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(continued on inside back cover)

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NUMERICAL STUDY OF FLAME PROPAGATION IN BOUNDARY LAYERS

Maria E. Calzada

Abstract. An incompressible, reactive, viscous flow reaches a hot circular cylinder causing a reaction to start. We observe the progress of the reaction, concentrating in the transition towards turbulence within the boundary layer. The problem is divided into the fluid dynamics effects and the combustion effects. The motion of the flow is resolved by the use of a hybrid vortex algorithm; the reactive part of the problem is modeled by a combination of a temperature equation and a flame advection and propagation algorithm that mimics Huyghens' principle. The following paper will explain these methods, how to implement them, and will present the numerical results of the experiment.

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Key Words and Phrases. Fluid Dynamics, Reactive Flows, Navier-Stokes Equations, Vortex Methods.

1. HYDRODYNAMICS EFFECTS

The motion of an incompressible, viscous flow is governed by the Navier-Stokes equations, which in vorticity transport form can be written as:

$$\begin{gathered} \partial_t \xi + (\vec{u} \cdot \vec{\nabla})\xi = Re^{-1}\Delta\xi \\ div(\vec{u}) = 0, \end{gathered} \tag{1}$$

where $\vec{u} = (u, v)$ is the velocity field, $\xi = curl(\vec{u}) = \partial_x v - \partial_y u$, Re is the Reynolds number associated with the flow, and Δ is the Laplacian operator. To observe turbulent effects we will choose the Reynolds number to be large.

We will solve the Navier-Stokes equations in an indirect way, by using a stream function ψ and a hybrid vortex algorithm. The stream function ψ has the properties that $u = \partial_y \psi$, $v = -\partial_x \psi$, and therefore $\Delta\psi = -\xi$. The idea of the vortex blobs method is to solve for ψ, given ξ, and then by differentiating ψ obtain $\vec{u} = (u, v)$.

Near the boundary of the cylinder, the Navier-Stokes equations can be approximated by Prandlt's boundary layer equations, which in vorticity transport form can be written as:

$$\begin{gathered} \partial_t \xi + (\vec{u} \cdot \vec{\nabla})\xi = \nu \partial_y^2 \xi \\ div(\vec{u}) = 0 \\ \xi = -\partial_y u. \end{gathered} \tag{2}$$

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These equations are only valid in a very thin region near the boundary of the cylinder due to the turbulent effects. In this region we will use the vortex sheets method to resolve the motion of the flow.

The vortex methods were originally developed by Alexander Chorin in the 1970's [1, 2]. They are Lagrangian methods best suited for flows at large Reynolds numbers. The vortex methods have been used successfully in many applications [3, 4, 5, 6, 7]. We will first describe the vortex blobs method and then the vortex sheets methods.

The vortex blobs method: The basis of the vortex blobs method is to solve an inviscid flow with no boundaries. Such a flow is regulated by Euler's equations in terms of vorticity transport in the entire plane:

$$\begin{aligned} \partial_t \xi + (\vec{u} \cdot \vec{\nabla})\xi &= 0 \\ div(\vec{u}) &= 0. \end{aligned} \tag{3}$$

Suppose we have a collection of N vortex blobs, approximations of point vortices, with circulations $\xi_1, \xi_2, \ldots, \xi_N$, located at the points $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ respectively, where $\vec{r} = (x, y)$. Assume there are no other contributions to the vorticity field. We can write the vorticity field in the following way:

$$\xi(\vec{r}) = \sum_{j=1}^{N} \xi_j \xi_o(\vec{r} - \vec{r}_j), \tag{4}$$

where ξ_o is the following smooth approximation to the Dirac delta function:

$$\xi_o(\vec{r}) = \begin{cases} \frac{1}{2\pi\sigma\|\vec{r}\|}, & \|\vec{r}\| < \sigma, \\ 0, & \|\vec{r}\| \geq \sigma. \end{cases} \tag{5}$$

σ is a cut-off to be determined. We can use this equation to find an appropriate stream function ψ, and by differentiation we can find the velocity field $\vec{u}_\xi$ that regulates the motion of the vortex blobs and the rest of the flow. Using this velocity field we are able to reproduce the successive motion of the blobs; for example, if we use Euler's method to move the blobs, their position at the next time step will be given by:

$$\vec{r}_i^{\,n+1} = \vec{r}_i^{\,n} + \Delta t \vec{u}_\xi(\vec{r}_i^{\,n}), \tag{6}$$

where $\vec{r}_i^{\,n} = \vec{r}_i(n\Delta t)$, and Δt is the time step.

We now make a correction to this flow by introducing boundaries. This creates the boundary condition $\vec{u} \cdot \vec{n} = 0$ ($\vec{n}$ is a unit vector normal to the boundary). We satisfy this boundary condition using a potential flow $\vec{u}_p = (u_p, v_p)$, solving the following problem: find φ so that $\partial_x \varphi = u_p$ and $\partial_y \varphi = v_p$ satisfying $\Delta\varphi = 0$, in D and $\partial_n \varphi = -\vec{u}_\xi \cdot \vec{n}$ on ∂D, where D is the computational domain, and ∂D is the boundary of D. As can be verified, this potential flow would cancel any normal component of velocity at the boundary of the cylinder introduced by the vortex blobs without affecting the vorticity of the flow. The position of the blobs is now approximated by:

$$\vec{r}_i^{\,n+1} = \vec{r}_i^{\,n} + \Delta t[\vec{u}_\xi(\vec{r}_i^{\,n}) + \vec{u}_p(\vec{r}_i^{\,n})]. \tag{7}$$

We make a second correction to this flow by finally introducing viscosity, that is $Re^{-1} \neq 0$; however, for the moment we disregard boundaries. The equations governing the flow are the full Navier-Stokes equations (1). We simulate this viscosity by adding a random walk to the blobs. Let $\vec{\eta}_i^{\,n} = (\eta_{x_i}^n, \eta_{y_i}^n)$, where $\eta_{x_i}^n$ and $\eta_{y_i}^n$ are normally distributed random variables with mean 0 and variance $2\Delta t/Re$. The motion of the blobs is now regulated by:

$$\vec{r}_i^{\,n+1} = \vec{r}_i^{\,n} + \Delta t[\vec{u}_\xi(\vec{r}_i^{\,n}) + \vec{u}_p(\vec{r}_i^{\,n})] + \vec{\eta}_i^{\,n}. \tag{8}$$

In the pure vortex blobs method we would consider the boundary at this point and would deal with the no-slip condition, $\vec{u} \cdot \vec{s} = 0$, caused by the viscosity ($\vec{s}$ is a unit tangent vector to the boundary). To cancel this tangential velocity component at the wall, vorticity would be created. However, in our hybrid algorithm, we utilize the Prandlt's boundary layer equations and the vortex sheets method to create vorticity at the boundary. We discuss this method next.

The vortex sheets method: The original vortex sheets method was designed using a flat plate as the boundary of the flow [2]. However, a simple change of variables allows us to use virtually the same equations when the boundary of the flow is a circular cylinder (see figure 1). All calculations for the sheets take place in the xy-plane while their real location is in the x_1y_1-plane, with relations $\theta = \frac{x}{2\pi r}$, $x_1 = (r+y)\cos\theta$, and $y_1 = (r+y)\sin\theta$. It is clear that the computational domain for the xy-plane needs to be periodical in the x direction with period $2\pi r$, i.e., when $x > 2\pi r$, x needs to be replaced by $x - 2\pi r$ to have $0 \le x \le 2\pi r$. Integrating Prandlt's boundary layer equations (2) and solving appropriately for u and v in a way that automatically satisfies the boundary condition $\vec{u} \cdot \vec{n} = 0$, we can obtain the following equations for the velocity field:

$$u(x,y) = U_\infty(x) - \int_y^\infty \xi dy. \tag{9}$$

$$v(x,y) = -\int_0^y \partial_x u dy. \tag{10}$$

In these equations $U_\infty(x)$ is the free stream velocity. Within the boundary layer, vorticity is carried in sheets. We seek to simulate this effect accurately by introducing vortex sheets S_i. They are parallel to the boundary, have length h and are centered at $r_i = (x_i, y_i)$. The intensity of a vortex sheet is defined to be $\xi_i = -(\vec{u} \cdot \vec{s}^{\,+} - \vec{u} \cdot \vec{s}^{\,-})$, that is the jump of the tangential velocity across the sheet. We choose notation similar to that of vortex blobs because that will ease the transition and coupling between the sheets and the blobs. In a way much like the motion of vortex blobs, we can advance the position of the vortex sheets using (9) and (10):

$$x_i^{n+1} = x_i^n + \Delta t u_i \tag{11}$$

$$y_i^{n+1} = y_i^n + \Delta t v_i + \eta_i, \tag{12}$$

where η_i's, drawn from a normal distribution with mean 0 and variance $2\nu\Delta t$. The random walk is only applied in the y direction because in the Prandlt's equations diffusion is neglected in the x direction.

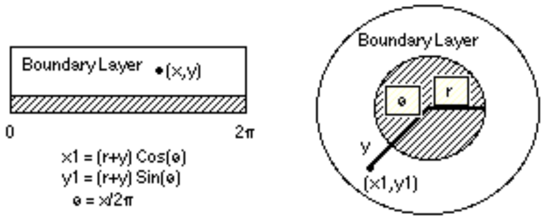

Figure 1. Illustration of Change of Variables for Sheets

To satisfy $\vec{u} \cdot \vec{s}$, we create sheet elements at the boundary in the following way: we take m segments of length h parallel to the boundary. We compute the tangential component of the velocity field at the centers of these segments, and imagine a vortex sheet of strength twice this velocity. We create sheets to cancel this velocity, incorporate the newly created sheets into the flow by a random walk, and let the sheets participate in the subsequent motion of the fluid.

Coupling the sheets and the blobs: Our hybrid vortex algorithm uses sheets near the boundary, where Prandlt's equations are valid, and blobs away from the boundary, where Prandlt's equations break down and the full Navier-Stokes equations regulate the flow. The boundary layer is proportional to $1/\sqrt{Re}$. From boundary layer theory we know that near the separation point the velocity gradient $(\vec{u} \cdot \vec{s})/(\vec{u} \cdot \vec{n})$ is very large. We use this criteria to assign a transition between sheets and blobs: Sheets having velocity gradient $(\vec{u}\cdot\vec{s})/(\vec{u}\cdot\vec{n})$ greater than a certain chosen parameter are close to the separation point and thus are turned into blobs. This parameter is determined by the geometry of the obstacle and by the mean value of the velocity gradients of all sheets. A sheet S_i with intensity ξ_i that leaves the boundary layer becomes a blob with circulation $h\xi_i$. As determined in [8] the cut-off σ is taken to be h/π. To ensure the consistency of the velocity across the boundary layer, imagine a sheet sitting on the edge of the boundary layer. It could be treated as a blob, and thus would be affected by all the other blobs. So all sheets should be affected by all blobs as well. But blobs are only affected by other blobs.

2. THE COMBUSTION SCHEME

In addition to the Navier-Stokes equations (1), the energy of the system is regulated by the following temperature equation:

$$\partial_t T + (\vec{u}\cdot\vec{\nabla})T = \Delta T + \Omega, \tag{13}$$

where Ω is a reaction term, and which is coupled to the Navier-Stokes equations through the velocity field $\vec{u}$. Equation (13) can be solved in a number of ways. We solved it by using operator splitting in the following way:

$$\partial_t T + (\vec{u}\cdot\vec{\nabla})T = 0 \tag{14}$$

$$\partial_t T = \Delta T \tag{15}$$

$$\partial_t T = \Omega \tag{16}$$

Equation (14) was solved using the random choice method presented in [6]; equation (15) was solved using Crank Nickolson [9]; and equation (16) was dealt with in the coupling of the SLIC algorithm with a flame dictionary to be discussed next.

SLIC and a flame dictionary: [10] discusses the construction of a flame dictionary that could be used to catalog the flame speed and preheat thickness of a given reaction. This dictionary can be coupled with different front tracking algorithms that would simulate the motion of the flame front. The current work couples the flame dictionary with a flame advection and propagation algorithm that simulates Huyghens' principle. Other front tracking algorithms like [11] can be used in conjunction with a flame dictionary.

SLIC (Simple Line Interface Calculation) was developed by Noh and Woodward [12] and has been successfully used to model a number of combustion problems. SLIC reconstructs the shape of the flame front using vertical and horizontal lines and moves it in a direction normal to itself using an appropriate velocity. Let the fluid occupy a domain D, with boundary ∂D, in two dimensions. The fluid subdomain $D_1(t) \subseteq D$ is burnt; the fluid in subdomain $D_2(t) = D - D_1(t)$ is unburnt. $D_1(t)$ is expanding; $D_2(t)$ is contracting. The boundary between $D_1(t)$ and $D_2(t)$ is an interface between the two fluids. This is the burning front that is both advected by the hydrodynamic velocity of the fluid and also moves with a speed $U(T)$, the flame speed, in a direction normal to itself. Superimpose a grid of mesh h on D, and consider a matrix $0 \leq f_{i,j}(n) \leq 1$ that at time n accounts for the proportion of burnt fluid contained in cell (i,j); we call this the matrix of burnt fractions. When called, SLIC reconstructs the shape of the burning front in each cell and moves it given the appropriate component of velocity. For more information on how SLIC reconstructs the flame front see [6].

The flame front is advected with the motion of the fluid. To mimic this SLIC reconstructs the front in sweeps. In a vertical sweep SLIC uses the vertical component of the velocity field from the vortex methods to move the flame front. In a horizontal sweep, SLIC uses the horizontal component of a velocity field from the vortex methods to move the flame front. At the end of the sweeps, $f_{i,j}(n)$ is updated to account for the new burnt fractions.

The flame front is also advanced within a direction normal to itself with a speed that depends on the chemical composition and on temperature. We use a flame dictionary of temperatures to obtain the appropriate flame speed for a given cell. We model the motion of the flame front in the direction normal to itself using the following numerical implementation of the Huyghens' principle: Consider eight angles equidistributed over the unit circle $\theta_l = l\pi/4, l = 1, \cdots, 8$. Move the flame front with the appropriate flame speed in each or those eight directions. Let $f_{ij}^{\theta_l}(n)$ be the new matrix of burnt fractions obtained by moving the flame front in each of the eight directions θ_l. Let $f_{ij}^{\theta_o}(n) = f_{ij}(n)$. We approximate the burnt region at time $n+1$ by

$$f_{ij}(n+1) = \max_{0 \leq l \leq 8} f_{ij}^{\theta_l}(n). \tag{17}$$

Note that under this numerical implementation of Huyghens principle having more than 8 equidistributed angles over the unit circle are unnecessary. Each cell (i,j) has exactly 8 immediate neighbors. The amount of mass transported from cell (i,j) to any of its neighbors is largest if the translation vector points from the center of cell (i,j) to the center of the neighboring cell. Those directions are the ones we are using in our implementation. Any other direction would not affect $f_{ij}(n+1)$.

We take care of equation (16) at this moment. The unburnt temperature of the fluid is given by T_u. When the fluid is burns, this temperature rises to $T_b = 1+T_u$. We mimic this by updating the temperatures of the cells that were newly burnt to the appropriate burnt temperature $1+T_u$ that related to their unburnt temperature. Care must be taken that the right unburnt temperature is used or the motion of the flame will be artificially accelerated.

3. NUMERICAL RESULTS OF EXPERIMENT

Figure 2 depicts the streamlines of a developing velocity field at different times, for $Re = 2000$. We can observe large vortices behind the circle and small vortices near the boundary of the cylinder due to the vorticity creation to satisfy the no-slip condition.

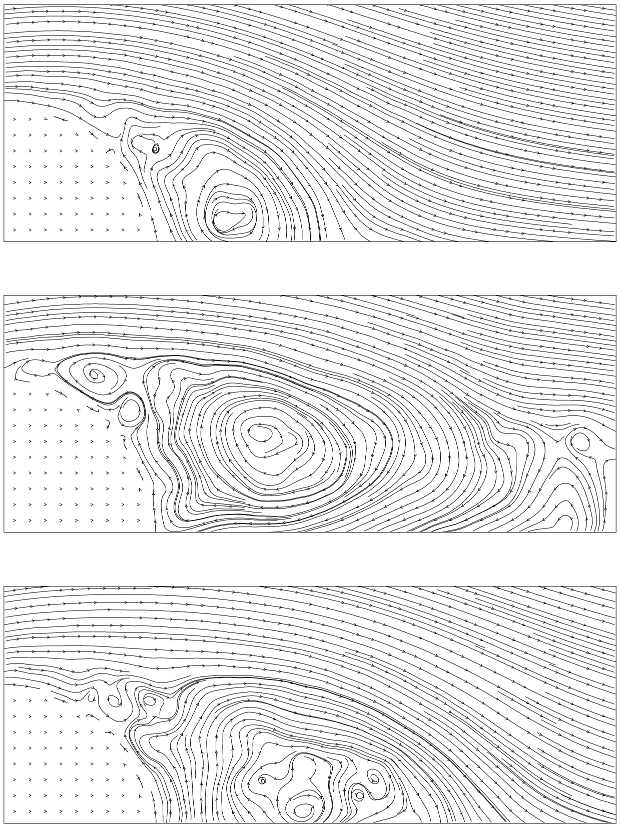

Figure 2. Typical Stream Lines at Different Time Steps

We ran the hybrid vortex procedures on a hot circular cylinder until we encountered well defined recirculation areas as the ones shown in Figure 2. We assumed that until that point the flow was not reactive. The fluid temperature was initially set at .05. The cylinder was initially set at .4. This differential in temperature caused a thermal boundary layer. The temperature diffused and was advected with the flow. After the flow developed nicely, a reactive fluid reached the cylinder and a spark ignited the fluid at the left edge of the cylinder. The reaction propagated as a flame front and moved with the flow and with the flame speed corresponding to the different temperatures.

Figure 3 represent temperature contour plots near the small recirculation areas of the circle were separation occurs. We observe jagged flame fronts due to the recirculation of the flow. We also observe how the recirculation both speeds up the burning process, due to more fluid being accessible to the front, and slows down the burning process, due to the burnt fluid being sent back onto itself.

The numerical simulation has very good qualitative agreement with experimental observations [13]. Other numerical methods agree with the current simulation [7]. The method described in [7] works very well; however it requires a large amount of memory to keep a dictionary of boundary value problems. This may become too expensive in certain settings. The current method uses a small dictionary of flame speeds; however it can prove to be slower than [7] in some cases. The quantitative modeling of the stochastic combustion parameters are left for a future study.

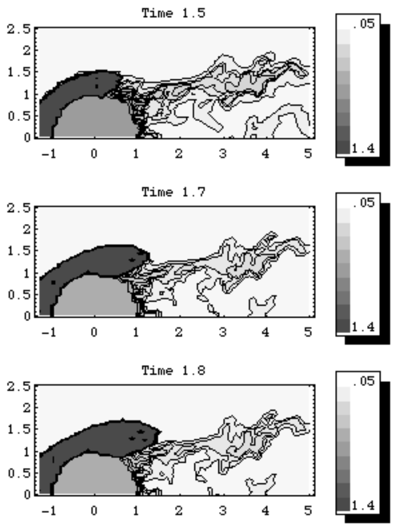

Figure 3. Temperature Profiles at Different Time Steps

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A HEURISTIC ALGORITHM FOR CONSTRUCTING OPTIMAL LOOKUP TABLES IN CIRCUIT DESIGN

Bing Shen and Ferenc Szidarovszky

Abstract. A heuristic algorithm is developed to find the optimum locations of interpolation points in constructing lookup tables in electronic circuit design. Cubic spline interpolators are used, and three versions of the algorithm are introduced. Computational results are reported to illustrate the effectiveness of the methodology.

1991 A.M.S. (MOS) subject classification Codes. 65D07,65D17.

Key Words and Phrases. Cubic splines, optimum location of nodes, heuristic algorithm.

1. INTRODUCTION.

Advances in integrated circuit technology have resulted in increased clock frequencies and higher speeds of operations. With an increased demand on smaller chip size and higher density in high performance systems, the analytical modeling became very difficult and complex. Even analytical modeling of passive elements such as resistors, capacitors, inductors is quite complicated and requires so much computer time in simulation that it is often replaced by modeling based on stored tables of relevant values. The tables can be filled using the data computed using analytical models or the data obtained via measurements. In the first case the advantage of a lookup table is that complex analytical models are used to build the tables for subsequent multiple, inexpensive use in the design. In the second case the tables can be constructed for the elements for which we do not have satisfactory analytical models. Examples of modeling high speed interconnections using lookup tables that can be classified as belonging to the first case of problems are described in You et al. (1996), Holt et al. (1992). In the analysis mode, given a set of dimensions the table is able to provide the electrical parameters as the output. In the synthesis mode, the table provides the dimensions and topology of interconnects for a given set of electrical parameters. The electrical parameters and line geometry are then determined by using specialized interpolation schemes based on tabulated data to get the desired values in a fraction of second and with limited memory size. This is achieved by using special, problem oriented interpolation algorithms that optimize response time and table size. Examples illustrating the modeling of the second case problems, where analytical approach is inadequate, are in Root et al. (1993), Detjens (1987). Lookup tables therefore have a significant practical importance in cases when the quantities in question cannot be computed easily. In looking for certain values it is very seldom the case that the required quantity is among the entries of the table. Since lookup tables represent the values of single or multi variable functions, any function value being between the entries requires the use of certain type of interpolation techniques. In the numerical analysis literature the following methods are most frequently applied for function approximation:

1. Least squares methods
2. Interpolation polynomials
3. Spline functions

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For our purposes spline functions are the most approximation techniques. Least squares methods require the assumption of a certain function type, and do not give back the original function exactly (Ross. 1987). In engineering applications no analytic function forms are usually given, and the assumption of any function type might result in high approximation errors. Interpolation polynomials are also questionable. In the case of a large number of points, the degree of (single or multi variable) interpolating polynomials becomes large, which leads to numerical instability and accumulation of roundoff errors. It is also well known that the increase in the number of the interpolating points (that is the number of table entries) does not necessarily result in smaller approximation error. As the famous Runge example, see for example Yakowitz and Szidarovszky (1989), shows the sequence of interpolation polynomials even might become divergent if the number of interpolation points converges to infinity. For our purpose spline approximations are the methods of choice, since (1) they interpolate; (2) no prior information on function types is needed; (3) at each segment they are low degree polynomials; (4) increasing number of interpolation points implies better approximation; (5) they are simple to compute. Spline functions are used by several authors in constructing lookup tables. For example, Root et al. (1993) have applied spline function interpolation in building tables for nonlinear model functions. Since there are several spline function types (with different approximation properties) know in the literature it is important to find the optimal spline function type that provides the best overall approximation for a given class of models based on tabulated values. A spline function type is considered to be optimal for a class of models if for any given error tolerance it requires the smallest number of tabulated values. That is, the optimal spline assures minimal computation cost in both the construction of the lookup table and its usage, i.e. in interpolating for the model values.

Therefore finding the best interpolation scheme requires the determination of the optimal locations of the nodes in order to minimize the overall error. In this paper a heuristic algorithm will be proposed, and some computational results will be reported. This problem has been first investigated by Chebyshev for interpolation polynomials, when he has introduced a spechial set of points and proved that uniform error distribution is achieved if and only if these points are selected as nodes (Yakowitz and Szidarovszky, 1989). Unfortunately no similar result are known for spline approximations. The optimal location problem can be mathematically formulated as a complicated nonlinear, nonconvex programming problem. Szidarovszky (1983) has developed a methodology and software package to find the best measurement locations for estimating the quantity and quality of mineral resources. A different than spline approximation scheme known as Kriging was then used, but that problem (arising in the mining industry) has an obvious analogy to the construction of lookup tables in electrical engineering.

2. THE ALGORITHMS.

Let f be given continuous function on a given finite interval $[a, b]$, and assume that the number $n+1$ of interpolation points is fixed. The algorithm starts with the equidistant distribution of the nodes, that is, $x_o = a$ and for $i = 1, 2, \ldots, n, x_i = a + (i-1)\frac{b-a}{n}$. The goodness of fit may be characterized by the maximum error. If p denotes the interpolator based on the given nodes, then this measure is defined as

$$E = \max_{x \in [a,b]} |f(x) - p(x)|$$

The algorithms are based on a simple procedure moving one of the points at a time optimally, by assuming that all others are fixed. Let x_k be one of the points. Since x_o is assumed to be the left endpoint of the interval and x_n is always the right endpoint, necessarily $k = 1, 2 \ldots, n-1$. The value of x_k is a point between x_{k-1} and x_{k+1}, and its optimal location can be determined by a discrete search. Selecting an $N > 1$, we can divide interval $[x_{k-1}, x_{k+1}]$ into N equal parts, and systematically checking each node for possible location of x_k, the best choice is obtained by comparison. Notice that this idea does not depend on the particular interpolating scheme.

The first algorithm is based on a cyclic application of the above simple procedure. Point x_1 is first located optimally, then the best location of x_2 is found, etc.. After x_{n-1} is relocated, then the best selection of x_1 is determined again, and so on. The procedre terminates, if in a whole cycle none of the points is moved to a new location.

The second algorithm is a refinement of the above procedure. After it terminates, we find the optimal locations of the pairs in order $(x_1, x_2), \ldots, (x_1, x_{n-1}), (x_2, x_3), \ldots, (x_2, x_{n-1}), \ldots, (x_{n-2}, x_{n-1})$. If at least one of the points is relocated, then this cycle is repeated again. The procedure terminates if no point changes location in an entire cycle. The third algorithm is a further refinement of the previous procedure. After it terminates the locations of all possible triples of nodes are determined in the order (x_1, x_2, x_3),(x_1, x_2, x_4), $\dots$, (x_1, x_2, x_{n-1}), $\dots$, $(x_{n-3}, x_{n-2}, x_{n-1})$. This procedure also terminates if in a whole cycle none of the points change location.

The above procedure can be applied for any given interpolation scheme as well as with any alternative measure of fit. For example, instead of minimizing the maximum error, we can minimize relative error

$$E* = \max_{x\in[a,b]} \frac{|f(x) - p(x)|}{max\{|f(x)|; |p(x)|\}}$$

or we can minimize the difference of the maximum and minimum errors

$$E** = \max_{x\in[a,b]} |f(x) - p(x)| - \min_{x\in[a,b]} |f(x) - p(x)|$$

3. COMPUTATIONAL EXPERIMENTS.

In our computer study a special interpolating cubic spline was selected, which is a sight modification of the natural cubic spline in the following way: Instead of requiring that $p''(a) = p''(b) = 0$, we assumed that these second derivatives are equal to the second derivative approximations at endpoints based on three first (or last) nodes. That is, we used the conditions

$$p''(a) = \frac{2f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{2f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{2f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

and

$$p''(b) = \frac{2f(x_n - 2)}{(x_{n-2} - x_{n-1})(x_{n-2} - x_n)} + \frac{2f(x_{n-1})}{(x_{n-1} - x_{n-2})(x_{n-1} - x_n)} + \frac{2f(x_n)}{(x_n - x_{n-1})(x_n - x_{n-2})}$$

Three functions, x^2, e^x, and $sin(\pi x/10)$ were considered and examined in interval $[0, 10]$. The number of nodes was also varied, the values $5, 10$, and 20 were selected for n. the above three algorithms were applied for the three functions, the tree values of n, and measures $E, E*, E**$. Hence we have overall $3 \times 3 \times 3 = 27$ results, which are presented in Tables 1, 2, and 3. The first number of each entry gives the corresponding measure of goodness of fit in the case of the initial node selection (that is, if the nodes are evenly distributed, and the second, third, and forth values correspond to algorithms 1, 2, and 3, respectively.

Table 1 shows the improvements most clearly. For smooth functions such as x^2 and $sin(\pi x/10)$, no large improvement can be observed, however for the exponential function it is significant. Since algorithms 2 and 3 are the further improvements of algorithm 1, the numbers in each entry of the table are decreasing. However this monotonicity does not hold in the other two cases, since the objective functions differ from the maximum error that is given in Tables 2 and 3. For the exponential case, between algorithms 1 and 2, and between algorithms 2 and 3, the improvements are much smaller than the improvement obtained by applying algorithm 1. This fact shows that it is important to use any one of the improving algorithms, and it does not make a significant difference which one is selected to be used.

4. CONCLUSIONS.

The problem of finding the optimal locations of nodes for any interpolation scheme is a complicated, nonconvex optimization problem. Heuristic algorithms are presented in this paper to find its approximate solution. The nonconvexity of the problem makes the algorithm development difficult, and there is no guarantee that the methods do not terminate at only local optima.

The numerical study illustrates, that the application of the algorithms being introduced in the paper reduces the measures of goodness of fit significantly compared to initial node selections, which shows the practical importance of the methodology.

The continuation of the algorithms with the simultaneous improvement of four nodes, then that of five, etc. nodes could result in further reduction of the overall errors, however in the case of large value of n, the application requires a large amount of computer time. Theoretically, if all the $(n - 1)$ interior points are simultaneously varied, then global optimum is reached, however computationally it is infeasible.

Table 1. Maximum Errors in Minimizing Maximum Error.

function	$n = 5$	$n = 10$	$n = 20$
x^2	0.134737	0.034302	0.008575
	0.134737	0.032265	0.008041
	0.116474	0.032265	0.008041
	0.116474	0.030876	0.007812
e^x	2048.475586	422.125000	72.188477
	189.112106	74.360107	30.498047
	189.112106	74.360107	30.498047
	168.601562	74.233749	30.498047
$sin(\pi x/10)$	0.014221	0.001932	0.000246
	0.011504	0.001320	0.000246
	0.011320	0.001320	0.000091
	0.010975	0.001168	0.000080

Value 1 = maximum error with evenly selected points
Value 2 = maximum error using algorithm 1
Value 3 = maximum error using algotithm 2
Value 4 = maximum error using algotithm 3

Table 2. Maximum Errors in Minimizing Maximum Relative Error.

function	$n = 5$	$n = 10$	$n = 20$
x^2	0.134737	0.034302	0.008575
	0.002655	0.033699	0.007423
	0.002655	0.033699	0.007423
	0.002655	0.029655	0.006660
e^x	2084.475586	422.125000	72.188477
	1427.731445	287.652344	55.441406
	1427.708984	287.652344	55.441406
	1464.326172	287.654297	55.480469
$sin(\pi x/10)$	0.014221	0.001932	0.000246
	0.023309	0.004641	0.000061
	0.019391	0.004641	0.000061
	0.019391	0.002897	0.000061

Value 1 = maximum error with evenly selected points
Value 2 = maximum error using algorithm 1
Value 3 = maximum error using algorithm 2
Value 4 = maximum error using algorithm 3

Table 3. Maximum Errors in Minimizing the difference Between Maximum and Minimum Errors.

function	$n = 5$	$n = 10$	$n = 20$
x^2	0.134737	0.034302	0.008575
	0.134737	0.026093	0.008041
	0.116983	0.026093	0.008041
	0.113804	0.026093	0.007812
e^x	2084.475586	422.125000	72.188477
	121.843750	74.350586	30.498047
	121.843750	74.360342	30.498047
	116.367188	74.202148	30.498047
$sin(\pi x/10)$	0.014221	0.001932	0.000246
	0.011504	0.001517	0.000246
	0.011320	0.001517	0.000091
	0.010975	0.001342	0.000080

Value 1 = maximum error using evenly selected points

Value 2 = maximum error using algorithm 1
Value 3 = maximum error using algorithm 2
Value 4 = maximum error using algorithm 3

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ESTIMATION OF CHANNEL HOLDING TIME DISTRIBUTION IN CELLULAR TELEPHONE SYSTEMS

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Abstract. In estimating the channel holding time distribution in cellular telephone systems, the pdf of the fraction of two non-negative random variables, distance and speed, has to be determined. Instead of the repeated application of numerical integration we show that the pdf of the holding time is the solution of an initial value problem of a non-linear ordinary differential equation, which can be solved very efficiently.

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Key Words and Phrases. Cellular telephone system, holding time distribution, transformed random variable.

1.INTRODUCTION.

Many traffic models have been proposed for the performance analysis of cellular telephone systems, mainly relied on simulation models [1]. When we make a traffic model into a mathematical expression, only a few models have a known closed-form solution under special conditions [2], and usually the traffic models cannot be solved in a closed-form. Therefore a simulation or approximating method has to be used to solve the problems.

The channel holding time of a mobile user in cellular telephone systems is important for system design and radio resource allocation. The distribution of this parameter can be computed by a double integral of the moving distance and speed of a mobile in a cell [2]. Although the cumulative distribution functions of the distance and the speed are derived as a closed-form solution, in most cases the double integrals are not solved analytically. In this paper, we propose a method for deriving the cannel holding time distribution and we apply this method to the analysis of cannel holding time for the traffic model of a soft handoff call which has non-closed-form solution. We also perform computer simulations to check the results of the non-closed-form application.

2. TRAFFIC MODEL OF SOFT HANDOFF.

We assume that the origination of a new call is uniformly distributed over a service area. The speed and the direction of the call are uniformly distributed, and they are unchanged during the call. We also assume that the speed and the direction are independent of each other. Soft handoff allows that a mobile station is connected to a new base station before it is disconnected from the old base station during handoff. Thus when a mobile station is in the soft handoff state, two cell sites are connected to use with diversity [3]. Due to this overlapped region, one more boundary exists in a soft handoff traffic model [4]. A circular-shaped cell is

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assumed for the traffic model for a simple analysis. In Fig. 1, the boundary B is the point that the received power level of a mobile is higher than its receiver threshold level, where an incoming soft handoff starts and the outgoing one stops. If the circular cell radius of boundary A is R_{eq}, the radius of outer boundary B of a soft handoff region is defined as R_{so} and it is

$$R_{so} \equiv (1+a)R_{eq}. \tag{2.1}$$

Here a is the soft handoff region factor in the interval $[0,1]$ and it indicates the ratio of overlapped area between cells. Three types of calls exist in the traffic model with a soft handoff scheme, which are a new call, an incoming call and a handoff region call [4]. In this paper, we will discuss only the channel holding time of a new call. The channel holding time of new call in a cell is defined as the time duration from the distant that a channel is occupied by the call to the instant that it is released by either completion of the cell or cross of the cell boundary. Thus it is either the call duration time or the residing time of a mobile in the cell, whichever is smaller. For the new call, the random variable T is defined as the residing time in the cell. In Fig. 1, the moving distance of these calls is illustrated. The cell site is assumed to be located at the center of a cell and is indicated by letter C. A new call (indicated by letter D) is generated at a point within the boundary A. The assigned channel is released when the mobile station is leaving the cell at the cell boundary B indicated by letter E. We assume that the direction of a mobile is uniformly distributed on the interval $[0, 2\pi]$. If we define the direction as the angle θ against the vector from the cell site to the mobile, by using symmetry, the random variable θ can be considered in interval $[0,\pi]$ with pdf

$$f_\theta(\theta) = \begin{cases} \frac{1}{\pi}, & \text{for} \quad 0 \le \theta \le \pi \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

$$f_r(r) = \begin{cases} \frac{2r}{R_{eq}^2}, & \text{for} \quad 0 \le r \le R_{eq} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The distance from a mobile station to cell boundary B is indicated by letter Z, which and a new random variable W are defined as

$$Z = \sqrt{R_{so}^2 - (r\sin\theta)^2 - r\cos\theta}, \qquad W = r\cos\theta. \tag{2.4}$$

The joint pdf of Z and W can be found as

$$f_{zw}(z,w) = \frac{2}{\pi R_{eq}^2}\frac{z+w}{\sqrt{R_{so}^2-(z+w)^2}} \qquad \text{for} \begin{cases} R_{so} - R_{eq} \le z \le R_{so} + R_{eq}, \\ \frac{-z^2+R_{so}^2-R_{eq}^2}{2z} \le w \le R_{so} - z \end{cases} \tag{2.5}$$

The marginal pdf of distance Z is then

$$\begin{aligned} f_z(z) &= \int_{-\infty}^{\infty} f_{zw}(z,w)dw \\ &= \begin{cases} \frac{2}{\pi R_{eq}^2}\sqrt{R_{so}^2 - (\frac{z^2+R_{so}^2-R_{eq}^2}{2z})^2}, & \text{for} \quad R_{so} - R_{eq} \le z \le R_{so} = R_{eq}. \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{2.6}$$

The speed of a mobile is uniformly distributed on the interval $[0, V_{max}]$ with pdf

$$f_v(v) = \begin{cases} \frac{1}{v_{max}}, & \text{for} \quad 0 \le v \le V_{max} \\ 0, & \text{elsewhere} \end{cases} \tag{2.7}$$

Then the residing time T is expressed by

$$T = \frac{Z}{V}. \tag{2.8}$$

The pdf of T is

$$f_T(t) = \int_{-\infty}^{\infty} f_Z(tw) f_V(w)|w|dw$$

$$= \begin{cases} \frac{2}{\pi R_{eq}^2 V_{max}} \int_0^{V_{max}} w\sqrt{R_{so}^2 - (\frac{z^2+R_{so}^2-R_{eq}^2}{2z})^2} dw, & \text{if} \quad \frac{R_{so}-R_{eq}}{V_{max}} \le t \le \frac{R_{so}+R_{eq}}{V_{max}} \\ \frac{2}{\pi R_{eq}^2 V_{max}} \int_0^{(\frac{R_{so}+R_{eq}}{t})} w\sqrt{R_{so}^2 - (\frac{z^2+R_{so}^2-R_{eq}^2}{2z})^2} dw, & \text{if} \quad t \ge \frac{R_{so}+R_{eq}}{V_{max}} \end{cases} \tag{2.9}$$

The above integrals do not have a closed-form solution. In the next section a practical method will be introduced for the efficient computation of $f_T(t)$ for a large set of t values.

3. Proposed integration method.

Assume that Z is a continuous random variable in the interval $[A, B]$, where A and B are positive, and V is a continuous random variable in the interval $[0, V_{max}]$, where $V_{max} > 0$ is given. Let $f_Z(z)$ and $f_V(v)$ denote the density functions of Z and V. From equation (2.8), notice first that the values of T are in the interval $[\frac{A}{V_{max}}, \infty)$, and if $F_T(t)$ denotes the cumulative distribution function of T, then

$$F_T(t) = \begin{cases} 0, & \text{if} \quad z < \frac{A}{V_{max}} \\ P\left\{\frac{Z}{V} \le t\right\} = P\left\{\frac{Z}{t} \le V\right\} & \text{otherwise} \end{cases} \tag{3.1}$$

This probability can be computed by a double integral. The region of integration is illustrated in Fig.2, which shows that we have to consider 2 separated cases. If $t \ge \frac{B}{V_{max}}$, then

$$\begin{aligned} F_T(t) &= \int_A^B \int_{\frac{z}{t}}^{V_{max}} f_Z(z) \cdot f_V(v) dv dz \\ &= \int_A^B f_Z(z)[1 - F_v(\frac{z}{t})]dz \\ &= 1 - \int_A^B f_Z(z) F_V(\frac{z}{t}) dz \end{aligned} \tag{3.2}$$

where $F_V(v)$ is the cumulative distribution function of V. If $\frac{A}{V_{max}} \le t \le \frac{B}{V_{max}}$, then

$$\begin{aligned} F_T(t) &= \int_A^{t\cdot V_{max}} \int_{\frac{z}{t}}^{V_{max}} f_Z(z) \cdot f_V(v) dv dz \\ &= \int_A^{t\cdot V_{max}} f_Z(z)[1 - F_V(\frac{z}{t})]dz \\ &= F_Z(t \cdot V_{max}) - \int_A^{t\cdot V_{max}} f_Z(z) F_V(\frac{z}{t}) dz \end{aligned} \tag{3.3}$$

where $F_Z(z)$ is the cumulative distribution function of Z. If the random variable V is uniformly distributed in the interval $[O, V_{max}]$, then

$$f_V(v) = \begin{cases} \frac{1}{V_{max}}, & \text{if} \quad 0 \le v \le V_{max} \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

$$F_V(v) = \begin{cases} \frac{v}{V_{max}}, & \text{if} \quad 0 \le v \le V_{max} \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

By using the special form of $F_V(v)$, a more efficient method can be suggested. The above integrals can be further simplified as follows: If $\frac{A}{V_{max}} \le t \le \frac{B}{V_{max}}$, then using integration by parts we have

$$\begin{aligned} F_T(t) &= F_Z(t \cdot V_{max}) - \int_A^{t \cdot V_{max}} \frac{z}{t \cdot V_{max}} f_Z(z) dz \\ &= F_Z(t \cdot V_{max}) - \frac{1}{t \cdot V_{max}} \left\{ [zF_Z(z)]_A^{t \cdot V_{max}} - \int_A^{t \cdot V_{max}} F_Z(z) dz \right\} \\ &= \frac{1}{t \cdot V_{max}} \int_A^{t \cdot V_{max}} F_Z(z) dz. \end{aligned} \tag{3.6}$$

Simple differentiation shows that

$$\begin{aligned} f_T(t) &= -\frac{1}{t^2 V_{max}} \int_A^{tV_{max}} F_Z(z) dz + \frac{1}{tV_{max}} F_Z(tV_{max}) \cdot V_{max} \\ &= -\frac{1}{t} F_T(t) + \frac{1}{t} F_Z(tV_{max}). \end{aligned} \tag{3.7}$$

If $t \ge \frac{B}{V_{max}}$, then from equation (3.2) we have

$$\begin{aligned} F_T(t) &= 1 - \int_A^B f_Z(z) \frac{z}{t \cdot V_{max}} dz \\ &= 1 - \frac{1}{t \cdot V_{max}} E_Z(z). \end{aligned} \tag{3.8}$$

Here the expected value $E_Z(z)$ is a constant that has to be computed only ones. By differentiation,

$$f_T(t) = \frac{1}{t^2 V_{max}} E_Z(z). \tag{3.9}$$

Since $f_T(t) = F_T'(t)$, an ordinary differential equation is obtained for $F_T(t)$ by equality (3.7). However F_Z is not known in a closed-form; therefore, the numerical solution of this differential equation is expensive, since at each step F_Z has to be determined by numerical integration. We can rewrite equation (3.7) as

$$t \cdot f_T(t) = -F_T(t) + F_Z(tV_{max}) \tag{3.10}$$

and further differentiations shows that

$$t \cdot f_T(t) + f_T(t) = -f_T(t) + f_Z(tV_{max}) \cdot V_{max} \tag{3.11}$$

which can be simplified as the initial-value problem

$$f_T'(t) = -\frac{2}{t} f_T(t) + \frac{V_{max}}{t} \cdot f_Z(tV_{max}), \qquad f_T(\frac{A}{V_{max}}) = 0 \tag{3.12}$$

where function f_Z can be easily computed. Notice also that $t > A/V_{max}$; therefore, no singularity occurs, and hence this initial-value problem can be solved very efficiently. For example, the fourth order Runge-Kutta method or any predictor-corrector formulae is useful. For details of such methods see for example [5], [6]. In summary, $f_T(t)$ can be obtained in the following way. In intervals $[0, A/V_{max}]$ and $[B/V_{max}, \infty]$ the function is given in a closed-form, equation (3.1) and (3.8), respectively. In the interval $[A/V_{max}, B/V_{max}]$ the initial-value problem (3.12) has to be solved. Because of roundoff errors, the solution trajectories obtained for different subintervals do not match exactly at the end points. At $t = A/V_{max}$ we have exact match, since the initial value is defined as zero at this point. From equation (3.9) we know that at $t = B/V_{max}$,

$$f_T(t) = \frac{V_{max}}{B^2} E_Z(z). \tag{3.13}$$

Therefore, if $\overline{f_T}$ denotes the computed trajectory in interval $[\frac{A}{V_{max}}, \frac{B}{V_{max}}]$, then the adjusted pdf can be obtained by equation

$$f_{adjusted}(t) = \overline{f_T}(t) \cdot \frac{V_{max} E_Z(z)}{B^2 \overline{f_T}(\frac{B}{V_{max}})}. \tag{3.14}$$

4. Application.

We illustrate now the above equation for the particular case of $R_{eq} = 1Km/Hr$ and $V_{max} = 60Km/Hr$. The solution is obtained by using the fourth-order Runge-Kutta method for solving the initial value problem (3.12), and then for $t > \frac{B}{V_{max}}$ using the closed-form representation (3.9). The pdf of T is illustrated in Figure 4 for different values of a. In order to show the effect of roundoff errors no adjustment was made.

5. Conclusions.

In this paper, we proposed a method for deriving the cannel holding time distribution by a numerical integration approach. Instead of using expensive integration routines repeatedly, for small values of t, the initial value problem of an ordinary differential equation is solved, and for larger values of t, a closed-form solution formula is obtained. The effect of roundoff errors can be eliminated by a special adjustment process making the proposed algorithm not only very effective but also very accurate. Using this methodology, we can determine the cannel holding time of the traffic models which have not a closed-form solution, such as a soft handoff model. The proposed method provides a useful tool for analyzing the traffic model of cellular telephone systems.

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GENERAL WAYS OF CONSTRUCTING ACCELERATING NEWTON-LIKE ITERATIONS ON PARTIALLY ORDERED TOPOLOGICAL SPACES

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Abstract. In this study, we examine the monotone convergence of Newton-like methods to a solution of an equation on a partially ordered topological space setting. In particular we provide sufficient conditions for the construction of accelerating sequences. This way the solution is obtained faster than in earlier results.

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Key Words and Phrases. Partially ordered topological space, accelerating iteration, nonlinear equation.

I. INTRODUCTION

In this study, we are concerned with the problem of approximating a solution z of the equation

$$F_1(x) = 0 \tag{1}$$

where F_1 is a continuous operator defined on a closed convex subset D of a partially ordered topological space (POTL-space) E_1 with values in a POTL-space E_2 [3], [4], [8].

Sufficient conditions for the monotone convergence of Newton-like methods to z have been found by several authors (see [1]–[4], [7], [8] and the references there). Here we first transform (1) into a fixed point problem of the form

$$x = G_1(x) \tag{2}$$

and then consider iterations, of the form

$$x_{n+1} = G_1(x_n) \quad (n \geq 0), \tag{3}$$

$$\bar{y}_{n+1} = G_2(x_n) \quad (n \geq 0), \tag{4}$$

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and

$$y_{n+1} = G_2(y_n) \quad (n \geq 0) \tag{5}$$

for some continuous operators $G_1, G_2: D \to E_2$ with $G_1(z) = G_2(z)$ whenever $F_1(z) = 0$. Under the conditions in the above-mentioned papers iteration $\{x_n\}$ $(n \geq 0)$ converges to z. We then provide sufficient conditions that show that iterations $\{\bar{y}_n\}$, $\{y_n\}$ $(n \geq 0)$ are accelerating sequences to z, in the sense that

$$y_n \leq \bar{y}_n \leq x_n \quad (n \geq 0) \quad \text{and} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} = \lim_{n\to\infty} \bar{y}_n = z. \tag{6}$$

Moreover in the case when $E_1 = E_2 = R$ our results can be reduced to weaker versions of the corresponding ones in [5].

Finally we provide an example to justify (6).

II. MONOTONE CONVERGENCE

We state and prove the following main result on the monotone convergence of Newton-like methods on partially ordered topological spaces.

Theorem. *Let F_1, F_2 be continuous operators defined on a closed convex subset D of a partially ordered topological space $E_1 = (E_1, \leq)$ (POTL-space) with values in a POTL-space $E_2 = (E_2, \leq)$. Assume:*

(a) *there exists points a, z, b of D such that $a \leq z \leq b$, $[a, b] \subseteq D$ and z is the only point in $[a, b]$ such that*

$$F_1(z) = F_2(z) = 0. \tag{7}$$

(b) *There exists operators $A_1(x)$, $A_2(x) \in L(E_2, E_1)$ $(x \in D)$ such that the Newton-like iterations*

$$x_{n+1} = x_n - A_1(x_n)F_1(x_n) \quad (n \geq 0) \tag{8}$$

$$y_{n+1} = y_n - A_2(y_n)F_2(y_n) \quad (n \geq 0) \tag{9}$$

are decreasing and converging to z for some starting points x_0, y_0 with $x_0 = y_0 \neq z$ and $z \leq x_0 \leq b$.

(c) *For all points $x \in [a, b]$ there exists an operator $L \in L(E_1, E_2)$ depending on F_1, F_2, z and x such that*

$$0 \leq L(x - z) = (G_1 - G_2)(x) - (G_1 - G_2)(z) \tag{10}$$

where

$$G_1(x) = x - A_1(x)F_1(x) \quad \text{and} \quad G_2(x) = x - A_2(x)F_2(x). \tag{11}$$

Then

(12) (i) *the estimate* $y_n \leq x_n$ *holds for all* $n \geq 0$ *such that* $x_n \neq z$.

(ii) $x_{n-1} \neq z$ *and* $x_n = z$ *imply* $y_n = z$.

Proof: (i) We will show estimate (12) using mathematical induction on $n \in N$. For $n = 0$ estimate (12) is true as equality. Using (7)–(11) for $n = 1$ we obtain

$$0 \leq L(x_0 - z) = (G_1 - G_2)(x_0) - (G_1 - G_2)(z) = x_1 - y_1$$

which shows (12) in this case. Assume that $y_k \leq x_k$ for $k = 1, 2, \ldots, n-1$. If $x_n \neq z$ then $x_{n-1} \neq z$ and from (7)–(11) we obtain

$$\begin{aligned} 0 \leq L(x_{n-1} - z) &= (G_1 - G_2)(x_{n-1}) - (G_1 - G_2)(z) \leq G_1(x_{n-1}) - G_2(y_{n-1}) \\ &= x_n - y_n, \end{aligned}$$

(since G_2 is increasing in $[z, b]$ and $\{y_n\} \subseteq [z, b]$ $(n \geq 0)$) which shows (12) for all $n \geq 0$. Part (ii) follows easily by applying the arguments used in part (i).

That completes the proof of the Theorem.

Remark 1. It is easily seen that for $E_1 = E_2 = R$ our theorem reduces to a weaker version of Theorem 1 in [5, p. 160] (see also [6]). A similar theorem can also be immediately stated for increasing sequences to z.

Remark 2. The condition that iteration $\{x_n\}$ $(n \geq 0)$ is decreasing (similarly for iteration $\{y_n\}$ $(n \geq 0)$) can be replaced by: $A_1(x)$ is a nonnegative linear operator and $0 \leq F_1(x)$ for $x \in [z, b]$.

Remark 3. Operator L appearing in (10) can be chosen as $L = [z, x; G_1 - G_2]$. That is as a divided difference of order one on D at the points z, x for the operator $G_1 - G_2$ [3], [4], [7], [8].

Remark 4. Several sufficient conditions for the monotone convergence of iterations $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ to z can be found in [3], [4], [7], [8] and the references there. In our theorem above we did not add such conditions to avoid repetitions and because we want to keep it as uncluttered and simple as possible.

Remark 5. Another choice for the operator L can be given by $L = A_2(x)[z, x; F_2] - A_1(x)[z, x; F_1]$. Indeed we can get from the identity

$$\begin{aligned} G_1(x) - G_2(x) &= x - A_1(x)F_1(x) - x + A_2(x)F_2(x) \\ &= A_2(x)(F_2(x) - F_2(z)) - A_1(x)(F_1(x) - F_1(z)) \\ &= (A_2(x)[z, x; F_2] - A_1(x)[z, x; F_1])(x - z) \\ &= L(x - z), \end{aligned}$$

which justifies the choice for L.

Remark 6. A reasonable choice for the operator F_2 can be given by

$$F_2(x) = A_3(x)F_1(x) + F_3(x,z)$$

for some $A_3(x) \in L(E_1, E_2)$ and $F_3: D \to E_2$ such that $F_2(z) = 0$ whenever $F_1(z) = 0$. Assume that $[z, x; F_1]$ is a divided difference of order one on D at the points $z, x \in D$ for the operator F_1. We can write

$$\begin{aligned} G_1(x) - G_2(x) &= x - A_1(x)F_1(x) - x + A_2(x)F_2(x) \\ &= A_2(x)(A_3(x)F_1(x) + F_3(x,z)) - A_1(x)F_1(x) \\ &= (A_2(x)A_3(x) - A_1(x))(F_2(x) - F_1(z)) + A_2(x)F_3(x,z) \\ &= (A_2(x)A_3(x) - A_1(x))[z, x; F_1](x - z) + A_2(x)F_3(x,z) \end{aligned}$$

and for $F_3(x,z) = F_4(x,z)(x-z)$, we get $G_1(x) - G_2(x) = L(x,z)$ where

$$L = [A_2(x)A_3(x) - A_1(x)][z, x; F_1] + A_2(x)F_4(x,z). \tag{13}$$

Natural conditions can be imposed on the operators appearing in the above approximation (13) so that (10) is satisfied. In [5] for $E_1 = E_2 = R$ operators A_3 and F_3 were chosen such that

$$A_3(x) = 1 \quad \text{and} \quad F_3(x,z) = -\frac{1}{2}F_1''(z)(x-z)^2.$$

Since z is an unknown point it was replaced by x_{n-1} and a new iterative process involving the second derivative of order three was found. With the exception of some special cases this iterative process has no practical value since they require an evaluation of the second Fréchet-derivative at each step which means a number of function evaluations proportional with the cube of the dimension of the space. In [3], [4] discretized versions of F_3 were considered on a POTL-space using divided differences of order one. Operator F_3 is now defined by

$$F_3(x_n, x_{n-1}) = -([x_{n-1}, x_n] - [x_{n-1}, x_{n-1}])(x_{n-1} - x_n) \quad (n \geq 1).$$

Several other variations of the above discretized scheme are also possible [3], [4], [7], [8].

We now complete this study by providing an example for the last remark.

Example. Let $E_1 = E_2 = R$, $[a,b] = [-\frac{1}{3}, \frac{1}{3}]$, $F_1(x) = x^2$ and $x_0 = .2$. Note that $z = 0$ is the only zero of equation $F_1(x) = 0$ on $[a,b]$. Consider iterations

$$x_{n+1} = x_n - x_n^2, \quad \bar{y}_{n+1} = x_n - 2x_n^2 \quad \text{and} \quad y_{n+1} = y_n - 2y_n^2 \quad (n \geq 0).$$

Iterations $\{\bar{y}_n\}$, $\{y_n\}$ $(n \geq 0)$ are accelerations of $\{x_n\}$ $(n \geq 0)$ with $y \leq \bar{y}_n \leq x_n$ $(n \geq 0)$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \bar{y}_n = \lim_{n\to\infty} x_n = z = 0$, which justifies (6).

In what follows we tabulate the first seven iterates for each iteration for comparison:

$$\begin{aligned} &x_0 = .2, \quad x_1 = .16, \quad x_2 = .1344, \quad x_3 = .1163366, \quad x_4 = .1028024, \\ &x_5 = .0922341, \quad x_6 = .083727, \quad x_7 = .0767168, \\ &\bar{y}_0 = .2, \quad \bar{y}_1 = .12, \quad \bar{y}_2 = .1088, \quad \bar{y}_3 = .0982733, \quad \bar{y}_4 = .0892682, \\ &\bar{y}_5 = .0816657, \quad \bar{y}_6 = .0752198, \quad \bar{y}_7 = .0697066, \\ &y_0 = .2, \quad y_1 = .12, \quad y_2 = .0912, \quad y_3 = .0745651, \quad y_4 = .0634452, \\ &y_5 = .0553946, \quad y_6 = .0492575, \quad y_7 = .0444049. \end{aligned}$$

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A NEW CONVERGENCE THEOREM FOR STEFFENSEN'S METHOD ON BANACH SPACES AND APPLICATIONS

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Abstract. We provide sufficient conditions for the convergence of Steffensen's method to a locally unique solution of a nonlinear operator equation on a Banach space. We use a fixed point theorem and a special choice of divided differences. Some applications of our results are also provided.

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I. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$f(x) = 0, \tag{1}$$

where f is a continuous operator defined on a closed convex domain D of a Banach space E_1 with values in a Banach space E_2.

We use Steffensen's method

$$x_{n+1} = x_n - [x_n, g(x_n); f]^{-1} f(x_n) \quad (n \geq 0) \tag{2}$$

to generate a sequence converging to x^*.

Here $[x, y; f] \in L(E_1, E_2)$ is a divided difference of order one for the operator f on the points $x, y \in D$. Operator $g: D \subseteq E_1 \to E_1$ is given and continuous on D. For $g(x) = x$ $(x \in D)$, method (2) reduces to Newton's method [3], [4] provided that f is Fréchet-differentiable on D. If $g(x_n) = x_{n-1}$ $(n \geq 0)$ we obtain secant method [3], [5] from (2). Several other choices for g are possible, including the most popular given by $g(x) = x - f(x)$ $(x \in D)$.

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We define the operator $P: D \subseteq E_1 \to E_2$ by

$$P(x) = x - [x, g(x); f]^{-1} f(x), \tag{3}$$

in which case (2) can also be written in the form

$$x_{n+1} = P(x_n) \quad (n \geq 0) \quad (x_0 \in D). \tag{4}$$

A solution x^* of equation (1) is said to be accessible from x_0 by Steffensen's method (2) if

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} P^n(x_0). \tag{5}$$

The region of accessibility of x^* by method (2) is defined to be the set of all x_0 such that (5) is true.

Let us define operator $L_f \in L(E_1, E_2)$ by

$$L_f(x) = P'(x) \quad (x \in D). \tag{6}$$

This operator is the degree of logarithmic convexity of f on x and is a measure of the convexity of the function.

Here we assume that the divided difference $[x, y; F]$ satisfies a certain Lipschitz condition that allows us to choose

$$[x, g(x); f] = \int_0^1 f'(x + t(g(x) - x))dt \quad \text{for all } x, g(x) \in D. \tag{7}$$

Steffensen's method (2) has been examined extensively under various assumptions (see [1]–[3], [6], [7] and the references there). In the above-mentioned references the authors have found a ball $U(x_0, p) = \{x \in E_1 \mid \|x - x_0\| \leq p\} \subseteq D$ such that $x^* \in U(x_0, p)$ and method (2) converges to x^* starting at $x_0 \in D$. Here using the fixed point theorem [3], [5] on (4), and (6) we provide sufficient convergence conditions for the convergence of Steffensen's method to $x^* \in U(x_0, q)$ for $q > p$. This way the region of convergence is enlarged.

II. CONVERGENCE ANALYSIS

Definition. The first order divided difference of f on the points $u, v \in E_1$ is an application $[u, v; f] \in L(E_1, E_2)$ which satisfies:

(i) $[u, v; f](v - u) = f(v) - f(u)$; (8)

(ii) if f is Fréchet differentiable at $u \in E_1$ then $[u, u; f] = f'(u)$.

Let a, b, c, α be nonnegative numbers, $x_0 \in D$ be such that the inverse of $[x_0, g(x_0); f]$ exists on D, and define parameters d, δ, β, ε by

$$d = a(1 + b), \ \delta = \|[x_0, g(x_0); f]^{-1} f(x_0)\|, \ \beta = a\|x_0 - g(x_0)\| \tag{9}$$

and

$$\varepsilon = \frac{\|x_0 - g(x_0)\|}{1 - b} \quad (\text{for } 0 \leq b < 1). \tag{10}$$

Moreover define real functions h_1, h_2, h_3, h_4 on $[0, +\infty)$ by

$$h_1(r) = r - \frac{1}{d}, \tag{11}$$

$$h_2(r) = 2d^2r^2 - d(3 + c - \beta)r + 1 - \beta - d\delta, \tag{12}$$

$$h_3(r) = 2d^2r^3 - d(c + \delta d - \beta + 3)r^2 + (1 - \beta + d\delta)r - \delta \tag{13}$$

and

$$h_4(r) = r - \varepsilon. \tag{14}$$

For our convergence theorem that follows we would like to be able to find an interval I which is a subset of the nonnegative axis such that

$$h_1(r) < 0, \tag{15}$$

$$h_2(r) > 0, \tag{16}$$

$$h_3(r) \geq 0, \tag{17}$$

$$h_4(r) \geq 0 \tag{18}$$

and

$$U(x_0, r) \subseteq D \tag{19}$$

for all $r \in I$.

The example at the end of this study shows that this can happen.

We state and prove the following semilocal convergence theorem for method (2):

Theorem 1. *Let $f: D \subseteq E_1 \to E_2$, $g: D \subseteq E_1 \to E_1$ be continuous operators, D a closed convex subset of E_1, and E_1, E_2 Banach spaces. Assume*:

(a) *f is twice and g once Fréchet-differentiable on D*;

(b) *there exist $a, b, c > 0$, $x_0 \in D$ such that*

$$\|[x_0, g(x_0); f]^{-1}([x, y; f] - [v, w; f])\| \leq a(\|x - v\| + \|y - w\|), \tag{20}$$

$$\|g(x) - g(y)\| \leq b\|x - y\|, \quad (b < 1), \tag{21}$$

$$\|[x_0, g(x_0); f]^{-1}(f(x) - f(y))\| \leq c\|x - y\| \tag{22}$$

and

$$\|[x_0, g(x_0); f]^{-1}([x, x, g(x); f] + [x, g(x), g(x); f]g'(x))\| \leq \alpha \tag{23}$$

for all $x, y, v, w \in D$;

(c) *conditions* (15)–(19) *hold on some interval $I \subseteq R^+ \cup \{0\}$*;

(d) *hypotheses* (c) *and* (d) *of the Proposition hold.*

Then the following hold:

(i) $$\|P'(x)\| \le c(r) < 1 \tag{24}$$

and

(25) $$r \ge \frac{\delta}{1-c(r)}$$

for all $r \in I$, *where*

(26) $$c(r) = \frac{1}{(1-dr)^2}[(1-dr)(dr+\beta) + \alpha(cr+\delta)].$$

(ii) P *has a unique fixed point* x^* *in* $U(x_0, r)$.

(iii) *Iteration* $x_{n+1} = P(x_n)$ $(n \ge 0)$ *converges to* x^*. *Moreover the following error bound holds for all* $n \ge 0$

(27) $$\|x_n - x^*\| \le c^n(r)r, \quad r \in I.$$

Proof. (i) Let $f(x), g(x) \in U(x_0, r)$ $(r \in I)$ whenever $x \in U(x_0, r)$. Using (3) and (7), we obtain

(28)
$$\begin{aligned} P'(x) &= I - ([x, g(x); f]^{-1})' f(x) - [x, g(x); f]^{-1} f'(x) \\ &= I - [x, g(x); f]^{-1}[x, g(x); f]'[x, g(x); f]^{-1} f(x) \\ &\quad - [x, g(x); f]^{-1} f'(x) \\ &= [x, g(x); f]^{-1}[x_0, g(x_0); f][x_0, g(x_0); f]^{-1} \Big\{ ([x, g(x); f] - f'(x)]) \\ &\quad - \int_0^1 f''(x + t(g(x) - x))(I + t(g'(x) - I))dt \cdot \\ &\quad \cdot [x, g(x); f]^{-1}[x_0, g(x_0); f][x_0, g(x_0); f]^{-1} f(x) \Big\}. \end{aligned}$$

By (15), (20), (21) and the estimate

$$\begin{aligned} \|[x_0, g(x_0); f]^{-1}([x, g(x); f] - [x_0, g(x_0); f])\| &\le a(\|x - x_0\| + \|g(x) - g(x_0)\|) \\ &\le a(r + br) = dr < 1, \end{aligned}$$

and from the Banach lemma on invertible operators [3], [5] it follows that $[x, g(x); f]^{-1}$ exists $(x \in U(x_0, r)$ $(r \in I))$ and

(29) $$\|[x, g(x); f]^{-1}[x_0, g(x_0); f]\| \le (1 - dr)^{-1}.$$

Condition (21) implies that $f'(x) = [x, x; f]$ $(x \in D)$ [3], [5], and

$$\|[x_0, g(x_0); f]^{-1}([x, g(x); f] - [x, x; f])\| \le a\|x - g(x)\|$$

(30)
$$\le a(\|x - x_0\| + \|x_0 - g(x_0)\| + \|g(x) - g(x_0)\|) \le a(r + \|x_0 - g(x_0)\| + br).$$

Moreover by (20)–(23) and (29), the second member inside the braces in (28) is bounded above by

$$\alpha(1-dr)^{-1}(cr+\delta). \tag{31}$$

Finally using majorizations (29), (30), and (31), (28) gives

$$\|P'(x)\| \leq c(r) \quad (r \in I),$$

where function $c(r)$ is given by (26). It now follows easily from (15) and (16) that $c(r) \in [0,1)$ which shows (24). Estimate (25) reduces to showing (17) which is true by hypothesis.

(ii) It follows from (18), (25) and (2) for $n=0$ that $\|x_1 - x_0\| \leq (1-c(r))r \leq r$, and

$$\|g(x_0) - x_0\| \leq (1-b)r < r,$$

which imply $x_1,\ g(x_0) \in U(x_0, r)$. For all positive integers smaller or equal to n assume that

$$x_n \in U(x_0, r) \quad \text{and} \quad \|x_n - x_0\| \leq (1-c^n(r))r \leq r.$$

Then we can get

$$\begin{aligned}\|g(x_n) - x_0\| &\leq \|g(x_n) - g(x_0)\| + \|g(x_0) - x_0\| \leq b\|x_n - x_0\| + \|g(x_0) - x_0\| \\ &\leq br + \|g(x_0) - x_0\| \leq r \quad \text{(by (18))}.\end{aligned}$$

That is $g(x_n) \in U(x_0, r)$. Using (4) and (24) we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|P(x_n) - P(x_{n-1})\| \leq \sup_{y \in [x_{n-1}, x_n]} \|P'(y)\|\, \|x_n - x_{n-1}\| \\ &\leq c(r)\|x_n - x_{n-1}\| \quad (r \in I).\end{aligned} \tag{32}$$

Therefore we can obtain in turn

$$\|x_{n+1} - x_n\| \leq c(r)\|x_n - x_{n-1}\| \leq \cdots \leq c^n(r)\|x_1 - x_0\| = c^n(r)(1-c(r))r,$$

and

$$\begin{aligned}\|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_0\| \leq c^n(r)(1-c(r))r + (1-c^n(r))r \\ &= (1-c^{n+1}(r))r \leq r.\end{aligned}$$

That is we have shown $x_n \in U(x_0, r)$ and $\|x_n - x_0\| \leq (1-c^n(r))r \leq r$ for all $n \in N$. From (32) it now follows that for $m, n \in N$

$$\|x_{n+m} - x_n\| \leq (1-c^m(r))c^n(r)r. \tag{33}$$

It follows from (33) that $\{x_n\}$ $(n \geq 0)$ is a Cauchy sequence in a Banach space E_1 and as such it converges to some limit x^*. Using the continuity of f and g, (4) gives

$P(x^*) = x^*$. To show uniqueness, assume $y^* \in U(x_0, r)$ and $P(y^*) = y^*$. Then by (24) we get

$$\begin{aligned} \|x^* - y^*\| = \|P(x^*) - P(y^*)\| &\le \sup_{z \in [x^*, y^*]} \|P'(z)\| \, \|x^* - y^*\| \\ &\le c(r)\|x^* - y^*\| \end{aligned}$$

which shows $x^* = y^*$.

Finally by letting $m \to \infty$ in (33) we obtain (27).

That completes the proof of Theorem 1.

The region of accessibility to x^* can be extended also to a ball around x_0 in a different way.

Theorem 2. *Under the hypotheses of Theorem 1 iteration $y_{n+1} = P(y_n)$, $y_0 \in U(x_0, r)$ $(r \in I)$ converges to x^*. Moreover the following error bounds hold for all $n \ge 0$*

$$\|x^* - y_n\| \le \frac{c^n(r)}{1 - c(r)} \|y_1 - y_0\|$$

and

$$\|x^* - y_n\| \le c^n(r)\|x^* - y_0\|.$$

Proof. For $x \in U(x_0, r)$ we obtain by (3) and (24) that

$$\|x_1 - p(x)\| = \|p(x_0) - p(x)\| \le c(r)\|x - x_0\| \le c(r)r.$$

That is P maps $U(x_0, r)$ into $U(x_1, c(r)r)$ with $U(x_1, c(r)r) \subseteq U(x_0, r)$. Hence $P\colon U(x_0, r) \to U(x_0, r)$ is a contraction, with contraction factor $c(r) < 1$.

The result follows from the contraction mapping principle [3], [5].

We complete our study with a simple example that shows that our convergence ball includes the ones found in earlier results [6], [7].

Example. Let $E_1 = E_2 = R$, $D = [-.1, .1]$, $x_0 = .01$ and define $f(x)\colon D \to E_2$ by

$$f(x) = x - x^2. \tag{34}$$

Then we have in turn

$$g(x) = x - f(x) = x - (x - x^2) = x^2,$$

$$[x, y; f] = \frac{f(x) - f(y)}{x - y} = 1 - (x + y), \ [x, x, g(x); f] = -1,$$

$$[x, g(x), g(x); f]g'(x) = \frac{2x(x+3)}{x - 1}, \quad (x \ne 0), \quad \alpha = 1.7061206,$$

$$a = \|[x_0, g(x_0); f]^{-1}\| = \left| \frac{1 - x_0}{1 - 2x_0 + x_0^3} \right| = 1.010203,$$

$$|g(x) - g(y)| = |x + y| \cdot |x - y| \Rightarrow b = .2,$$

$$|f(x)-f(y)| = |1-(x+y)|\,|x-y| \Rightarrow c = d = 1.4546923,$$

$$\delta = \beta = .010001 \quad \text{and} \quad \varepsilon = .012375.$$

The hypotheses of Theorem 1 are satisfied if $h_1(r) = r - .6874306 < 0$, $h_2(r) = 4.2322594r^2 - 6.465451r + .9754506 > 0$, $h_3(r) = 4.2322594r^3 - 6.4866216r^2 + 1.0045605r - .010001 \geq 0$ and $h_4(r) = r - .012375 \geq 0$ for r belonging in a suitable subset I of the nonnegative axis, and such that $U(x_0, r) \subseteq D$. It is simple algebra to show that these conditions are satisfied if we set

$$I = [.012375,\ .09]. \tag{35}$$

It follows from Theorem 2 and (35) that iteration $y_{n+1} = P(y_n)$ converges to a solution $x^* = 0$ of equation $f(x) = 0$, where f is given by (24) provided that

$$y_0 \in [-.08,\ .1] = I_A. \tag{36}$$

To compare our results with the ones obtained in [6, p. 858] (similarly for [7]), we first note that no uniqueness result was given there. Moreover the convergence ball is $U(.01,\ .039802)$. That means convergence is assured for

$$y_0 \in [-.029802,\ .049802] = I_P$$

and by (36)

$$I_P \subseteq I_A.$$

Hence we showed that there exist infinitely many new starting points $y_0 \in I_A - I_P$ for which the convergence of iteration $y_{n+1} = P(y_n)$ is assured via Theorem 2, whereas this is not the case for Theorem 1 in [6, p. 858].

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A LOCAL CONVERGENCE THEOREM FOR THE INEXACT NEWTON METHOD AT SINGULAR POINTS

Ioannis K. Argyros

Abstract. If Newton's method is used to find a solution of an equation on a Banach space, and the Fréchet-derivative is singular at the solution, the convergence of Newton iterates to the solution is linear rather than quadratic. In this study we employ the inexact Newton method to restore quadratic convergence, especially in cases when Newton's or the corrected Newton's method cannot achieve this or cannot even be used.

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Key Words and Phrases. Banach space, Newton method, singular points, quadratic convergence.

I. INTRODUCTION

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0 \tag{1}$$

where F is an operator defined on a closed convex subset D of a Banach space E_1 with values on a Banach space E_2.

Newton's method

$$y_{n+1} = y_n - F'(y_n)^{-1}F(y_n) \quad (n \geq 0) \tag{2}$$

is the most popular way of generating a sequence $\{y_n\}$ $(n \geq 0)$ converging to x^*. According to the well-known Newton-Kantorovich theorem [3], [5], [7] method (2) converges quadratically to x^* provided $\|x_0 - x^*\|$ is small and the Fréchet derivative $F'(x^*)$ is an invertible operator. The solution x^* is called isolated in this case. A solution x^* at which $F'(x^*)$ is not invertible is called singular. Method (2) converges to x^*, only linearly in this case [3], [5], [7].

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In [6] and the references there the convergence behavior of method (2) has been studied. Roughly speaking the Banach space E_1 is written as a direct, $N \oplus E$, sum of the null space N of $F'(x^*)$ and an appropriate complementing space E. Then, when the iterates converge, the components of the iterate error in N converges linearly, while the E-component of the error converges with asymptotic order 2. This technique is considered rather complicated, and most importantly it is assumed that the iterates can be computed exactly. However this is not the case most of the time. That is why we propose the study of the inexact Newton method of the form

$$x_{n+1} = x_n - P'(x_n)^{-1}F(x_n) - z_n, \tag{3}$$

where

$$F(x) = P(x) + Q(x) \quad (x \in D), \tag{4}$$

$F(x^*) = 0$, $P'(x^*)$ is invertible and $P, Q\colon D \to E_2$ are Fréchet-differentiable operators on D.

The importance of studying the inexact Newton method (3) comes from the fact that many commonly used variants of Newton's method can be considered procedures of this type. Indeed approximation (3) characterizes any iterative process in which corrections are taken as approximate solutions of Newton equations. If for example an equation on the real line is solved, $F(x_n) \geq 0$ $(n \geq 0)$, and $P'(x_n)^{-1}$ $(n \geq 0)$ overestimates the derivative, y_n is always larger than the corresponding Newton iterate. In such cases, a positive z_n $(n \geq 0)$ correction term is appropriate. Points $z_n \in D$ $(n \geq 0)$ are determined in such a way that iteration $\{x_n\}$ $(n \geq 0)$ converges to a solution x^* of equation (1). Sufficient conditions for the convergence of method (3) to x^* have been given in [1]–[4], [7] under various assumptions. Here as in [6] and the references there we will assume that x^* exists but we will not construct N and E which are very expensive to produce in general. A local convergence theorem is given an special choices of the points z_n $(n \geq 0)$ that restore quadratic convergence in some cases will be given.

Finally a numerical example is also provided to show that quadratic convergence can be restored in the case when Newton's method or the corrected Newton's method cannot even be used.

II. CONVERGENCE ANALYSIS

Let F, Q be Fréchet-differentiable on $D \subseteq E_1$, $x^* \in D$ with $F(x^*) = 0$, $P'(x^*)$ invertible on D, a, b, c, c_n $(n \geq 0)$ nondecreasing functions on the nonnegative axis, $R \geq 0$ such that:

$$\|(F'(x^*) - Q'(x^*))^{-1}[F'(x + t(x^* - x)) - F'(x)]\| \leq a(\|x^* - x\|)\|x^* - x\|t, \tag{5}$$

$$\|(F'(x^*) - Q'(x^*))^{-1}(Q'(x^*) - Q'(x))\| \leq b(\|x - x^*\|), \tag{6}$$

and

$$\|(F'(x^*) - Q'(x^*))^{-1}B_n\| \leq c_n(\|x^* - x_n\|)\|x_n - x^*\| \leq c(\|x^* - x_n\|)\|x_n - x^*\|, \tag{7}$$

where

(8) $B_n = (Q'(x_n) - F'(x_n))z_n - Q'(x_n)(x_n - x^*) \quad (n \geq 0)$ with x_n given by (3),

and for all $x, x_n \in U(x^*, r) = \{x \in E_1 \mid \|x - x^*\| \leq r\} \subseteq U(x^*, R) \subseteq D$, $t \in [0,1]$.

It is convenient to define on the nonnegative axis the functions f_1 and f_2 by

$$f_1(r) = 1 - b(r) - a(r)r, \tag{9}$$

and

$$f_2(r) = 3a(r)r + 2c(r) + 2b(r) - 2. \tag{10}$$

We can now state and prove the following local convergence result concerning the inexact Newton method (3):

Theorem. *Assume F, Q, P be Fréchet-differentiable operators defined on a closed convex subset D of a Banach space E_1 with values in a Banach space E_2, $x^* \in D$ with $F(x^*) = 0$, $P'(x^*)$ invertible on D, a, b, c, c_n $(n \geq 0)$ nondecreasing functions on the nonnegative axis and $R \geq 0$ such that:*

(a) *conditions* (5)–(7) *are satisfied*;

(b) *there exists an interval $[0, d]$, $d < R$ such that*

$$f_1(r) > 0, \tag{11}$$

and

$$f_2(r) < 0, \tag{12}$$

for all $r \in [0, d]$.

(c) $U(x_0, d) \subseteq D$.

Then the inexact Newton method $\{x_n\}$ $(n \geq 0)$ generated by (3) *is well defined remains in $U(x^*, d)$ for all $n \geq 0$ and converges to x^* provided that $x_0 \in U(x^*, d)$. Moreover the following error bounds hold for all $n \geq 0$*

$$\|x_{n+1} - x^*\| \leq \frac{1}{1 - b(\|x_n - x^*\|) - a(\|x_n - x^*\|)\|x_n - x^*\|}\bar{\varepsilon}_n \tag{13}$$

$$\leq \frac{1}{1 - b(d) - a(d)d}\varepsilon_n, \tag{14}$$

where

$$\bar{\varepsilon}_n = \left[\frac{1}{2}a(\|x_n - x^*\|) + c_n(\|x_n - x^*\|)\right]\|x_n - x^*\| \tag{15}$$

and

$$\varepsilon_n = \left[\frac{1}{2}a(d) + c(d)\right]\|x_n - x^*\|. \tag{16}$$

Proof: By hypothesis the point x_0 belongs in $U(x^*, d)$. Assume $x_k \in U(x^*, d)$ for $k = 0, 1, 2, \ldots, n$. Then using (6), (7), the Banach Lemma on invertible operators [7], the approximation

$$F'(x_k) - Q'(x_k) = F'(x_k) - Q'(x_k) - (F'(x^*) - Q'(x^*)) + (F'(x^*) - Q'(x^*))$$

and the estimate

$$\begin{aligned}
&\|(F'(x^*) - Q'(x^*))^{-1}[(F'(x_k) - F'(x^*)) - (Q'(x_k) - Q'(x^*))]\| \\
&\qquad \leq a(\|x_k - x^*\|)\|x_k - x^*\| + b(\|x_k - x^*\|) \leq a(d)d + b(d) < 1 \quad \text{(by (12))},
\end{aligned}$$

it follows that $P'(x_k)^{-1}$ exists and

$$\begin{aligned}
&\|P'(x_k)^{-1}(F'(x^*) - Q'(x^*))\| \leq [1 - b(\|x_k - x^*\|) - a(\|x_k - x^*\|)\|x^* - x_k\|]^{-1} && (17)\\
&\qquad\qquad\qquad\qquad \leq [1 - b(d) - a(d)d]^{-1}. && (18)
\end{aligned}$$

Using (3) and (8) we obtain the approximation

$$\begin{aligned}
x_{k+1} - x^* &= x_k - x^* - P'(x_k)^{-1}F(x_k) - z_k \\
&= [F'(x_k) - Q'(x_k)]^{-1}[F'(x^*) - Q'(x^*)][F'(x^*) - Q'(x^*)]^{-1} \cdot \qquad (19)\\
&\quad \cdot \left\{ \int_0^1 [F'(x_k + t(x^* - x_k)) - F'(x_k)](x^* - x_k)dt + B_k \right\}.
\end{aligned}$$

Using (5)–(7), (17) and (18) we obtain (13) and (14). Moreover by (12) and (14)

$$\|x_{k+1} - x^*\| \leq \varepsilon\|x_k - x^*\| \leq \cdots \leq \varepsilon^{k+1}\|x_0 - x^*\| \leq \varepsilon^{k+1} d \tag{20}$$

for

$$\varepsilon = \frac{a(d) + 2c(d)}{2[1 - b(d) - a(d)d]} \in [0, 1). \tag{21}$$

It follows from (20) and (21) that

$$x_k \in U(x^*, d) \quad \text{for all } k \geq 0 \text{ and } \lim_{k \to \infty} x_k = x^*.$$

That completes the proof of the Theorem.

Remark 1. We note that in the proof of the Theorem we have not used (or assumed) that $F'(x^*) = 0$.

Remark 2. Set $z_n = 0$ $(n \geq 0)$, $E_1 = E_2 = \mathbf{R}$ and assume x^* is a zero of multiplicity m for F. Choose $Q(x) = (1 - m)F(x)$ to obtain using (4) that $P(x) = mF(x)$. In this case (3) reduces to the corrected Newton's method which converges quadratically to x^* [5], [7].

Remark 3. Choose $z_n = P'(x_n)^{-1}Q'(x_n)(x^* - x_n)$ $(n \geq 0)$ then by (7) we can choose $c_n = c = 0$ $(n \geq 0)$. Moreover if $a(t) = qt$, $t \in [0, d]$ for some $q \geq 0$ we obtain by (13) that the order of convergence of method (3) is quadratic.

Remark 4. Let $z_n = L_n(x_n - x_{n-1})$ $(n \geq 1)$, for some bounded linear operators L_n, $z_0 = 0$, $x_0, x_1 \in U(x^*, d)$, $\|L_n\| \leq \|x_n - x^*\|$ $(n \geq 0)$ and c_n given by

$$\begin{aligned} c_n(\|x_n - x^*\|) = s_1(\|x_n - x^*\|)\|x_n - x^*\|^{p_1} + s_2(\|x_n - x^*\|)\|x_n - x_{n-1}\|^{p_2} \\ + \; s_3(\|x_n - x^*\|)\|x_n - x^*\|^{p_3}\|x_n - x_{n-1}\|^{p_4} + s_4(\|x_n - x^*\|), \end{aligned}$$

for all $n \geq 1$ where s_1, s_2, s_3, s_4 are nondecreasing functions on $[0, d]$ and p_1, p_2, p_3 and p_4 are nonnegative numbers. Moreover assume there exists h such that

$$\|(F'(x^*) - Q'(x^*))^{-1}Q'(z)\| \leq h \quad \text{for } z = x^* \text{ or } x_0.$$

Furthermore assume $F'(x^*) = 0$. Then from the approximation $B_n = (Q'(x_n) - F'(x_n))L_n(x_n - x_{n-1}) - Q'(x_n)(x_n - x^*)$, we obtain

$$\begin{aligned} \|(F'(x^*) - Q'(x^*))^{-1}B_n\| \leq a(\|x_n - x^*\|)\|L_n\|\,\|x_n - x^*\| \cdot \|x_n - x_{n-1}\| \\ + \; (b(\|x_n - x^*\|) + h)\|L_n\|\,\|x_n - x_{n-1}\| \\ + \; (b(\|x_n - x^*\| + h)\|x_n - x^*\|. \end{aligned}$$

Hence we can take $s_1(r) = 0$, $s_2(r) = b(r) + h$, $c_3(r) = a(r)$, $c_4(r) = b(r) + h$, $r \in [0, d]$, $p_2 = p_3 = p_4 = 1$ any $p_1 > 0$, and

$$c(r) = \max\{\bar{c}(r), h\}, \quad r \in [0, d]$$

where

$$\bar{c}(r) = s_1(r)r^{p_1} + 2^{p_2}s_2(r)r^{p_2} + 2^{p_4}s_3(r)r^{p_3+p_4} + s_4(r), \quad r \in [0, d].$$

Remark 5. Our Theorem can be used to solve autonomous differential equations satisfying

$$P'(x) = T(F(x))$$

for some continuous operator T. Since $P'(x^*) = T(F(x^*)) = T(0)$ the inverse of $P'(x^*)$ can be computed without knowing the solution x^* [3], [7].

Remark 6. Our results can be extended to include a wider class of inexact Newton methods of the form

$$x_{n+1} = x_n - P_1(x_n)^{-1}F(x_n) - z_n \quad (n \geq 0) \tag{22}$$

related to

$$v_{n+1} = v_n - A(v_n)^{-1}F(v_n) \quad (n \geq 0),$$

where

$$A(x) = P_1(x) + Q_1(x) \quad (x \in D)$$

and $A(x)$, $P_1(x)$, $Q_1(x) \in L(E_1, E_2)$ $(x \in D)$, with $A(x)$ being an approximation to $F'(x)$ $(x \in D)$. The conclusions of the theorem will hold for method (22) assuming the following changes: In (5) replace $F'(x^*)$, $Q'(x^*)$ and $F'(x)$ by $A(x^*)$, $Q_1(x^*)$ and $A(x)$ respectively; in (6) replace $F'(x^*)$, $Q'(x^*)$ and $Q'(x)$ by $A(x^*)$, $Q_1(x^*)$ and $Q_1(x)$ respectively; in (7) replace $F'(x^*)$, $Q'(x^*)$ by $A(x^*)$ and $Q_1(x^*)$ respectively and in (8) replace $Q'(x_n)$, $F'(x_n)$ by $Q_1(x_n)$ and $A(x_n)$ $(n \geq 0)$ respectively. Then exactly as in the proof of the theorem using the approximation

$$x_{n+1} - x^* = P_1(x_n)^{-1}[F(x^*) - F(x_n) - A(x_n)(x^* - x_n) + B_n]$$

we can show all the conclusions of the theorem. Comments similar to the ones made in Remarks 1–5 can also be made for method (22).

We will finish this study with an example related with Remark 3.

Example. Let $E_1 = E_2 = \mathbf{R}$, $x^* = 0$, $R = d \in [0, \frac{1}{3})$, $U(x^*, d) = D$, consider the operator F on D given by

$$F(x) = x^2. \tag{23}$$

Choose

$$z_n = (1 - 2x_n)^{-1}(x_n - x^*) \quad (n \geq 0) \quad \text{for } |x_0| \in U(0, d). \tag{24}$$

and

$$Q(x) = x \quad (x \in D). \tag{25}$$

Using (5)–(10), (23), (24) and (25) we can choose $a(r) = 2$, $b(r) = c(r) = 0$ for all $r \in [0, d]$. Moreover we have $B_n = 0$ $(n \geq 0)$, $f_1(r) = 1 - 2r > 0$ and $f_2(r) = 6r - 2 < 0$ for all $r \in [0, d]$. Furthermore $q = 0$ in Remark 3. Hence all hypotheses of the Theorem are satisfied and (13) becomes

$$x_{n+1} - x^* = (2x_n - 1)^{-1}(x_n - x^*)^2 \quad (n \geq 0),$$

which shows quadratic convergence of sequence $\{x_n\}$ $(n \geq 0)$ to x^*. We also note that $x^* = 0$ is a zero of multiplicity 2 for F given by (23).

We observe that as it can easily be seen Newton's method cannot be used since $F'(x_n)^{-1} \neq 0$ $(n \geq 0)$ and $y_{n+1} = y_n - F'(y_n)^{-1}F(y_n)$ cannot be computed in any neighborhood of zero, for any starting point y_0. The corrected Newton's method $w_{n+1} = w_n - 2F'(w_n)^{-1}F(w_n)$ $(n \geq 0)$ converges quadratically to x^* [3], [5], [7], provided that $z_n = 0$ $(n \geq 0)$, which is not true in general.

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