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Sufficient conditions for semilocal convergence of a fourth order multipoint iterative method for solving equations in Banach Spaces

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Abstract

We present sufficient conditions on the initial pivot for local convergence of a fourth order multipoint iterative method for solving nonlinear systems of equations. We do this by means of a system of majorizing real sequences, each one controlling either the norm of the operators involved or the error estimation. We use a technique similar to the one introduced in [2] from the reference list.

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1 Introduction

Consider the equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator from a Banach space X in a Banach space Y. We consider the following two-step iteration

$$\begin{aligned}
y_n &= x_n - [F'(x_n)]^{-1}F(x_n) = x_n - \Gamma_n F(x_n),\\
z_n &= x_n + \frac{2}{3}(y_n - x_n) = x_n - \frac{2}{3}\Gamma_n F(x_n)\\
H(x_n, y_n) &= \Gamma_n \left[F'(z_n) - F'(x_n)\right],\\
x_{n+1} &= y_n - \frac{3}{4}\left[I + \frac{3}{2}H(x_n, y_n)\right]^{-1} H(x_n, y_n)(y_n - x_n)
\end{aligned} \tag{2}$$

for generating approximate solutions to (1).

We study the convergence of the above two-step iteration to a solution of (1). This method was introduced by Jarratt [8], [9] following the basic idea of increasing the order of convergence and avoiding the computation of the second Fréchet derivative in Newton's method, so that a two-point scheme is applied. (This idea was first found and developed by Ostrowski [11] and Traub [12] for functions of one variable).

Observe that (2) can be viewed as an extension of the classical Newton method. This method is computationally competitive for scalar equations and for systems of very few equations, which was Jarratt's original purpose (see[8],[9],[12]).

For this study, we use the technique considered in [2],[3] for one-point iterative processes that consists of constructing a system of real sequences, such that the convergence of the iterative process in Banach spaces follows from the analysis of these real sequences. The use of real sequences simplifies the study of iterations in Banach spaces by means of the majorizing principle [10], [1].

In Section 2, we construct a system of real sequences which are analysed and some properties are given. Next, in Section 3, we establish a convergence theorem and also give an explicit expression for the error bound of the method (2). Finally, a well-known integral equation is considered and we reach better error bounds than the ones obtained by other authors.

2 General Recurrence Relations

Let F be a nonlinear three times Fréchet-differentiable operator defined on some open convex subset Ω of a Banach space X with values in another Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Let us assume that

(c$_1$) $\|\Gamma_0\| \leq \alpha$,

(c$_2$) $\|\Gamma_0 F(x_0)\| \leq \beta$,

(c$_3$) $\|F''(x)\| \leq M, \quad x \in \Omega$,

(c$_4$) $\|F'''(x)\| \leq N, \quad x \in \Omega$,

(c$_5$) $\|F'''(x) - F'''(y)\| \leq K\|x - y\|, \quad x, y \in \Omega, \quad K \geq 0$.

Let us denote $\theta = M\alpha\beta, \omega = N\alpha\beta^2$ and $\delta = K\alpha\beta^3$. We define the sequences

$$a_0 = 1 = c_0, \quad b_0 = \frac{2}{3}\theta, \quad d_0 = \frac{2-\theta}{2(1-\theta)}$$

and

$$a_{n+1} = \frac{a_n}{1 - \theta a_n d_n}, \quad b_{n+1} = \frac{2}{3}\theta a_{n+1} c_{n+1},$$

$$c_{n+1} = \frac{2^3(2-3b_n)^4}{(4-3b_n)^4}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3 + \frac{17}{108}\delta + \frac{\omega a_n}{3(2-3b_n)}\theta\right] a_{n+1}\, d_n^4,$$

$$d_{n+1} = \frac{4-3b_{n+1}}{4-6b_{n+1}} c_{n+1},$$

for all $n \geq 0$.

We then prove the following statements:

[I$_n$] $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leq a_n \alpha$,

[II$_n$] $\|H(x_n, y_n)\| \leq b_n$,

[III$_n$] $\|\Gamma_n F(x_n)\| \leq c_n \beta$,

[IV$_n$] $\|x_{n+1} - x_n\| \leq d_n \beta$, for all $n \geq 0$. We assume for all $n \geq 0$ that

$$x_n, y_n \in \Omega, \quad \theta a_n d_n < 1 \quad \text{and} \quad \theta < \frac{1}{2},$$

and use mathematical induction on n. It is easy to check the case when $n = 0$ by initial hypothesis. We assume that $[\mathbf{I}_n]$–$[\mathbf{IV}_n]$ are true for a fixed $n \geq 1$.

$[\mathbf{I}_{n+1}]$: Under the above hypotheses, we have:

$$\|I - \Gamma_n F'(x_{n+1})\| \leq \|\Gamma_n\| \|F'(x_n) - F'(x_{n+1})\| \leq a_n \alpha M \|x_{n+1} - x_n\| \leq \theta a_n d_n < 1$$

and, by the Banach lemma [10], Γ_{n+1} exists and

$$\|\Gamma_{n+1}\| \leq \frac{\|\Gamma_n\|}{1 - \|I - \Gamma_n F'(x_{n+1})\|} \leq \frac{a_n \alpha}{1 - \theta a_n d_n} = a_{n+1}\alpha.$$

$[\mathbf{II}_{n+1}]$:

$$\|H(x_{n+1}, y_{n+1})\| \leq \|\Gamma_{n+1}\| \|F'(x_{n+1} + \frac{2}{3}(y_{n+1} - x_{n+1})) - F'(x_{n+1})\|$$

$$\leq \frac{2\theta}{3} a_{n+1} c_{n+1} = b_{n+1}.$$

$[\mathbf{III}_{n+1}]$: By means of Taylor's theorem, we have

$$F(x_{n+1}) = \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2$$

$$+ \int_0^1 [F''(x_n + t(y_n - x_n))(1-t) - \frac{1}{2}F''(x_n + \frac{2}{3}t(y_n - x_n))]dt(y_n - x_n)^2$$

$$+ \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{2}{3}t(y_n - x_n)) - F''(x_n + t(y_n - x_n))\right] dt(y_n - x_n)$$

$$* \left[I + \frac{3}{2}H(x_n, y_n)\right]^{-1} \Gamma_n \int_0^1 F''(x_n + \frac{2}{3}t(y_n - x_n))dt(y_n - x_n)^2$$

and

$$\left\|\int_0^1 \left[F''(x_n + t(y_n - x_n))(1-t) - \frac{1}{2}F''(x_n + \frac{2}{3}t(y_n - x_n))\right] dt\right\|$$

$$\leq \left\|\int_0^1 \int_0^1 [F'''(x_n + st(y_n - x_n)) - F'''(x_n)]\, ds\, t\,(1-t)\, dt(y_n - x_n)\right\|$$

$$+\frac{1}{3}\left\|\int_0^1\int_0^1 [F'''(x_n+\frac{2}{3}st(y_n-x_n))-F'''(x_n)]ds\,t\,dt(y_n-x_n)\right\|$$
$$\leq \frac{17}{216}K\|y_n-x_n\|^2.$$

Hence

$$\|\Gamma_{n+1}F(x_{n+1})\| \leq \|\Gamma_{n+1}\|\|F(x_{n+1})\|$$
$$\leq \frac{1}{2}a_{n+1}\alpha\left[M\|x_{n+1}-y_n\|^2+\left(\frac{17}{108}K+\frac{MN}{3(2-3b_n)}\|\Gamma_n\|\right)\|\Gamma_n F(x_n)\|^4\right]$$
$$\leq \frac{a_{n+1}}{2}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3+\frac{17}{108}\delta+\frac{a_n\omega}{3(2-3b_n)}\theta\right]c_n^4\beta$$
$$=\frac{8(2-3b_n)^4}{(4-3b_n)^4}\left[\frac{a_n^2}{(2-3b_n)^2}\theta^3+\frac{17}{108}\delta+\frac{a_n\omega}{3(2-3b_n)}\theta\right]a_{n+1}d_n^4\beta=c_{n+1}\beta. \quad (3)$$

$[\mathbf{IV}_{n+1}]$:

$$\|x_{n+2}-x_{n+1}\| \leq \|\frac{3}{4}\left[I+\frac{3}{2}H(x_{n+1},y_{n+1})\right]^{-1}H(x_{n+1},y_{n+1})\|\|y_{n+1}-x_{n+1}\|$$
$$\leq \frac{4-3b_{n+1}}{4-6b_{n+1}}c_{n+1}\beta=d_{n+1}\beta.$$

This completes the induction and shows that the previous statements $[\mathbf{I}_n]$–$[\mathbf{IV}_n]$ hold for all $n\in\mathbb{N}$.

3 Semilocal convergence analysis

We now prove the convergence of sequence (2) from an analysis of the sequences $\{a_n\},\{b_n\},\{c_n\}$ and $\{d_n\}$. We first need the following technical lemma, whose proof is trivial.

Lemma 3.1 *Let us define the following real functions:*

$$f(x)=\frac{27x^4(2-3x)^2}{[9x^2-24x+8]^2},$$
$$g(x,y)=f(x)\left[\frac{1}{(2-3x)^2}+\frac{17}{108}\frac{A}{y^2}+\frac{1}{3}\frac{B}{y(2-3x)}\right], \quad (4)$$

where $A, B \in \mathbb{R}_+$, *and*

$$q(x) = \frac{(2-3x)^2}{[9x^2-24x+8]^2},$$

$$p(x) = \frac{(3x-1)(27x^3-144x^2+192x-64)}{9x^3(2-3x)^2}. \tag{5}$$

Then

(i) $f : [0, \frac{1}{3}] \to \mathbb{R}$ *is an increasing function and* $f(0) = 0$,

(ii) f' *is increasing and* $f'(0) = 0$,

(iii) $g(x, y) < g(x, 1)$ *for* $y \geq 1$ *and* $x \in \left[0, \frac{1}{3}\right)$,

(iv) *the function* $g_0(x) = g(x, 1)$ *is increasing for* $x \in \left[0, \frac{1}{3}\right)$,

(v) *the function* $p(x)$ *is decreasing for* $x \in \left[0, \frac{1}{3}\right)$,

(vi) *the function* $q(x)$ *is increasing for* $x \in \left[0, \frac{1}{3}\right)$.

With these notations, we can write

$$b_{n+1} = g(b_n, a_n), \quad n \geq 0, \tag{6}$$

where g is defined in (4). Next, we give some properties for the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$.

Theorem 3.2 *Let* $\theta \in (0, 1/2)$, *and*

$$0 \leq \delta < \frac{27(2\theta-1)(\theta^3-8\theta^2+16\theta-8)}{17(1-\theta)^2},$$

$$0 \leq \omega < \frac{3(2\theta-1)(\theta^3-8\theta^2+16\theta-8)}{4\theta(1-\theta)^2} - \frac{17\delta}{36\theta}. \tag{7}$$

Then

(i) *the sequence* $\{b_n\}$ *is decreasing,*

(ii) $\theta a_n d_n < 1$ *for all* $n \geq 0$,

(iii) $a_n \geq 1$ *for all* $n \geq 0$,

(iv) *the sequence* $\{a_n\}$ *is increasing.*

Proof. We prove for all $n \geq 0$:

(i$_n$) $b_n \geq b_{n+1}$,

(ii$_n$) $\theta a_n d_n < 1$,

(iii$_n$) $a_n \geq 1$,

(iv$_n$) $a_n \leq a_{n+1}$.

We use induction on n. For $n = 0$, taking into account (4), **(i$_0$)** is equivalent to

$$\frac{17}{36}\frac{\delta}{\theta^3} + \frac{\omega}{\theta^2} < \frac{(3b_0 - 1)(27b_0^3 - 144b_0^2 + 192b_0 - 64)}{9b_0^3(2 - 3b_0)^2} = p(b_0).$$

This follows from (6). As $\theta < 1/2$, $a_0 = 1$ and $d_0 < 2$, **(ii$_0$)** is trivial and **(iii$_0$)** and **(iv$_0$)** follow immediately. We now suppose that **(i$_k$)**–**(iv$_k$)** hold for $k = 1, \ldots, n-1$. By (5), lemma 3.1 **(v)**, **(i$_{n-1}$)** and **(iv$_{n-1}$)** we obtain **(i$_n$)**.

(ii$_n$) is immediate from

$$\theta a_n d_n = \frac{3b_n(4 - 3b_n)}{4(2 - 3b_n)}$$

and $b_n \leq 1/3$. Finally, **(iii$_n$)** and **(iv$_n$)** follow by recurrence and the induction is concluded. ∎

Theorem 3.3 *Under the assumptions of theorem 3.2, if we denote* $\gamma = b_1/b_0$*, then* $\gamma < 1$,

$$b_n \leq \frac{\gamma^{\frac{4^{n+1}}{3}}}{\gamma^{1/3}} b_0, \quad n \geq 1. \tag{9}$$

In addition, $\lim_n b_n = 0$ *and* $\sum_{n=0}^{\infty} b_n < +\infty$.

Proof. By conditions (7) and (8), $b_1 < b_0$ and $\gamma = b_1/b_0 < 1$.

In addition, as $b_1 = \gamma b_0 < b_0$, and by Lemma 3.1 (v) and (vi) it follows that $b_2 = g(b_1, a_1) \leq \gamma^4 b_1$, since $a_1 \geq a_0$. Next, by recurrence, we infer that $b_n \leq \gamma^4 b_{n-1}$ and consequently (9) is satisfied for all $n \geq 1$. It is then clear that $\lim_n b_n = 0$. Finally, $\sum_{n=0}^{\infty} b_n < +\infty$ as a consequence of $\sum_{n=0}^{\infty} b_n \leq \frac{b_0}{\gamma^{\frac{1}{3}}} \sum_{n=0}^{\infty} \gamma^{\frac{4^{n+1}}{3}}$.
■

Theorem 3.4 *Under the assumptions of theorem 3.2, there exists an $L \in \mathbb{R}$, $L \geq 1$, such that $a_n \leq L$ for all $n \geq 0$.*

Proof. We can write

$$a_{n+1} = \prod_{i=0}^{n} \left[1 + \frac{3b_i(4 - 3b_i)}{9b_i^2 - 24b_i + 8} \right]$$

from the definition of the sequence $\{a_n\}$. Taking into account that

$$u(x) = \frac{(4 - 3x)}{9x^2 - 24x + 8}$$

is an increasing function for $x \in [0, \frac{1}{3})$, we obtain that

$$3b_i u(0) \leq 3b_i u(b_i) \leq 9b_i$$

for all $i \geq 0$. Then, as a consequence of the fact that $\sum_{n=0}^{\infty} b_n < +\infty$, it follows that

$$\prod_{n=0}^{\infty} (1 + 3b_n u(b_n)) < +\infty$$

and the proof is concluded. ■

Theorem 3.5 *Under the assumptions of theorem 3.2, we have*

$$d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)} \frac{\gamma^{\frac{4^{n+1}}{3}}}{\gamma^{1/3}} b_0$$

for all $n \geq 0$ and $\gamma = b_1/b_0$.

Proof. Note that

$$d_n = \frac{3b_n}{4\theta a_n}\frac{4-3b_n}{2-3b_n}.$$

As the function $r(x) = (4-3x)/(2-3x)$ is increasing for all x, we get $2 = r(0) \leq r(b_n) \leq r(b_0) = (2-\theta)/(1-\theta)$. Hence

$$d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)}b_n$$

since $a_n \geq 1$ for all $n \geq 0$.

The proof is completed by theorem 3.3. ∎

Remark. Note that $\sum\limits_{n=0}^{\infty} d_n < +\infty$ follows from the previous theorem and from taking into account that

$$\frac{6}{4\theta L}b_n \leq d_n \leq \frac{3(2-\theta)}{4\theta(1-\theta)}b_n$$

since $1 \leq a_n \leq L$ for all $n \geq 0$.

We are now ready to prove the main convergence theorem. Let us denote $\overline{B(x,r)} = \{y \in X; \|y-x\| \leq r\}$ and $B(x,r) = \{y \in X; \|y-x\| < r\}$.

Theorem 3.6 *Let F be a nonlinear operator which is three times Fréchet differentiable in an open convex domain Ω of a Banach space X with values in another Banach space Y. Let us suppose that $\Gamma_0 = F(x_0)^{-1}$ exists at some $x_0 \in \Omega$. Assume $(\mathbf{c_1}) - (\mathbf{c_5})$ and $\overline{B(x_0, R\beta)} \subset \Omega$ where $R = \lim\limits_n \sum\limits_{i=0}^{n} d_i$. Let us denote $\theta = M\alpha\beta, \omega = N\alpha\beta^2$ and $\delta = K\alpha\beta^3$. If $\theta \in (0,1/2)$, and the conditions in (7) are verified, then the sequence $\{x_n\}$ given by (2) is well-defined, $x_n, y_n \in B(x_0, R\beta)$ for all $n \geq 0$, and converges to a solution $x^* \in \overline{B(x_0, R\beta)}$ of equation (1). Moreover, the solution x^* is unique in $B(x_0, \frac{2}{\alpha M} - R\beta)$. Furthermore, we also have the following error estimates:*

$$\|x^* - x_n\| \leq \sum_{i \geq n} d_i\beta \leq \beta\frac{(2-\theta)}{2(1-\theta)}\frac{1}{\gamma^{1/3}}\sum_{i\geq n}\gamma^{4^{i+1}/3} \tag{10}$$

where $\gamma = b_1/b_0$.

Proof. Under the assumptions mentioned above, it follows immediately from the previous theorems that sequence (2) is convergent. We also have

$$\|x_n - x_0\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_1 - x_0\| \leq \sum_{i=0}^{n-1} d_i\beta < R\beta.$$

Following a similar method, we obtain that

$$\|y_n - x_0\| < R\beta$$

and since $z_n = x_n + \frac{2}{3}(y_n - x_n)$, then $x_n, y_n, z_n \in B(x_0, R\beta) \subset \Omega_0$.

If x^* is the limit of the sequence (2), taking into account (3) and the continuity of F, we get that $F(x^*) = 0$.

In addition, we have $\|x_{n+1} - x_n\| \leq d_n\beta$ and consequently, for $p \geq 0$,

$$\begin{aligned}\|x_p - x_0\| &\leq \|x_p - x_{p-1}\| + \|x_{p-1} - x_{p-2}\| + \cdots + \|x_1 - x_0\| \\ &\leq (d_{p-1} + d_{p-2} + \cdots d_0)\beta.\end{aligned}$$

By letting $p \to \infty$, we obtain $\|x^* - x_0\| \leq R\beta$ and the error estimates given in (10).

Finally, to show the uniqueness, let us assume that $y^* \in B(x_0, \frac{2}{M\alpha} - R\beta)$ is another solution of $F(x) = 0$. According to [7]

$$0 = \Gamma_0[F(y^*) - F(x^*)] = \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*).$$

Then we only have to prove that the operator $\left[\int_0^1 F'(x^* + t(y^* - x^*))\, dt\right]^{-1}$ exists and therefore $y^* = x^*$. ∎

Example. Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, with the max-norm and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - 1 - \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt,$$

with $s \in [0,1]$, $x \in C[0,1]$ and $0 < \lambda \leq 2$. Integral equations of this kind (called Chandrasekhar equations) arise in elasticity or neutron transport problems [4].

For $\lambda = 1/4$, it is easy to prove that

$$[F'(x)(y)](s) = y(s) - \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}y(t)\,dt - \frac{1}{4}y(s)\int_0^1 \frac{s}{s+t}x(t)\,dt,$$

$$[F''(x)(yz)](s) = -\frac{1}{4}\left[z(s)\int_0^1 \frac{s}{s+t}.y(t)\,dt + y(s)\int_0^1 \frac{s}{s+t}.z(t)\,dt\right].$$

and

$$[F'''(x)(yzu)](s) = 0$$

and starting at $x_0 = x_0(s) = s$, we have

$$F(x_0)(s) = -\frac{s}{4}\log\frac{s+1}{s},$$

$$\|F(x_0)(s)\| = \frac{\log 2}{4}.$$

The existence of Γ_0 will now be proved and we will calculate the parameters appearing in Theorem 3.6

$$\|[I - F'(x_0)]\| = \max_{s\in[0,1]} |y(s) - F'(x_0)y(s)| \le \frac{\log 2}{2} = 0.3465\ldots < 1.$$

The Banach lemma gives the existence of Γ_0 and we obtain:

$$\|\Gamma_0\| \le \frac{1}{1-(\log 2/2)} = \alpha = 1.5303\ldots$$

$$\|\Gamma_0 F(x_0)\| \le \frac{(\log 2/4)}{(\log 2/2)} = \beta = 0.2651\ldots,$$

$$\|F''(x)\| \le 0.3465\ldots = M, \qquad N = K = 0.$$

So $\theta = M\alpha\beta = 0.1406312\ldots$ and $\omega = \delta = 0$, and the conditions of Theorem 3.6 are fulfilled.

To compare the super-Halley and the Jarratt methods, we give an upper bound C to the number $10^{16}\|x^* - x_2\|$, where x_2 is the second iterate obtained by using these methods. We have that C is 31 for the super-Halley method and 0.143799 for the Jarratt method. Notice that the above operator is quadratic and the super-Halley method has order four for this kind of operators [5], [6].

Conclusion: The Jarratt and super-Halley methods both have order four as we can see. However, the error bound that we have for the Jarratt method is much better than the one obtained for the super-Halley method.

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