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DIFFERENTIATED SHIFT-INVARIANT MULTIVARIATE INTEGRAL OPERATORS

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Abstract. In a recent paper [1] the author, together with H. Gonska, introduced some wavelet-type, very general multivariate integral operators over $\mathbb{R}^d$, $d \geq 1$, and studied their various properties such as shift-invariance, global smoothness preservation and optimality, convergence to the unit, differentiation, and preservation of continuous multivariate probability distribution functions. These operators are introduced through a convolution-like iteration of another very general multivariate operator with a general multivariate scaling type function. In this paper the author gives sufficient conditions, so that the *partial derivatives of the above described multivariate operators* possess most of the above listed nice properties of their originals. In particular, a sufficient condition is given so that a corresponding "global smoothness preservation" sharp multivariate inequality is attained. The corresponding convergence of differentiated operators to the differentiated unit operator is given with rates of convergence.

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Main Results.

Let $X := C_U(\mathbb{R}^d)$, $d \geq 1$ be the space of uniformly continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$.

For $f \in X$, define the first order modulus of continuity of f by

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|\leq\delta}} |f(x)-f(y)|, \delta > 0,$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. It is known [1] that $\omega_1(f,\delta) < \infty$, $\forall\delta > 0$. Here $C(\mathbb{R}^d)$ denotes the space of continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Let $\{\ell_k\}_{k\in\mathbb{Z}}$ be a sequence of linear operators from $C(\mathbb{R}^d)$ into itself such that

$$\ell_k(f,x) = \ell_0(f(2^{-k}\cdot),x), \quad \text{all } x \in \mathbb{R}^d,\ f \in C(\mathbb{R}^d). \tag{1}$$

Assume that

$$\ell_0 : C^{(\rho)}(\mathbb{R}^d) \to C^{(\rho)}(\mathbb{R}^d), \quad \rho \in \mathbb{N} \quad \text{be fixed}, \tag{2}$$

where $C^{(\rho)}(\mathbb{R}^d)$ is the space of ρ-times continuously differentiable functions.

In this article we consider *only* $f \in C^{(\rho)}(\mathbb{R}^d)$ such that $\frac{\partial^\rho f}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_m}^{j_m}} \in X$, where fixed $i_1,\dots,i_m \in \{1,\dots,d\}$ be such that $i_1 < \cdots < i_m$, and fixed $j_1,\dots,j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$.

From now on we will denote by

$$f^\partial := \frac{\partial^\rho f}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_m}^{j_m}}, \quad f \in C^{(\rho)}(\mathbb{R}^d), \tag{3}$$

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where the mixed partial is defined as above.

Let φ be a bounded real valued function of compact support $\subseteq \times_{i=1}^{d}[-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \geq 0$, φ is Lebesgue measurable and

$$\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{d-\text{fold}} \varphi(x_1 - u_1, x_2 - u_2, \ldots, x_d - u_d)\, du_1 \ldots du_d = 1,$$

$$\text{all } (x_1, \ldots, x_d) \in \mathbb{R}^d \tag{4}$$

((4) same as

$$\int_{\mathbb{R}^d} \varphi(x-u)\, du = 1, \text{all } x \in \mathbb{R}^d;$$

where $x = (x_1, \ldots, x_d)$, $u = (u_1, \ldots, u_d)$). One can easily prove that

$$\int_{\mathbb{R}^d} \varphi(u) du = 1. \tag{5}$$

Example. For $i = 1, \ldots, d$ take the characteristic function

$$\varphi_i(x_i) := \chi_{[-\frac{1}{2},\frac{1}{2})}(x_i) = \begin{cases} 1, & x_i \in \left[-\frac{1}{2}, \frac{1}{2}\right) \\ 0, & \text{else.} \end{cases}$$

Define

$$\varphi^*(x) := \prod_{i=1}^{d} \varphi_i(x_i), \quad \text{all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Note that φ^* is bounded, supp $\varphi^* \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]^d$, $\varphi^* \geq 0$ and φ^* is Lebesgue measurable. Also

$$\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{d-\text{fold}} \prod_{i=1}^{d} \varphi_i(x_i - u_i)\, du_1 \ldots du_d = 1,$$

i.e.,

$$\int_{\mathbb{R}^d} \varphi^*(x-u) du = 1, \quad \text{all } x \in \mathbb{R}^d.$$

Let $\{\mathcal{L}_k\}_{k \in \mathbb{Z}}$ be the sequence of linear operators acting on $C^{(\rho)}(\mathbb{R}^d)$ and defined as follows

$$\mathcal{L}_0(f, x) := \int_{\mathbb{R}^d} \ell_0(f, u) \varphi(x-u)\, du \tag{6}$$

and

$$\mathcal{L}_k(f, x) := \mathcal{L}_0(f(2^{-k} \cdot); 2^k x), \tag{7}$$

all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}$. I.e.,

$$\mathcal{L}_k(f, x) = \int_{\mathbb{R}^d} \ell_k(f, u) \varphi(2^k x - u)\, du. \tag{8}$$

For our convenience we shall use the notation

$$\int_{-\vec{a}}^{\vec{a}} \bullet = \int_{\times_{i=1}^{d}[-a_i, a_i]} \bullet. \tag{9}$$

Furthermore, we obtain that

$$\begin{aligned} \mathcal{L}_k^{\partial}(f, x) &= 2^{k\rho} \cdot \int_{-\vec{a}}^{\vec{a}} \ell_k^{\theta}(f, 2^k x - u) \varphi(u)\, du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^{\partial}(f, 2^k x - u) \varphi(u)\, du \\ &= 2^{k\rho} \cdot \int_{\mathbb{R}^d} \ell_k^{\partial}(f, u) \varphi(2^k x - u)\, du, \text{all } x \in \mathbb{R}^d,\ k \in \mathbb{Z}. \end{aligned} \tag{10}$$

Assumption. Take $f \in C^{(\rho)}(\mathbb{R}^d)$, $\rho \in \mathbb{N}$ fixed, such that $f^\partial \in X$. For fixed $\alpha > 0$ we assume that

$$\sup_{\substack{u,y\in\mathbb{R}^d \\ \|u-y\|_\infty \le \alpha}} |\ell_0^\partial(f,u) - f^\partial(y)| \le \omega_{1,\infty}\left(f^\partial, \frac{m\alpha+n}{2^r}\right), \tag{11}$$

is true for f as above, $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$, where $\omega_{1,\infty}$ is the modulus of continuity ω_1 defined with respect to $\|\cdot\|_\infty$.

Here we present our results without proofs due to their length. We intend to show proofs elsewhere.

Definition 1. Let $f_\alpha(\cdot) := f(\cdot + \alpha)$, $\alpha \in \mathbb{R}^d$ and ϕ be an operator. If $\phi(f_\alpha) = (\phi f)_\alpha$, then ϕ is called a *shift invariant operator.*

Proposition 1. *Assume that*

$$\ell_0^\partial\left(f(2^{-k}\cdot+\alpha); 2^k u\right) = \ell_0^\partial\left(f(2^{-k}\cdot); 2^k(u+\alpha)\right), \tag{12}$$

all $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}^d$ fixed, all $u \in \mathbb{R}^d$; any $f \in C^{(\rho)}(\mathbb{R}^d)$; $\rho \in \mathbb{N}$ be given. Then $\mathcal{L}_k^\partial$, any $k \in \mathbb{Z}$ is a shift invariant operator.

Next we establish the property of "*global smoothness preservation*" of operators $\mathcal{L}_k^\partial$, $k \in \mathbb{Z}$.

Theorem 1. *For any $f \in C^{(\rho)}(\mathbb{R}^d)$, such that $f^\partial \in X$, $\rho \in \mathbb{N}$ fixed, and any $u \in \mathbb{R}^d$ assume that*

$$|\ell_0^\partial(f, x-u) - \ell_0^\partial(f, y-u)| \le \omega_1(f^\partial, \|x-y\|), \quad \text{all } x, y \in \mathbb{R}^d, \tag{13}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. Then

$$\omega_1((\mathcal{L}_k f)^\partial, \delta) \le \omega_1(f^\partial, \delta), \tag{14}$$

any $\delta > 0$.

Optimality of (14) follows.

Theorem 2. *Let $i_1, \ldots, i_m \in \{1, \ldots, d\}$ be such that $i_1 < \cdots < i_m$, and $j_1, \ldots, j_m \in \mathbb{N}$ be such that $\sum_{r=1}^m j_r = \rho$. Consider $g_{j_1}(x) := \frac{x^{j_1+1}}{(j_1+1)!}$, and $g_{j_r}(x) := \frac{x^{j_r}}{j_r!}$, $r = 2, \ldots, m$; all $x \in \mathbb{R}$. Denote by $f^\partial := \frac{\partial^\rho f}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}}$, for any $f \in C^{(\rho)}(\mathbb{R}^d)$. Denote also by $pr_{i_r} : \mathbb{R}^d \ni (x_1, \ldots, x_d) \to x_{i_r}$, $r = 1, \ldots, m$ the projection onto the i_r coordinate. For all $x, y, u \in \mathbb{R}^d$ assume that*

$$\begin{aligned} &\left(\ell_0\left(\left(\sum_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^\partial (2^k x - u) \\ &- \left(\ell_0\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)(2^{-k}\cdot)\right)\right)^\partial (2^k y - u) = 2^{-k\rho}\cdot(x_{i_1} - y_{i_1}). \end{aligned} \tag{15}$$

Then

$$\omega_1\left(\left(\mathcal{L}_k\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)\right)^\partial; \delta\right) = \omega_1\left(\left(\prod_{r=1}^m g_{j_r} \circ pr_{i_r}\right)^\partial; \delta\right), \tag{16}$$

for any $\delta > 0$; $k \in \mathbb{Z}$. That is establishing that (14) is sharp!

The convergence of $\frac{\partial^\rho \mathcal{L}_k}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}} (\sum_{r=1}^m j_r = \rho)$ to $\frac{\partial^\rho}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_m}^{j_m}}$ operator, as $k \to +\infty$, with rates given in the following.

Theorem 3. *For $f \in C^{(\rho)}(\mathbb{R}^d)$, $d \ge 1$, such that $f^\partial \in C_U(\mathbb{R}^d)$, under the assumption (11), it holds*

$$|\mathcal{L}_k^\partial(f, x) - f^\partial(x)| \le \omega_{1,\infty}\left(f^\partial, \frac{ma+n}{2^{k+r}}\right), \tag{17}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $k, r \in \mathbb{Z}$, and $a := \max(a_i)$, $i = 1, \ldots, d$.

Again, due to their length we intend to present applications of the above results elsewhere.

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