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Helly property associated with the discrete planes.

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Abstract

Discrete planes are basic concepts in discrete geometry and digital topology. In this note we introduce a discrete version of the Helly theorem, and we show that only 8-connected grids have the Helly property.

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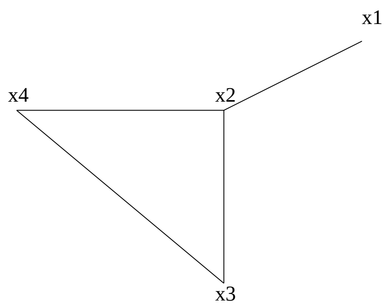


Figure 1: Example of a graph

1 Introduction

The growth of computer sciences has highlighted discrete mathematics. Discrete geometry and digital topology which are derived from classical mathematical fields, are the focus of much research works. These fields are crucial to image analysis. One major concept of discrete geometry and digital topology is the notion of grid graph or discrete plane. In this note we associate the helly property with these planes.

2 Preliminaries

2.1 Graph

Recall that a *graph* is a couple $G = (V; E)$, where V is a set of elements called *vertices*, and E is a set of unordered pairs of members of V called *edges*.
Two vertices are *adjacent* if they are linked by an edge.
Given a graph G, we denote by $\Gamma(x)$ the *neighborhood* of a vertex x, i. e. the set formed by all vertices adjacent to x is defined by:

$$\Gamma(x) = \{y \in V, \{x, y\} \in E\}$$

Figure 1 is an example of a graph.

The number of neighborhood vertices x is the *degree of* x , (denoted by dx). If dx is constant, the graph is called *regular.* A graph is *loopless* if it does not contain an edge of type (x, x).
A *chain* in a graph is a sequence of distinct vertices $(v_1, v_2, \dots v_p)$ linked by an edge. A *cycle* is a chain such as the first vertex and the last vertex are the same. We note a cycle with the length equal to n, C_n.
A graph is *connected* if any pair of vertices is linked by a chain.
A cycle is *centered* if there exists a vertex adjacent to all the vertices from the cycle.
A *sun* S_3 is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ such that :

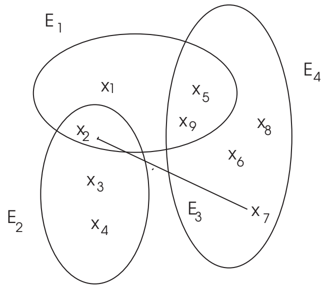


Figure 2: Representation of the hypergraph. In this example we have S the set of vertices with $\mathbf{S} = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9\}$. The set of hyperedges is E with $E =\{\ E_1 = \{x_1, x_2, x_5, x_9\},\ E_2 = \{x_2, x_3, x_4\},\ E_3 = \{x_2, x_7\},\ E_4 = \{x_5, x_6, x_7, x_8, x_9\}\}$, the star centered on x_2 is $H(x_2) = \{E_1,\ E_2,\ E_3\}$, with degree $dx_2 = 3$ and the star centered on x_4 is $H(x_4) = \{E_2\}$, with degree $dx_4 = 1$.

- a_1, a_2, a_3, a_1 is a cycle.
- b_i has only two neighbors a_i et a_{i+1} (i mod 3)

Figure 4 shows a cycle C_4 and a sun S_3.

2.2 Hypergraph

A *hypergraph* H on a set $\mathbf{S}$ is a family $(E_i)_{i\in I}$ of non-empty subsets of $\mathbf{S}$ called hyperedges with:

$$\bigcup_{i\in I} E_i = \mathbf{S}$$

$$\forall i \in I \quad I = \{1, 2, \ldots, n\} \quad n \in \mathbb{N}$$

Let us note :

$$H = (\mathbf{S}; (E_i)_{i\in I})$$

One calls *n-family* any sub-family from $(E_i)_{i\in I}$ with n edges.
For $x \in \mathbf{S}$, a *star* of H (with center x) is the set of hyperedges which contains x, and is called $H(x)$. The *degree of* x is the cardinality of the star $H(x)$ denoted by :

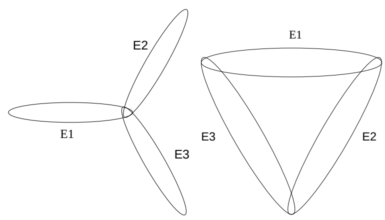
$$dx = card(H(x)).$$

We can represent a hypergraph by the figure 2.

A hypergraph has the *Helly property* if each family of hyperedges intersecting two by two (intersecting family) has a non-empty intersection (belonging to a star).

Figure 3 shows two examples of intersecting hyperedges :

- Fig. 3(a) : A family of three hyperedges intersecting two by two with a non-empty intersection
- Fig. 3(b) : A family of three hyperedges intersecting two by two with an empty intersection



(a) star (b) triangle

Figure 3: Suppose that the hypergraphs have only three hyperedges, then the first hypergraph has the Helly property.

Remark. *A loopless graph with no isolated vertex is therefore a particular hypergraph with every hyperedge having cardinality 2.*

A hypergraph is *regular* if any edge has the same cardinality.

2.2.1 Neighborhood hypergraph

Given a graph G, the hypergraph having the vertices of G as vertices and the neighborhood of these vertices as hyperedges (including these vertices), is called the *neighborhood hypergraph* of G. To each graph we can associate a neighborhood hypergraph:

$$H_G = (\mathbf{S}, (E_x = \{x\} \cup \Gamma(x)))$$

The neighborhood hypergraph associated with the example in figure 1 is:

$$\mathbf{S} = \{x_1, x_2, x_3, x_4, \}$$

$$E_{x_1} = \{x_1, x_2\},\ E_{x_2} = \{x_1, x_2, x_3, x_4\},$$

$$E_{x_3} = \{x_2, x_3, x_4\},\ E_{x_4} = \{x_2, x_3, x_4\}$$

A *discrete plane* (resp a *grid graph*) is a loopless regular connected graph such that :
The vertices are the points of a regular lattice $\mathbf{X}$ from $\mathbb{R}^2$ which can be either square or triangular or hexagonal.
The set of edges is generated thanks to classical distances :

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

and

$$d_2(x,y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{\frac{1}{2}}$$

On a square lattice the grid associated with d_1 (resp d_∞) is the 4-connected grid (resp 8-connected grid), one notes it Γ^4 (resp Γ^8). On a triangular lattice the grid associated with d_2 is the 6-connected grid, one note it Γ^6. On an hexagonal lattice one defines the 3-connected grid thanks to d_2, which is noted by Γ^3.

Generally the grid is defined by :

$$\Gamma(x) = \{y \in \mathbf{X}, \mathbf{d}(\mathbf{x},\mathbf{y}) = \mathbf{1}\}$$

On a grid, one defines the neighborhood graph (or neighborhood relation) in this way:

$$\Gamma_p(x) = \{y \in \mathbf{X}, \mathbf{1} \leq \mathbf{d}(\mathbf{x},\mathbf{y}) \leq \mathbf{p}(\mathbf{x})\}$$

where p is a mapping from $\mathbf{X}$ in $\mathbf{N}^*$ which gives the size of the neighborhood of x. This graph leads to the following neighborhood hypergraph:

$$H_{\Gamma_p} = (\mathbf{X}, (\{x\} \cup \Gamma_p(x)))$$

If $p(x)$ is a constant n, the hypergraph is uniform, it is noted by:

$$H_n = (X, (E_x = \{x\} \cup \Gamma_n(x)))$$

3 Grid graph and Helly property

Theorem 1. *Let H_{Γ_p} be a neighborhood hypergraph associated with a discrete plane.*
H_{Γ_p} has the Helly property if and only if the discrete plane is an 8-connected grid.

Proof. The condition is sufficient. Indeed we will prove the following lemma:

Lemma 1. *The neighborhood hypergraphs associated with the grid* Γ^8 *have the Helly property.*

Proof. Let $(E_i)_{i\in I}$ be an intersecting n-family, and (x_i, y_i) the coordinates of the vertex i which is the generator of E_i. The relation

$$E_i \cap E_j \neq \emptyset$$

can be expressed in the following way:

$$|x_i - x_j| \leq p_i + p_j \textit{ and } |y_i - y_j| \leq p_i + p_j$$

with $p(x_i, y_i) = p_i$ et $p(x_j, y_j) = p_j$. The first inequality implies that :

$$x_i - p_i \leq x_j + p_j \quad \forall\, i, j \in I$$

consequently

$$x^- = Max(x_i - p_i) \leq Min(x_j + p_j) = x^+$$

In the same way, we have:

$$y^- = Max(y_i - p_i) \leq Min(y_j + p_j) = y^+$$

The result comes from the equality:

$$\bigcap_{i\in I} E_i = \{(x, y), x^- \leq x \leq x^+ \text{ et } y^- \leq x \leq y^+\}$$

□

To prove that the condition is necessary , it is sufficient to show that the uniform hypergraphs associated with the grid different from Γ^8 do not have the Helly property. Indeed it is sufficient to find a configuration of edges which does not verify the Helly property and to vary the size of the edges from the configuration.

Let H be a neighborhood hypergraph associated with a graph. If this graph contains either a no centered cycle with a length equal to four, five, six, or a non-centered sun S_3, obviously the neighborhood hypergraph does not have the Helly property. Figure 4 gives two examples from C_4 and S_3.

Lemma 2. *The hypergraphs associated with the grid* Γ^4 *do not have the Helly property.*

Proof. Now we will show that there exists an intersecting 4-family which is not a star. To do this, one builts four rectangular triangles on the grid by the following algorithm

- Let (x, a, u) be a rectangular triangle such that $d_1(x, a) = n-1$, $d_1(a, u) = n$, from the Montanari theorem [6] $d_1(x, u) = 2n - 1$

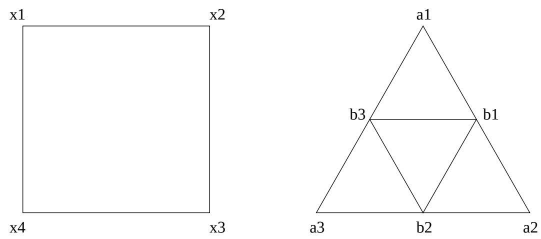


Figure 4: In the first figure we have $H_{C_4} = \{E_{x1} = \{x1, x2, x4\}; E_{x2} = \{x1, x2, x3\}; E_{x3} = \{x3, x2, x4\}; E_{x4} = \{x4, x1, x3\}$. These edges intersecte two by two but $\bigcap_{i=1}^{i=4} E_{xi} = \emptyset$. Second figure shows a S_3, we have $\bigcap_{i=1}^{i=3} E_{ai} = \emptyset$ but these edges intersecte two by two.

- Let (u, b, v) be a rectangular triangle such that $d_1(u, b) = n-1$, $d_1(b, v) = n$, one has $d_1(u, v) = 2n - 1$
- Let (v, c, w) be a rectangular triangle such that $d_1(v, c) = n-1$, $d_1(c, w) = n$, one has $d_1(v, w) = 2n - 1$
- Let (w, d, x) be a rectangular triangle such that $d_1(w, d) = n-1$, $d_1(d, x) = n$, one has $d_1(w, x) = 2n - 1$

Consequently, one obtains a cycle (x, u, v, w, x) with a length equal to four which is not centered, so the family has an empty intersection. □

Lemma 3. *The neighborhood relation Γ_n associated with the grid Γ^6 contains a sun S_3 which is not centered. So, H_n the neighborhood hypergraph associated with this grid does not verify the Helly property.*

Proof. Let a_1, a_2, a_3 be three vertices with: $d_2(a_i, a_{i+1}) = 2n$ $(i = 0, 1, 2 \; mod \; 3)$, such that these vertices form an equilateral triangle (this triangle exists because it is the union of equilateral unity triangles). $2n$ being even, there exists a vertex b_i between a_i and b_i such that: $d_2(a_i, b_i) = d_2(b_i, a_{i+1}) = n$, so a_i is adjacent to b_i which is adjacent to a_{i+1}. Moreover, since $d_2(a_i, b_i) = d_2(b_{i-1}, a_i) = n$, and since the b_i are the middle of the sides of the equilateral triangle , one has : $d_2(b_i b_{i+1}) = n$ $(i = 0,\ 1,\ 2 \; mod \; 3)$. From this algorithm, one obtains a no centered sun S_3, thanks to : a_1, a_2, a_3, b_1, b_2, b_3. □

Lemma 4. *Let H_n be a hypergraph associated with the grid Γ^3:*

- *$n = 1$, Γ_n contains a non-centered cycle with a length equal to six. We note C_6.*
- *$n \geq 2$, Γ_n a non-centered sun S_3.*

The hypergraphs H_n do not have the Helly property.

Proof. If $n = 1$: it is obvious.

Suppose $n \geq 2$: Let x be a vertex, a_1, a_2, a_3, the vertices such that $d_2(x, a_1) = 1$, $d_2(x, a_2) = 1$, $d_2(x, a_3) = 1$. The three vertices a_1, a_2, a_3 form a cycle C_3 centered in x. Let a_i, a_{i+1} $(mod 3)$, there exists a vertex b_i such that $d_2(b_i, a_i) = n$ and $d_2(b_i, a_{i+1}) = n$.

One obtains a cycle $(a_1\ a_2\ a_3\ b_1\ b_2\ b_3)$ with : $(a_1\ a_2\ a_3)$ which is a cycle C_3, b_i is only adjacent to a_i and a_{i+1}. So $(a_1\ a_2\ a_3\ b_1\ b_2\ b_3)$ form a non-centered sun S_3 □

4 Conclusion

The main applications of discrete geometry and digital topology are in the image analysis and the pattern recognition. The introduction of the Helly property in image processing allows both to generalize the region growing segmentation [3], [1] and to give a new class of image filters [2].

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