descent to solve differential equations was first introduced by Cauchy in [2] and modifications such as conjugate gradient and variable metric methods, [6], were later introduced to speed up the convergence of numerical implementations. Sobolev descent is a systematic preconditioning technique where gradients are based on Sobolev spaces determined by the problem at hand rather than on Euclidean space.

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The method was introduced by J. W. Neuberger in [13] along with sufficient conditions for convergence and a complete discussion of Soblev descent may be found in [14]. The extension of the technique to singular ordinary differential equations by utilizing weighted Sobolev spaces based on the problem at hand, [11], compliments and extends his work. In [10] a convergence proof is given for discrete spaces such as those in this paper. Existence and uniqueness arguments for singular problems in Sobolev spaces are given in [18] and [3]. For a paper concerning Sobolev gradients which are constructed based on the problem at hand, consider [16]. Problem specific applications of Sobolev descent are given in, [4], [5], [7], [12] and [17]. General references for Sobolev spaces are [1] and [8].

2 Preliminaries

Let $I = [0,1]$, $\Omega \subset \Re^2$, and $L = \left(L^2_\Omega, \langle\cdot,\cdot\rangle_L\right)$. Define $\pi_1,\ \pi_2 : \Re \times \Re \to \Re$ by $\pi_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha$ and $\pi_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta$. Suppose $a, b \in C^1_\Omega$ vanish at only finitely many points in Ω, $u \in C^1_\Omega$, and $\vec{v} = (v_1, v_2)$ is a vector valued function on $\Re^2$. We make the following definitions:

$$\begin{aligned}
&\nabla u = (u_x, u_y) = \left(\tfrac{\partial u}{\partial x}, \tfrac{\partial u}{\partial y}\right)\\
&\nabla \cdot \vec{v} = (v_1)_x + (v_2)_y\\
&\nabla_w u = (au_x, bu_y)\\
&\nabla_w \cdot \vec{v} = a(v_1)_x + b(v_2)_y\\
&\Delta u = \nabla^2 u = \nabla \cdot \nabla u = u_{xx} + u_{yy}\\
&\Delta_w u = \nabla^2_w u = \nabla_w \cdot \nabla_w u = a(au_x)_x + b(bu_y)_y
\end{aligned} \tag{1}$$

$$G_w = \overline{\left\{ \begin{pmatrix} u \\ \nabla_w u \end{pmatrix} : u \in C^1_\Omega \right\}}^{L\times(L\times L)}.$$

Definition 1 *H_w is the collection of functions which appear as first coordinates of elements of G_w.*

Theorem 1 *If $a, b \in C^1_\Omega$ and a, b vanish at only finitely many points in Ω then G_w is a function.*

Proof. Let $\mathcal{P}$ denote the polynomials on Ω. Suppose $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ w \end{pmatrix} \in G_w$. The linearity of G_w implies that $\begin{pmatrix} 0 \\ v-w \end{pmatrix} \in G_w$. Since there exists a sequence, $(u_k)_{k\in\mathbf{N}} \in C^1_\Omega$ such that $\begin{pmatrix} u_k \\ \nabla_w u_k \end{pmatrix} \to \begin{pmatrix} 0 \\ v-w \end{pmatrix}$, it suffices to show that given

$\left(\begin{pmatrix} 0 \\ f \\ g \end{pmatrix}\right) \in G_w$, we have $\begin{pmatrix} f \\ g \end{pmatrix} = 0$ in $L \times L$. Let $\begin{pmatrix} f \\ g \end{pmatrix} \in G_w$ and $(u_k)_{k\in\mathbf{N}}$ such that $\begin{pmatrix} u_k \\ \nabla_w u_k \end{pmatrix} \to \left(\begin{pmatrix} 0 \\ f \\ g \end{pmatrix}\right)$. Let $p, q \in \mathcal{P}$, $F = \begin{pmatrix} p \\ q \end{pmatrix}$, and $F_w = \begin{pmatrix} ap \\ bq \end{pmatrix}$.

$$\begin{aligned}
\left|\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L\times L}\right| &\le \left|\left\langle \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L\times L}\right| + \left|\left\langle \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L\times L}\right| \\
&\le \left\| \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix} \right\|_{L\times L} \|F\|_L + \left| \int_\Omega \nabla_w u_k \cdot F \right| \\
&= \left\| \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix} \right\|_{L\times L} \|F\|_L + \left| \int_\Omega \nabla u_k \cdot F_w \right| \\
&\le \left\| \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix} \right\|_{L\times L} \|F\|_L + \left| \int_{\partial\Omega} u_k (F_w \cdot N) \right| + \left| \int_\Omega u_k (\nabla \cdot F_w) \right| \\
&\le \left\| \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} a(u_k)_x \\ b(u_k)_y \end{pmatrix} \right\|_{L\times L} \|F\|_L + \left| \int_{\partial\Omega} u_k (F_w \cdot N) \right| + \|u_k\|_L \, \|\nabla \cdot F_w\|_L
\end{aligned}$$

As $k \to \infty$, all three summands tend to zero, from which it follows that $\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L\times L} = 0$ for all $p, q \in \mathcal{P}$, a dense subset of L. *q.e.d.*

Define three operators on H_w by

$$\begin{aligned}
E_0(u) &= u \\
E_1^a(u) &= \pi_1 G_w u \\
E_2^b(u) &= \pi_2 G_w u.
\end{aligned}$$

Thus, $G_w u = \left(E_1^a u, E_2^b u\right)$ is the generalized weighted gradient. If $u \in C^1_\Omega$ then $G_w u = \nabla_w u$. Define an inner product on H_w by

$$\langle u, v\rangle_{H_w} = \langle u, v\rangle_L + \langle E_1^a(u), E_1^a(v)\rangle_L + \left\langle E_2^b(u), E_2^b(v)\right\rangle_L$$

The case $a \equiv 1 \equiv b$ has been studied, [14]. Without regard to boundary conditions, the spaces under consideration are H and H_w where H represents the unweighted Sobolev space obtained when $a \equiv 1 \equiv b$.

Theorem 2 *G_w is a closed, bounded, non-expansive, densely defined operator on H_w.*

Proof. G_w is closed by definition. Let $\mathcal{P}$ denote the polynomials on Ω. Since, $\mathcal{P}$ is dense in L, and $\mathcal{P} \subset G_w \subset L$, G_w is densely defined. For any $u \in H_w$,

$$\frac{\|G_w u\|_L}{\|u\|_{H_w}} = \frac{\|G_w u\|_L}{\|u\|_L + \|G_w u\|_L} \le 1,$$

thus G_w is bounded and non-expansive from H_w to L. *q.e.d.*

Theorem 3 *H_w is a Hilbert space.*

Proof. Let $(u_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in H_w. Thus $\begin{pmatrix} u_n \\ G_w u_n \end{pmatrix}_{n\in\mathbf{N}}$ is a Cauchy sequence in G_w which is complete as a closed subspace of the complete space, $L \times L$. Let $\begin{pmatrix} f \\ g \end{pmatrix} \in G_w$ such that $\begin{pmatrix} u_n \\ G_w u_n \end{pmatrix}_{n\in\mathbf{N}} \to \begin{pmatrix} f \\ g \end{pmatrix}$. Now, $(u_n)_{n\in\mathbf{N}} \to f$ in H_w. *q.e.d.*

In order to develop gradients depending on the constraints we define the following spaces dependent upon the boundary conditions.

Definition 2 *Suppose B is a linear operator representing the boundary conditions and put $L^0 = \{u \in L : Bu = 0\}$ and $H_w^0 = H_w \cap L^0$.*

Denote by $\pi_L : L \to L^0$ the orthogonal projection under $\langle \cdot, \cdot \rangle_L$ and denote by $\pi_{H_w} : H_w \to H_w^0$ the orthogonal projection under $\langle \cdot, \cdot \rangle_{H_w}$.

The next definition is used to define gradients based on $L, L^0, H, H^0, H_w, and H_w^0$.

Definition 3 *If S is any Hilbert space, $J : S \to \Re$ is a bounded linear functional, and $s \in S$, define $\nabla_S J(s)$ to be the unique element in S such that $J'(s)(r) = \langle \nabla_s J(s), r \rangle_S$ for all $s \in S$.*

The following theorem, [12], guarantees convergence in the continuous setting. Related convergence results for the discrete case are offered in [9] and [13].

Theorem 4 *(J. W. Neuberger) Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $G \in L(\mathcal{H}, \mathcal{K})$. Suppose $g \in \mathcal{K}$, $v \in \mathcal{H}$, $Gv = g$, and $\phi(u) = \frac{1}{2}\|Gu - g\|^2$ for every $u \in \mathcal{H}$. If $x \in \mathcal{H}$ and z is the function on $[0, \infty)$ so that*

$$z(0) = x, z'(t) = -(\nabla\phi)(z(t)), t \geq 0$$

then $u = lim_{n\to\infty} z(t)$ exists and $Gu = g$.

3 An Example

Consider the class of problems,

$$\begin{array}{l} a, b \in C_\Omega^2 \\ \Omega = [0,1] \times [0,1] \\ au_x + bu_y = 0 \\ u(x,x) = 2x^2 \ \forall \ x \in [0,1]. \end{array} \tag{2}$$

Define the functional,

$$J(u) = \frac{1}{2} \int_\Omega (au_x + bu_y)^2 \, d\Omega.$$

The gradient on which descent is denoted by $\nabla_{H_w^0} J$ and is the unique element in the Hilbert space H_w^0 which satisfies, $J'(u)(h) = \left\langle \left(\nabla_{H_w^0} J\right)(u), h\right\rangle$ for all $u, h \in H_w^0$ as constructed via Definition 3. For all $u \in H_w, h \in H_w^0$ we have,

$$\begin{aligned} J'(u)(h) &= \langle (\nabla_H J)(u), h\rangle \\ &= \langle (\nabla_H J)(u), \pi_{H_w} h\rangle \\ &= \langle \pi_{H_w} (\nabla_H J)(u), h\rangle \end{aligned}$$

Hence for all $u, h \in H_w^0$, $\left\langle \left(\nabla_{H_w^0} J\right)(u), h\right\rangle = \langle \pi_{H_w} (\nabla_H J)(u), h\rangle$ and $\left(\nabla_{H_w^0} J\right)(u) = \pi_{H_w} (\nabla_H J)(u)$.

For $u \in H_w$ define

$$J(u) = \frac{1}{2}\int_\Omega (E_1^a u + E_2^b u)^2 \, d\Omega.$$

The iteration is now,

$$u^{k+1} = u^k - \delta_k \left(\nabla_{H_w^0} J\right)(u^k)$$

where δ_k is given by the smallest positive real number that minimizes $\alpha(\delta_k) = J\big(u^k - \delta_k (\nabla_{H_w} J)(u^k)\big)$. Setting $\alpha'(\delta_k) = 0$ and solving yields the optimal step size,

$$\delta_k = \frac{\| (\nabla_{H_w} J)(u^k)\|_{H_w}^2}{\left\langle (\nabla_{H_w} J)^2(u^k), (\nabla_{H_w} J)(u^k)\right\rangle_{H_w}}.$$

3.1 Numerical Considerations

Subdivide Ω into n pieces along each axis. Order the grid starting in the lower left hand corner at $(0,0)$ so that $u_{i,j} = u(\frac{1}{n}(i-1), \frac{1}{n}(j-1))$.

Discrete versions of u, $E_0(u)$, $E_1^a(u)$, and $E_2^b(u)$ are subscripted as matrices, but are treated as vectors. Their definitions follow:

$$u = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{n+1,n+1} \end{pmatrix}, \quad E_0(u) = \begin{pmatrix} \{E_0(u)\}_{1,1} \\ \{E_0(u)\}_{2,1} \\ \vdots \\ \{E_0(u)\}_{n+1,n+1} \end{pmatrix}, \quad E_w(u) = \begin{pmatrix} E_0(u) \\ E_1^a(u) \\ E_2^b(u) \end{pmatrix}.$$

$$\left\{ E_0 \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{n+1,n+1} \end{pmatrix} \right\}_{i,j} = \frac{1}{4}(u_{i,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1})$$

$$\left\{ E_1^a \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{n+1,n+1} \end{pmatrix} \right\}_{i,j} = \frac{n}{4}(a_{i,j} + a_{i+1,j})(-u_{i,j} + u_{i+1,j} - u_{i,j+1} + u_{i+1,j+1})$$

$$\left\{ E_2^b \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{n+1,n+1} \end{pmatrix} \right\}_{i,j} = \frac{n}{4}(b_{i,j} + b_{i,j+1})(-u_{i,j} - u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1})$$

The discretized versions of the spaces L and H_w are $\left(\Re^{(n+1)^2}, \langle\cdot,\cdot\rangle_L\right)$, and $\left(\Re^{(n+1)^2}, \langle\cdot,\cdot\rangle_{H_w}\right)$ where $\langle\cdot,\cdot\rangle_L$ denotes the Euclidean inner product and $\langle u, v\rangle_{H_w} = \langle E_0 u, E_0 v\rangle_L + \langle E_1^a u, E_1^a v\rangle_L + \langle E_2^b u, E_2^b v\rangle_L$.

Discretizing our functional yields,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega (au_x + bu_y)^2 \, d\Omega \\ &= \frac{1}{2}\int_\Omega (E_1^a u + E_2^b u)^2 \, d\Omega \\ &\cong \frac{1}{2n^2}\sum_{i,j}^{n}(\{E_1^a u\}_{i,j} + \{E_2^b u\}_{i,j})^2 \end{aligned}$$

The discrete analog to $(\nabla_L J)$ is $\nabla J = \begin{pmatrix} \frac{\partial J}{\partial u_{1,1}} \\ \frac{\partial J}{\partial u_{2,1}} \\ \vdots \\ \frac{\partial J}{\partial u_{n+1,n+1}} \end{pmatrix}$. Four summands from our functional, J, contribute to $\frac{\partial J}{\partial u_{p,q}}$.

$$\begin{aligned} \frac{\partial J}{\partial u_{p,q}} &= \frac{1}{2n^2}\frac{\partial}{\partial u_{p,q}}\Bigg\{ \left[(E_1^a u)_{p-1,q-1} + \left(E_2^b u\right)_{p-1,q-1}\right]^2 \\ &\qquad + \left[(E_1^a u)_{p,\ q-1} + \left(E_2^b u\right)_{p,\ q-1}\right]^2 \\ &\qquad + \left[(E_1^a u)_{p-1,q} + \left(E_2^b u\right)_{p-1,q}\right]^2 \\ &\qquad + \left[(E_1^a u)_{p,\ q} + \left(E_2^b u\right)_{p,\ q}\right]^2 \Bigg\} \\ &= \frac{1}{n^2}\Bigg\{ \left[(E_1^a u)_{p-1,q-1} + \left(E_2^b u\right)_{p-1,q-1}\right]\left[\frac{n}{4}\left(a_{p-1,q-1} + a_{p,q-1}\right) + \frac{n}{4}\left(b_{p-1,q-1} + b_{p-1,q}\right)\right] + \\ &\qquad \left[(E_1^a u)_{p,\ q-1} + \left(E_2^b u\right)_{p,\ q-1}\right]\left[-\frac{n}{4}\left(a_{p,q-1} + a_{p+1,q-1}\right) + \frac{n}{4}\left(b_{p,q-1} + b_{p,q}\right)\right] + \end{aligned}$$

$$\left[\left(E_1^a u\right)_{p-1,q} \quad + \left(E_2^b u\right)_{p-1,q} \quad \right]\left[\ \frac{n}{4}\left(a_{p-1,q}+a_{p,q}\right)-\frac{n}{4}\left(b_{p-1,q}+b_{p-1,q+1}\right)\right]+$$
$$\left[\left(E_1^a u\right)_{p-1,q} \quad + \left(E_2^b u\right)_{p-1,q} \quad \right]\left[-\frac{n}{4}\left(a_{p,q}+a_{p+1,q}\right)-\frac{n}{4}\left(b_{p,q}+b_{p,q+1}\right)\right] \ \Big\}$$

The two inner products are related by, $\langle u,v\rangle_{H_w} = \langle Au,v\rangle_L$ where $A = (E_0)^t E_0 + (E_1^a)^t E_1^a + (E_2^b)^t E_2^b$.

The canonical perturbation space will be $\Re_0^{(n+1)^2} = \left\{x \in \Re^{(n+1)^2} | Bx = 0\right\}$ where B is the linear operator representing our boundary conditions. Let $\pi_{H_w} : \Re^{(n+1)^2} \to \Re_0^{(n+1)^2}$ denote the orthogonal projection under $\langle\cdot,\cdot\rangle_{H_w}$ and $\pi_L : \Re^{(n+1)^2} \to \Re_0^{(n+1)^2}$ denote the orthogonal projection under $\langle\cdot,\cdot\rangle_{L.}$

Applying the Reisz Representation Theorem twice and using the self-adjoint property of projections repeatedly we have for every $u \in \Re^{(n+1)^2}, h \in \Re_0^{(n+1)^2}$,

$$\begin{aligned}
\langle \pi_L(\nabla J)(u),h\rangle_L &= \langle(\nabla J)(u),\pi_L h\rangle_L \\
&= \langle(\nabla J)(u),h\rangle_L \\
&= J'(u)(h) \\
&= \langle\left(\nabla_{H_w} J\right)(u),h\rangle_{H_w} \\
&= \langle\left(\nabla_{H_w} J\right)(u),\pi_{H_w} h\rangle_{H_w} \\
&= \langle\pi_{H_w}\left(\nabla_{H_w} J\right)(u),h\rangle_{H_w} \\
&= \left\langle\left(\nabla_{H_w^0} J\right)(u),h\right\rangle_{H_w} \\
&= \left\langle A\left(\nabla_{H_w^0} J\right)(u),h\right\rangle_L \\
&= \left\langle A\left(\nabla_{H_w^0} J\right)(u),\pi_L h\right\rangle_L \\
&= \left\langle \pi_L A\left(\nabla_{H_w^0} J\right)(u),h\right\rangle_L
\end{aligned}$$

Consequently, $\pi_L A\left(\nabla_{H_w^0} J\right)(u) = \pi_L\left(\nabla_L J\right)(u)$ is valid for every $u \in \Re^{(n+1)^2}$ and provides a linear system allowing us to solve for $\left(\nabla_{H_w^0} J\right)(u)$.

Solving for $x_{i,j} = \left\{\left(\nabla_{H_w^0} J\right)(u)\right\}_{i,j}$ indicates that $x_{i,j}$ receives contributions from eight of the nine non-zero bands of A that correspond to the eight "neighbors" of $x_{i,j}, (i.e. x_{i-1,j-1}, x_{i-1,j}, ..., x_{i+1,j+1})$. We solve the system via SOR.

3.2 Results

The results in this section are for the equation,

$$\begin{array}{l} yu_x - xu_y = 0 \\ u(x,x) = 2x^2 \ \forall \ x \in I \end{array} \tag{3}$$

which has solution $u(x,y) = x^2 + y^2$. Tables 1 and 2 demonstrate two features of the algorithm. First observe the significant improvements in the obtainable

Table 1: Singular Partial Differential Equation

$\mathbf{yu_1 - xu_2 = 0}$			$\mathbf{u(x,x) = 2x^2}$	$\mathbf{Grid = 10 \times 10}$	
Gradient	Iterations	Seconds	Div. Dif. Err.	Avg. Abs. Err.	Max. Abs. Err.
L	$3,031$	$0:10$	10^{-5}	10^{-4}	7.1×10^{-4}
H	$1,315$	$0:18$	10^{-5}	10^{-4}	6.6×10^{-4}
H_w	25	$0:1$	10^{-4}	10^{-4}	4.2×10^{-4}

Table 2: Singular Partial Differential Equation

$\mathbf{yu_1 - xu_2 = 0}$			$\mathbf{u(x,x) = 2x^2}$	$\mathbf{Grid = 50 \times 50}$	
Gradient	Iterations	Min:Sec	Div. Dif. Err.	Avg. Abs. Err.	Max. Abs. Err.
L	$46,282$	$62:20$	10^{-5}	10^{-4}	2.1×10^{-3}
H	$1,334$	$10:23$	10^{-4}	10^{-4}	2.1×10^{-3}
H_w	25	$5:04$	10^{-4}	10^{-5}	7.6×10^{-4}

accuracy by utilizing the weighted steepest descent. Second observe that the results in Table 1 are based on a 10 x 10 grid and yet are representative of the results (except for run time) in Table 2 which are based on a 50 x 50 grid.

The three rows represent descent based on the Eucliden, the non-weighted and the weighted gradients respectively. The three columns represent the divided difference error, the average absolute error, and the maximum absolute error. If u represents the computed solution, then the divided difference error is given by,

$$\sum_{i,j}^{n} |\{E_1^a u\}_{i,j} + \{E_2^b u\}_{i,j}|/(n+1)^2,$$

while the average absolute error is given by,

$$\sum_{i,j}^{n} |u_{i,j} - 2\frac{1}{n}(i-1)\frac{1}{n}(j-1)|/(n+1)^2.$$

4 Conclusions

The algorithm gives good results on a small number of divisions making the algorithm a candidate for multi-grid methods.

Improvements to the algorithm as presented would be to incorporate multi-grid techniques, appropriate SOR constants, and conjugate gradient technology.

A "C" code for the algorithm may be obtained from the author along with a Mathematica code for computation of the matrices involved.

Preliminary investigations indicate that the method extends to second order equations such as:

$$\begin{array}{l} f(u) = \lambda u + u^3 \\ \Omega = Cl(B[0,1]) \\ \Delta u + f(u) = 0 \\ u|_{\partial\Omega} = 0. \end{array} \tag{4}$$

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A MONOTONE CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS USING HYPOTHESES ON DIVIDED DIFFERENCES OF ORDER TWO

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Abstract. In this study we approximate a solution of a nonlinear operator equation on a partially ordered topological space by enclosing it by two monotonically convergent Newton-like sequences. In contrast to earlier results we use hypotheses on bilinear instead of on linear operators. This way we establish different sufficient convergence conditions for Newton-like methods than before. Hence, our results have theoretical and practical value.

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Key Words and Phrases. Newton-like method, monotone convergence, partially ordered topological space, linear-bilinear operator.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a closed convex subset D of a partially ordered topological space E_1 (POTL-space) with values in a (POTL-space) E_2. We refer the reader to [5], [10] for definitions of the concepts introduced here.

So far Newton-like methods have been used to enclose monotonically x^*, under hypotheses on linear operators (see (20) and (21)). For a survey of such results we refer the reader to [5].

Here, we provide new sufficient convergence conditions for the monotone convergence of Newton-like methods, but we use hypotheses on bilinear (see (3) and (4)) as well as linear operators. This way our convergence conditions differ from earlier ones [1]–[8], [10]. Hence, our results have theoretical and practical value.

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Theorem 1. *Consider an operator $F\colon D \subset E_1 \to E_2$, where E_1 is a regular POTL-space, and E_2 is a POTL-space. Let x_0, y_0, y_{-1} be three points of D such that*

$$x_0 \le y_0 \le y_{-1},\ [x_0, y_{-1}] \subseteq D,\ F(x_0) \le 0 \le F(y_0). \tag{2}$$

Denote by D_1, D_2, D_3 the sets

$$D_1 = \{(x,y) \in E_1^2; x_0 \le x \le y \le y_0\},$$
$$D_2 = \{(z, y_{-1}) \in E_1^2; x_0 \le z \le y_0\},$$

and $D_3 = D_1 \cup D_2$.

Assume:

(a) *operator F has nonnegative, symmetric, and continuous divided differences of order one and two, denoted by $[x,y;F] \in L(E_1,E_2)$ and $[x,y,z;F] \in L(E_1, L(E_1, E_2))$ at the points (x,y) and (x,y,z) $(x,y,z \in D_3)$ respectively;*

(b) *there exists operator $A\colon D_3 \to L(E_1,E_2)$ having a continuous nonsingular nonnegative left subinverse for all $(x,y) \in D_3$; and an operator $B \in L(E_2, L(E_1, E_2))$ such that*

$$0 \le B \le [x,y,z;F] \tag{3}$$

and

$$[y,z;F] - B(z-x) - A(y,w) \le 0 \tag{4}$$

for all $x,y,z,w \in D_3$ with $x \le y \le z \le w$.

Then, there exist two sequences $\{x_n\}$ $(n \ge 1)$, $\{y_n\}$ $(n \ge 1)$, and two points $x^*, y^* \in E_2$ such that for all $n \ge 0$, and for $A_n := A(y_n, y_{n-1})$ $(n \ge 0)$:

$$F(y_n) + A_n(y_{n+1} - y_n) = 0, \tag{5}$$
$$F(x_n) + A_n(x_{n+1} - x_n) = 0, \tag{6}$$
$$F(x_n) \le 0 \le F(y_n), \tag{7}$$

$$x_0 \le x_1 \le \cdots \le x_n \le x_{n+1} \le y_{n+1} \le y_n \le \cdots \le y_1 \le y_0, \tag{8}$$

and

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*, \tag{9}$$

with

$$x^* \le y^*. \tag{10}$$

Moreover, if operators A_n $(n \ge 0)$ are inverse nonnegative, then any solution of the equation $F(x) = 0$ from the interval $[x_0, y_0]$ belongs to the interval $[x^*, y^*]$.

Proof. Let L_0 be a continuous nonsingular nonnegative left subinverse of A_0. Define operator $P\colon [0, y_0 - x_0] \to E_1$ by

$$P(x) = x - L_0(F(x_0) + A_0(x)). \tag{11}$$

Since L_0, F and A_0 are continuous, so is operator P. From $x \le y$, and $L_0 A_0 \le I$, we can get $(I - L_0 A_0)(y - x) \ge 0$, which shows $P(x) \le P(y)$. That is operator P is also isotone. By (2) and (5) we have respectively:

$$\begin{aligned} P(0) &= -L_0 F(x_0) \ge 0, \\ P(y_0 - x_0) &= y_0 - x_0 - L_0 F(y_0) + L_0(F(y_0) - F(x_0) - A_0(y_0 - x_0)). \end{aligned} \tag{12}$$

But using (3) and (4), we get

$$\begin{aligned} &F(y_0) - F(x_0) - A_0(y_0 - x_0) \\ &\quad = F(y_0) - F(x_0) - [y_{-1}, y_0; F](y_0 - x_0) \\ &\qquad + [y_{-1}, y_0; F](y_0 - x_0) - A_0(y_0 - x_0) \\ &\quad = ([x_0, y_0; F] - [y_{-1}, y_0; F])(y_0 - x_0) \\ &\qquad + ([y_{-1}, y_0] - A_0)(y_0 - x_0) \\ &\quad = -([y_{-1}, y_0, x_0; F] - B)(y_{-1} - x_0)(y_0 - x_0) \\ &\qquad + [[y_{-1}, y_0; F] - A_0 - B(y_{-1} - x_0)]\,(y_0 - x_0) \le 0. \end{aligned} \tag{13}$$

Approximation (12) because of (2) and (12) gives

$$P(y_0 - x_0) \le y_0 - x_0.$$

According to the famous Kantorovich theorem concerning fixed points on regular POTL-spaces [5], [6], operator P has a fixed point $w \in [0, y_0 - x_0]$. Set $x_1 = x_0 + w$ to obtain

$$F(x_0) + A_0(x_1 - x_0) = 0, \quad x_0 \le x_1 \le y_0. \tag{14}$$

Using (5), (6) and (14) we obtain

$$\begin{aligned} F(x_1) &= F(x_1) - F(x_0) - A_0(x_1 - x_0) \\ &= ([x_0, x_1; F] - A_0)(x_1 - x_0) \\ &= ([x_1, x_0; F] - [y_0, x_1; F])(x_1 - x_0) + ([x_1, y_0; F] - A_0)(x_1 - x_0) \\ &= ([y_0, x_1, x_0; F] - B)(x_0 - y_0)(x_1 - x_0) \\ &\quad + [[x_1, y_0; F] - A_0 - B(y_0 - x_0)]\,(x_1 - x_0) \le 0. \end{aligned} \tag{15}$$

Similarly, we define operator $Q\colon [0, y_0 - x_1] \to E_1$ by

$$Q(x) = x + L_0(F(y_0) - A_0(x)).$$

Arguing as above we conclude that Q is continuous and isotone.

By (2) and (6) we have respectively,

$$\begin{aligned} Q(0) &= L_0 F(y_0) \ge 0, \\ Q(y_0 - x_1) &= y_0 - x_1 + L_0 F(x_1) + L_0(F(y_0) - F(x_1) - A_0(y_0 - x_1)). \end{aligned} \tag{16}$$

But using (3) and (4) we get

$$
\begin{aligned}
(17)\qquad F(y_0)-F(x_1)-A_0(y_0-x_1) &= \left[[y_0,x_1;F]-[y_{-1},y_0;F]\right](y_0-x_1)\\
&\quad+\ ([y_{-1},y_0;F]-A_0)(y_0-x_1)\\
&= ([y_{-1},y_0,x_1;F]-B)(x_1-y_{-1})(y_0-x_1)\\
&\quad+\left[[y_0,y_{-1};F]-A_0-B(y_{-1}-x_1)\right](y_0-x_1)\le 0.
\end{aligned}
$$

Using (2) and (16), approximation (17) gives

$$Q(y_0-x_1)\le y_0-x_1.$$

By Kantorovich's theorem we deduce the existence of a point $z\in[0,y_0-x_1]$ such that $z=Q(z)$. Set $y_1=y_0-z$ to obtain

$$(18)\qquad F(y_0)+A_0(y_1-y_0)=0,\quad x_1\le y_1\le y_0.$$

Using (18) and (9) we get

$$
\begin{aligned}
(19)\qquad F(y_1) &= F(y_1)-F(y_0)-A_0(y_1-y_0)\\
&= ([y_0,y_1;F]-A_0)(y_1-y_0)\\
&= [y_{-1},y_0,y_1;F](y_1-y_{-1})(y_1-y_0)\ge 0.
\end{aligned}
$$

Proceeding by induction on $n\ge 0$, we can show that there exist two sequences $\{x_n\}$ $(n\ge 1)$ and $\{y_n\}$ $(n\ge 1)$ satisfying (5)–(8), in a regular space E_1. It follows that there exist $x^*,y^*\in E_1$ such that $x^*=\lim_{n\to\infty}x_n$, and $y^*=\lim_{n\to\infty}y_n$. We obviously have $x^*\le y^*$.

If $x_0\le u\le y_0$ and $F(u)=0$, we can have

$$
\begin{aligned}
A_0(y_1-u) &= A_0(y_0)-F(y_0)-A_0(u)=A_0(y_0-u)-(F(y_0)-F(u))\\
&= \left[[y_0,y_{-1};F]-[u,y_0;F]\right](y_0-u)\\
&= [u,y_0,y_{-1};F](y_{-1}-u)(y_0-u)\ge 0,
\end{aligned}
$$

and

$$
\begin{aligned}
A_0(x_1-u) &= A_0(x_0)-F(x_0)-A_0(u)=A_0(x_0-u)-(F(x_0)-F(u))\\
&= \left[([y_0,y_{-1};F]-[y_0,x_0;F])+([y_0,x_0;F]-[x_0,u;F])\right](x_0-u)\\
&= \left[[x_0,y_0,y_{-1};F](y_{-1}-x_0)+[u,x_0,y_0;F](y_0-u)\right](x_0-u)\le 0.
\end{aligned}
$$

Hence, if operator A_0 is inverse nonnegative it follows $x_1\le u\le y_1$. Proceeding by induction, we deduce $x_n\le u\le y_n$ $(n\ge 0)$. Hence, we conclude, $x^*\le u\le y^*$.

That completes the proof of the Theorem. □

Remark 1. A possible choice for operator B can be $B=[x,y,z;F]$, in which case (3) is satisfied, whereas (4) can give

$$[y,z;F]\le[x,y,z;F](z-x)+A(y,w)=[y,z;F]-[x,y;F]+A(y,w),$$

or

$$[x, y; F] \le A(y, w). \tag{20}$$

In the special case, when $A(y,w) = [y, w; F]$ $(y, w \in D_3)$ (secant method), it follows from (20) that (4) is satisfied if

$$[x, y; F] \le [w, y; F] \text{ for all } x, y, w \in D_1 \text{ with } x \le y \le w. \tag{21}$$

Monotone convergence results have been proved in [1]–[8], [10] for Newton-like methods based on the assumption

$$[x, y; F] \le A(w, z) \text{ for all } (x, y),\ (y, w) \in D_1,\ (w, z) \in D_3,$$

which is condition (20), in this case.

Natural conditions and their proofs under which the points x^* and y^* are solutions of the equation $F(x) = 0$ can be found in [1]–[8], [10]. However, here for completion we list some of them.

Proposition. *Assume*:

(a) *hypotheses of Theorem 1 hold*;
(b) *operator F is continuous at x^* and y^*.*

If, one of the following conditions hold:

(i) $x^* = y^*$,

(ii) E_1 *is normal and there exists an operator* $H\colon E_1 \to E_2$, *with* $H(0) = 0$ *having an isotone continuous inverse at the origin such that* $A_n \le H$ *for sufficiently large* n;

(iii) *operators* A_n $(n \ge 0)$ *are equicontinuous*;

(iv) E_2 *is normal and there exists an operator* $G\colon E_1 \to E_2$ *with* $G(0) = 0$, *continuous at the origin with* $A_n \le G$ *for sufficiently large* n.

Then, we deduce

$$F(x^*) = F(y^*) = 0.$$

The regularity of the space E_1 assumed in Theorem 1 is essentially used to show sequences $\{x_n\}$, $\{y_n\}$ $(n \ge 0)$ are well defined and convergence can be dropped. Instead of iterative procedure (5)–(6) consider the explicit form of it

$$y_{n+1} = y_n - L_n F(y_n), \quad (n \ge 0), \tag{22}$$

$$x_{n+1} = x_n - L'_n F(x_n), \quad (n \ge 0), \tag{23}$$

where L_n and L'_n are nonnegative subinverses of A_n $(n \ge 0)$. By dropping the regularity assumption on E_1 we cannot prove convergence. However, we can prove in the theorem that follows that for any solution $u \in [x_0, y_0]$ of equation $F(u) = 0$, the following enclosure estimates hold for all $n \ge 0$:

$$x_n \le x_{n+1} \le u \le y_{n+1} \le y_n \quad (n \ge 0).$$

Such a result is important in cases where the existence of the solution can be proved by other means and we just want to enclose it monotonically [1], [2]–[5], [7].

Theorem 2. *Let $F\colon D \subseteq E_1 \to E_2$ be an operator between two POL-spaces E_1 and E_2. Assume*:

(a) *there exist points $x_0, y_0, y_{-1} \in D$ satisfying (2)*;
(b) *hypotheses (a) of Theorem 1 hold*;
(c) *define D_1, D_2, D_3 by Theorem 1 and assume there exists an operator $A\colon D_3 \to L(E_1, E_2)$ having a nonnegative subinverse for any $(v, w) \in D_3$; and an operator $B \in L(E_1, L(E_1, E_2))$ such that hypotheses (3) and (4) hold.*

Then, iterative algorithm (22)–(23) generates two sequences having properties (5)–(18).

Moreover for any solution $u \in [x_0, y_0]$ of the equation $F(x) = 0$, we have

$$x_n \le u \le y_n \quad (n \ge 0). \tag{24}$$

Proof. We show that (24) is true using induction on $n \ge 0$. First for $n = 0$, we have $x_0 \le y_0$, $F(x_0) \le 0 \le F(y_0)$,

$$\begin{aligned} &L_0 \ge 0,\ L_0' \ge 0,\ I \ge A_0 L_0,\ I \ge L_0 A_0, \\ &I \ge A_0 L_0',\ \text{and } I \ge L_0' A_0. \end{aligned} \tag{25}$$

We can write in turn

$$\begin{aligned} y_0 - y_1 &= L_0 F(y_0) \ge 0, \\ y_1 - x_0 &= y_0 - x_0 - L_0 F(y_0) \ge y_0 - x_0 - L_0(F(y_0) - F(x_0)) \\ &\ge L_0(A_0(y_0 - x_0) - (F(y_0) - F(x_0))) \ge 0, \quad \text{(by (13))}, \\ x_1 - x_0 &= -L_0' F(x_0) \ge 0, \\ y_0 - x_1 &= y_0 - x_0 + L_0' F(x_0) \ge y_0 - x_0 - L_0'(F(y_0) - F(x_0)) \\ &\ge L_0'[A_0(y_0 - x_0) - (F(y_0) - F(x_0))] \ge 0, \quad \text{(by (13))}. \end{aligned}$$

That is points $x_1, y_1 \in [x_0, y_0]$. Moreover using (3), (4), (22), (23), (25) and the proof of Theorem 1 we can also have in turn

$$\begin{aligned} F(y_1) &= F(y_1) + A_0(y_0 - y_1 - L_0 F(y_0)) = F(y_1) - A_0 L_0 F(y_0) + A_0(y_0 - y_1) \\ &\ge F(y_1) - F(y_0) + A_0(y_0 - y_1) \ge 0, \quad \text{(by (19))}, \\ F(x_1) &= F(x_1) - A_0(x_1 - x_0 + L_0' F(x_0)) = F(x_1) - A_0 L_0' F(x_0) - A_0(x_1 - x_0) \\ &\le F(x_1) - F(x_0) - A_0(x_1 - x_0) \le 0, \quad \text{(by (15))}, \\ y_1 - x_1 &\ge y_1 - x_1 + L_0 F(x_1) = y_0 - x_1 - L_0(F(y_0) - F(x_1)) \\ &\ge L_0(A_0(y_0 - x_1) - (F(y_0) - F(x_1)) \ge 0, \quad \text{(by (19))}. \end{aligned}$$

Hence, we obtain $x_0 \le x_1 \le y_1 \le y_0$, and $F(x_1) \le 0 \le F(y_1)$.

Let $u \in [x_0, y_0]$ with $F(u) = 0$. We get

$$\begin{aligned} y_1 - u &= y_0 - u - L_0 F(y_0) + L_0 F(u) \ge L_0(A_0(y_0 - u) - (F(y_0) - F(u))) \ge 0, \\ u - x_1 &= u - x_0 + L_0' F(x_0) - L_0' F(u) \ge L_0'(A_0(u - x_0) - (F(u) - F(x_0)) \ge 0. \end{aligned}$$

That is, we get $x_1 \leq u \leq y_1$. Proceeding by induction it follows that $x_n \leq u \leq y_n$.

That completes the proof of the Theorem. □

We state the following proposition for completion whose proof can be found in [5], [7], [10].

Proposition 2. *Under the hypotheses of Theorem 2, assume*:

(a) E_2 *is a POTL-space and* E_1 *is a normal POTL-space*;
(b) $\lim_{n\to\infty} x_n = x^*$, *and* $\lim_{n\to\infty} y_n = y^*$;
(c) F *is continuous at* x^* *and* y^*, *and*
(d) *there exist two continuous, nonnegative, nonsingular operators* L *and* L' *such that* $L_n \geq L$ *and* $L'_n \geq L'$ *for sufficiently large* n. *Then, we deduce* $F(x^*) = F(y^*) = 0$.

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