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UPPER BOUND ON HANKEL DETERMINANT FOR BOUNDED TURNING FUNCTION ASSOCIATED WITH SĂLĂGEAN-DIFFERENCE OPERATOR

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Abstract. By making use of Sălăgean-difference operator we introduce a new function class $\mathcal{R}_\lambda^\beta(\alpha)$ which generalizes the class of functions of bounded turning of order alpha. We investigate upper bounds on the third Hankel determinant for the class $\mathcal{R}_\lambda^\beta(\alpha)$. Our results generalize the results of earlier researchers in this direction.

1 Introduction and Definitions

Let $\mathbb{R}$ and $\mathbb{C}$ be denote the set of real and complex numbers respectively. Denote by $\mathcal{A}$, the class of of all functions h of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ satisfying condition $h(0) = h'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consist of univalent functions. A function $h \in \mathcal{A}$ said to be of bounded turning if and only if $\Re(h'(z)) > 0$ for any $z \in \Delta$. We denote such class of functions by $\mathcal{R}$. For a function $h \in \mathcal{A}$, we define a linear operator $D_\lambda^\beta : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$D_\lambda^0 h(z) = h(z),$$

$$\begin{aligned} D_\lambda^1 h(z) =& zh'(z) + \frac{\lambda}{2}[h(z) - h(-z) - 2z] \qquad (\lambda \in \mathbb{R}) \\ =& z + \sum_{n=2}^{\infty} \left[n + \frac{\lambda}{2}\left(1 + (-1)^{n+1}\right) \right] a_n z^n, \\ D_\lambda^2 h(z) =& D^1(D_\lambda^1 h(z)). \end{aligned}$$

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In general, for $\beta \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$,

$$D_\lambda^\beta h(z) = D_\lambda^1(D_\lambda^{\beta-1}h(z)) = z + \sum_{n=2}^{\infty}\left[n + \frac{\lambda}{2}(1 + (-1)^{n+1})\right]^\beta a_n z^n \quad (z \in \Delta). \quad (1.2)$$

The operator D_λ^β is known as the Sălăgean-difference operator in literature (see [17, 18]). This operator is a modified Dunkel operator of complex variables (see[8, 16]). When $\lambda = 0$, $D_\lambda^\beta = D_0^\beta = D^\beta$ is known as the Sălăgean differential operator (see[44]).

Example 1. *Let*

$$h(z) = ze^{\frac{z}{2}} = z + \sum_{n=2}^{\infty}\frac{z^n}{2^{n-1}(n-1)!}.$$

Then

$$D_1^1 h(z) = z + z^2 + \frac{z^3}{2} + \frac{z^4}{12} + \frac{z^5}{64} + \cdots.$$

Example 2. *Let*

$$h(z) = z\left(1 - \frac{z}{5}\right)^{-2} = z + \sum_{n=2}^{\infty}\frac{n}{5^{n-1}}z^n.$$

Then

$$D_1^1 h(z) = z + \frac{4}{5}z^2 + \frac{12}{25}z^3 + \frac{16}{125}z^4 + \cdots.$$

In 1976, Noonan and Thomas [32] defined q^{th} Hankel determinant of function h for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2}\cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q}\cdots & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinant plays an important role in the study of singularities (see [7, 10]). It is useful in showing that a function of bounded characteristic in Δ, i.e. a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational (see [5]). Pommerenke [37] proved that the Hankel determinant of univalent functions satisfy $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}, \quad (n = 1, 2, \cdots; \quad q = 2, 3, \cdots)$, where $\beta > \frac{1}{4000}$ and K depends only on q. Later, Hayman [15] proved that $|H_2(n)| < An^{\frac{1}{2}}, \quad (n = 1, 2, \cdots)$, (A is an absolute constant) for areally mean univalent functions. The study of $|H_q(n)|$ for various subfamilies of $\mathcal{A}$ are of interest for many researchers (see [11, 32, 38]. Finding the upper bounds of the Hankel determinants whose elements are the coefficients of univalent and multivalent functions for different values of q and n is an interesting area of research in the geometric function theory. For $q = 2$, $n = 1$, $a_1 = 1$ and $q = n = 2$, $a_1 = 1$, the Hankel determinant respectively reduce to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

The Hankel determinant $H_2(1)$ is popularly known as the Fekete-Szegö functional. Fekete-Szegö [12] gave a sharp estimate of non-linear functional $|a_3 - \mu a_2^2|$ for μ real. It is a combination of the two coefficients which describes the area problem posed earlier by Gronwall [14]. The problem of calculating $max_{h\in\mathcal{F}}|H_2(1)|$ for various compact subfamilies $\mathcal{F}$ of $\mathcal{A}$ was considered by various researchers (see [4, 22, 23, 29, 34, 35, 36]).

Recent research has focused on $H_2(2)$ for various subclasses of $\mathcal{S}$. Janteng et al.[19, 20] derived the exact bounds for $H_2(2)$ for the class of starlike functions ($\mathcal{S}^*$), the class of convex functions ($\mathcal{C}$) and the class of functions whose derivatives have positive real parts ($\mathcal{RT}$) in Δ . The bounds obtained for these three classes are $|H_2(2)| \leq 1$, $|H_2(2)| \leq \frac{1}{8}$, $|H_2(2)| \leq \frac{4}{9}$ respectively. Lee et al.[25] investigated $H_2(2)$ in the general class $\mathcal{S}^*(\phi)$ of starlike functions with respect to a given function ϕ and in particular obtained the results when $f \in \mathcal{S}^*(\alpha)$, the class of starlike functions of order α ($|H_2(2)| \leq (1-\alpha)^2$), the class $\mathcal{S}^*_L$ of lemniscate starlike functions ($|H_2(2)| \leq \frac{1}{16}$) and the class $\mathcal{S}^*_\beta$ of strongly starlike functions of order β ($|H_2(2)| \leq \beta^2$). Krishna and Ramreddy [24] generalized the result of Janteng et al.[20] giving the sharp bound of $H_2(2)$ in the class of starlike and convex functions of α.

Zaprawa [50] showed that if $f \in \mathrm{T}$, the class of typically real functions, then $|H_2(2)| \leq 9$. Ramreddy and Krishna [40] obtained the Hankel detrminant for starlike and convex functions with respect to symmetric points. Using Owa and Srivastava operator [33] Ω_z^δ $(0 \leq \delta \leq 1)$, Mishra and Gochhayat [28] introduced the class

$$\mathcal{R}_\delta(\gamma, \alpha) = \left\{ h \in \mathcal{A} : \Re\left(e^{i\gamma}\frac{\Omega_z^\delta h(z)}{z}\right) > \alpha cos\gamma, \quad (|\gamma| < \frac{\pi}{2}, \quad 0 \leq \alpha \leq 1) \right\}$$

and obtained the sharp upper bounds for $H_2(2)$. Apart from these, many researchers obtained the upper bounds for various subclasses of univalent analytic functions (see [2, 6, 21, 30, 42, 43, 47, 49]).

In this paper, we focus on third Hankel determinant for $q = 3$ and $n = 1$, denoted by $H_3(1)$ given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5. \end{vmatrix}$$

For $h \in \mathcal{A}$ and $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

An application of triangle inequality yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{1.3}$$

Recently, Babalola (see[1]) obtained the the sharp upper bound of $H_3(1)$ for the functions belongs to the class $\mathcal{S}^*, \mathcal{C}$ and $\mathcal{RT}$. Krishna et al. [48] introduced the generalized class $\mathcal{RT}(\alpha)$ as $\Re(h'(z)) > \alpha \quad (z \in \Delta)$ and obtained the bound on $H_3(1)$. Further, Bansal et al. [3] and Raza and Malik [41] obtained the bound $H_3(1)$ for certain subclass of univalent functions. Very recently, making use of Hohlov operator, Gochhayat et al.[13] introduced the class $\mathcal{R}_{a,b}^c$ and obtained the sharp bounds for $H_2(2)$ and $H_3(1)$ in terms of Gauss hypergeometric function. For recent results on third Hankel determinant see [31, 39, 45, 46].
Motivated by the above researchers, we introduced the subclass of univalent function by making use of Sălăgean-difference operator D_λ^β as follows:

Definition 3. *A function $h \in \mathcal{A}$ given by (1.1) is in the class $\mathcal{R}_\lambda^\beta(\alpha)$ if it satisfy the condition*

$$Re\left[\frac{D_\lambda^\beta h(z)}{z}\right] > \alpha \quad (0 \leq \alpha \leq 1,\ \beta \in \mathbb{N}_0,\ \lambda \in \mathbb{R},\ z \in \Delta). \tag{1.4}$$

It may be noted that by taking $\lambda = 0$ and $\beta = 1$ the class $\mathcal{R}_0^1(\alpha) = RT(\alpha)$ studied by Krishna et al. [48] (also, see [28]). Also, letting $\lambda = 0$, $\beta = 1$ and $\alpha = 0$ we obtain the class $\mathcal{RT}$ studied by Babalola [1]. Further, if we take $\beta = 0$ in the class $\mathcal{R}_\lambda^\beta(\alpha)$, we get the class $\mathcal{R}_0(0,\alpha) = \mathcal{R}(\alpha)$ studied by Mishra and Gochhayat [28]. In this paper, following a method of classical analysis derived by Libera and Zlotkiewicz [26, 27], we obtain the upper bounds of $H_3(1)$ for the function belonging to the class $R_\lambda^\beta(\alpha)$.

2 Preliminary Lemmas

Let $\mathcal{P}$ denote the class of functions denoted by p such that

$$p(z) = 1 + d_1z + d_2z^2 + \cdots \tag{2.1}$$

which are regular in the open unit disk Δ and satisfy $\Re(p(z)) > 0$ for any $z \in \Delta$. Here $p(z)$ is called the Caratheodory function (see[9]).
To prove our main results we need the following lemmas.

Lemma 4. *(see[9]) If $p \in \mathcal{P}$ is of the form (2.1), then*

$$|d_n| \leq 2 \quad (n \in \mathbb{N}). \tag{2.2}$$

The equality holds for the function

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n.$$

Lemma 5. *(see [26, 27]) If $p \in \mathcal{P}$ is of the form (2.1), then*

$$2d_2 = d_1^2 + (4 - d_1^2)x, \tag{2.3}$$

and

$$4d_3 = d_1^3 + 2(4 - d_1^2)d_1 x - (4 - d_1^2)d_1 x^2 + 2(4 - d_1^2)(1 - |x|^2)z, \tag{2.4}$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$.

3 Main Results

Theorem 6. *Let the function h given by (1.1) be in the class $R_\lambda^\beta(\alpha)$ $(\lambda \in \mathbb{R},\ \beta \in \mathbb{N}_0,\ 0 \leq \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2(1-\alpha)}{2^\beta}, \qquad |a_3| \leq \frac{2(1-\alpha)}{(3+\lambda)^\beta},$$

$$|a_4| \leq \frac{2(1-\alpha)}{4^\beta}, \qquad |a_5| \leq \frac{2(1-\alpha)}{(5+\lambda)^\beta}.$$

Proof. Let $h \in \mathcal{R}_\lambda^\beta(\alpha)$. Then there exists an analytic function $p \in \mathcal{P}$ in the unit disk Δ with $p(0) = 1$ and $\Re(p(z)) > 0$ such that

$$\frac{D_\lambda^\beta h(z)}{z} = \alpha + (1-\alpha)p(z). \tag{3.1}$$

Using the series expansion for $D_\lambda^\beta h(z)$ and $p(z)$ in (3.1), we get

$$\begin{aligned} &1 + 2^\beta a_2 z + (3+\lambda)^\beta a_3 z^2 + 4^\beta a_4 z^3 + (5+\lambda)^\beta a_5 z^4 + \cdots \\ &= 1 + (1-\alpha)d_1 z + (1-\alpha)d_2 z^2 + (1-\alpha)d_3 z^3 + (1-\alpha)d_4 z^4 \cdots . \end{aligned} \tag{3.2}$$

Equating the coefficient of various powers of z, z^2, z^3 and z^4 on both the sides of (3.2), we obtain

$$a_2 = \frac{(1-\alpha)d_1}{2^\beta}, \quad a_3 = \frac{(1-\alpha)d_2}{(3+\lambda)^\beta}, \quad a_4 = \frac{(1-\alpha)d_3}{4^\beta}, \quad a_5 = \frac{(1-\alpha)d_4}{(5+\lambda)^\beta}. \tag{3.3}$$

Application of triangle inequality to (3.3) and followed by the Lemma 4 give the desire estimate. This completes the proof of Theorem 6. □

Remark 7. *Taking $\lambda = 0$, $\beta = 1$ in the above theorem we obtain the coefficient bounds for the class $\mathcal{RT}(\alpha)$ as $|a_n| \leq \frac{2(1-\alpha)}{n} \quad (n \geq 2)$.*

Theorem 8. *Let the function $h(z)$ given by (1.1) be in the class $\mathcal{R}_{\lambda}^{\beta}(\alpha)$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4(1-\alpha)^2}{(3+\lambda)^{2\beta}}.$$

Proof. Putting the values of a_2, a_3 and a_4 from (3.3) in the functional $|a_2a_4 - a_3^2|$, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \left|\frac{(1-\alpha)d_1}{2^\beta}.\frac{(1-\alpha)d_3}{4^\beta} - \frac{(1-\alpha)^2d_2^2}{(3+\lambda)^{2\beta}}\right| \\
&= \frac{(1-\alpha)^2}{2^{3\beta}(3+\lambda)^{2\beta}}\left|(3+\lambda)^{2\beta}d_1d_3 - 2^{3\beta}d_2^2\right| \\
&= \frac{(1-\alpha)^2}{2^{3\beta}(3+\lambda)^{2\beta}}\left|e_1d_1d_3 - e_2d_2^2\right|, \qquad (3.4)
\end{aligned}$$

where

$$e_1 = (3+\lambda)^{2\beta} \quad \text{and} \quad e_2 = 2^{3\beta}. \qquad (3.5)$$

Substituting the values of d_2 and d_3 from (2.3) and (2.4) of Lemma 5 on the right hand side of (3.4), we get

$$\begin{aligned}
|e_1d_1d_3 - e_2d_2^2| &= \Bigg|\frac{e_1d_1}{4}\left\{d_1^3 + 2d_1(4-d_1^2)x - d_1(4-d_1^2)x^2 + 2(4-d_1^2)(1-|x|^2)z\right\} \\
&\quad - \frac{e_2}{4}\left\{d_1^2 + x(4-d_1^2)\right\}^2\Bigg| \\
&= \Bigg|\frac{e_1d_1^4}{4} + \frac{e_1d_1^2(4-d_1^2)x}{2} - \frac{e_1d_1^2(4-d_1^2)x^2}{4} + \frac{e_1d_1(4-d_1^2)(1-|x|^2)z}{2} \\
&\quad - \frac{e_2d_1^4}{4} - \frac{e_2(4-d_1^2)^2x^2}{4} - \frac{e_2d_1^2(4-d_1^2)x}{2}\Bigg|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
4|e_1d_1d_3 - e_2d_2^2| = &|(e_1-e_2)d_1^4 + 2e_1d_1(4-d_1^2)z + 2(e_1-e_2)d_1^2x(4-d_1^2) \\
&-(4-d_1^2)x^2(e_1d_1^2 + e_2(4-d_1^2)) - 2e_1d_1(4-d_1^2)|x|^2z|. \qquad (3.6)
\end{aligned}$$

Using the fact that $|z| \leq 1$ and $|xa + yb| \leq |x||a| + |y||b|$ where $x, y, a, b \in \mathbb{R}$ in the expression (3.6), after simplifying we get

$$\begin{aligned} 4|e_1d_1d_3-e_2d_2^2| &\leq \Big|(e_1-e_2)d_1^4+2e_1d_1(4-d_1^2)+2(e_1-e_2)d_1^2(4-d_1^2)|x| \\ &- \{e_1d_1^2+e_2(4-d_1^2)+2e_1d_1\}(4-d_1^2)|x|^2\Big| \\ &= \Big|(e_1-e_2)d_1^4+2e_1d_1(4-d_1^2)+2(e_1-e_2)d_1^2(4-d_1^2)|x| \\ &- \{(d_1+2)((e_1-e_2)d_1+2e_2)\}(4-d_1^2)|x|^2\Big|. \qquad (3.7)\end{aligned}$$

By Lemma 4, $|d_1| \leq 2$. Suppose that $d_1 = d$ and we may assume without restriction that $d \in [0, 2]$. Using the well-known results $(d_1+a)(d_1+b) \geq (d_1-a)(d_1-b)$ where $a, b \geq 0$ on the right hand side of (3.7) upon simplification give

$$\begin{aligned} 4|e_1d_1d_3-e_2d_2^2| \leq &\Big|(e_1-e_2)d_1^4+2e_1d_1(4-d_1^2)+2(e_1-e_2)d_1^2(4-d_1^2)|x| \\ &-(d_1-2)\big\{(e_1-e_2)d_1-2e_2\big\}(4-d_1^2)|x|^2\Big|. \qquad (3.8)\end{aligned}$$

Applying triangle inequality to the right hand side of (3.8), replacing $|x|$ by ρ and putting the values of e_1 and e_2 from (3.5) in (3.8) we get

$$\begin{aligned} 4|e_1d_1d_3-e_2d_2^2| \leq &\big[(3+\lambda)^{2\beta}-2^{3\beta}\big]d^4+2(3+\lambda)^{2\beta}d(4-d^2)+2\big[(3+\lambda)^{2\beta}-2^{3\beta}\big]d^2(4-d^2)\rho \\ &+(d-2)\big\{[(3+\lambda)^{2\beta}-2^{3\beta}]d-2^{3\beta+1}\big\}(4-d^2)\rho^2 \\ =&[(3+\lambda)^{2\beta}-2^{3\beta}]d^4+\big[2(3+\lambda)^{2\beta}d+2[(3+\lambda)^{2\beta}-2^{3\beta}]d^2\rho \\ &+(d-2)\{[(3+\lambda)^{2\beta}-2^{3\beta}]d-2^{3\beta+1}\}\rho^2\big](4-d^2) \\ =&G(d,\rho)(say) \quad (0 \leq \rho = |x| \leq 1). \qquad (3.9)\end{aligned}$$

Now, we maximize the function $G(d, \rho)$ on the close interval region $[0, 2] \times [0, 1]$. Differentiating G partially with respect to ρ we get

$$\frac{\partial G}{\partial \rho} = 2\Big[\big((3+\lambda)^{2\beta}-2^{3\beta}\big)d^2+(d-2)\big\{\big((3+\lambda)^{2\beta}-2^{3\beta}\big)d-2^{3\beta+1}\big\}\rho\Big](4-d^2). \quad (3.10)$$

For $0 < \rho < 1$ and for fixed d with $0 < d < 2$ we observe from (3.10) that $\frac{\partial G}{\partial \rho} > 0$. Therefore $G(d, \rho)$ is an increasing function of ρ and hence it cannot have the maximum value in the interior of the close region $[0, 2] \times [0, 1]$. Hence, for fixed $d \in [0, 2]$, we have

$$maxG(d, \rho) = G(d, 1) = H(d)(say),$$

where

$$\begin{aligned}H(d) &= (e_1-e_2)d^4+\Big[2e_1d+2(e_1-e_2)d^2+(d-2)\{(e_1-e_2)d-2e_2\}\Big](4-d^2)\\ &= (e_1-e_2)d^4+8e_1d-2e_1d^3+8(e_1-e_2)d^2-2(e_1-e_2)d^4+(d-2)\\ &\qquad \big\{(e_1-e_2)d-2e_2\big\}(4-d^2)\\ &= (e_1-e_2)d^4+8e_1d-2e_1d^3+8e_1d^2-8e_2d^2-2(e_1-e_2)d^4\\ &\qquad +\big[(e_1-e_2)(d^2-2d)-2e_2(d-2)\big](4-d^2)\\ &= -2(e_1-e_2)d^4-12e_2d^2+8e_1d-2e_1d^3+8e_1d^2+4e_1d^2\\ &\qquad -4e_2d^2-8e_1d+8e_2d-8e_2d+16e_2+2e_1d^3-2e_2d^3+2e_2d^3\\ &= -2(e_1-e_2)d^4-16e_2d^2+12e_1d^2+16e_2\\ &= -2(e_1-e_2)d^4+(12e_1-16e_2)d^2+16e_2. \end{aligned} \tag{3.11}$$

Now,

$$H'(d) = -8(e_1-e_2)d^3+2(12e_1-16e_2)d. \tag{3.12}$$

From (3.12), we observe that $H'(d)\leq 0$ for every $d\in[0,2]$. Therefore, $H(d)$ is a decreasing function of d in the interval $d\in[0,2]$ whose maximum value occurs at $d=0$. From (8)

$$max_{0\leq d\leq 2}H(d) = H(0) = 16e_2. \tag{3.13}$$

From (3.9) and(3.13) we have

$$|e_1d_1d_3-e_2d_2^2| \leq \frac{16e_2}{4} = 4e_2. \tag{3.14}$$

Using (3.14) in (3.4) we obtain

$$|a_2a_4-a_3^2| \leq \frac{(1-\alpha)^2 4e_2}{2^{3\beta}(3+\lambda)^{2\beta}} = \frac{4(1-\alpha)^2}{(3+\lambda)^{2\beta}}. \tag{3.15}$$

This completes the proof of Theorem 8. □

Remark 9. *For $\lambda=0$ and $\beta=1$, our result in Theorem 8 coincides with the result of Krishan and Ramreddy [48] (also, see [28]).*

Remark 10. *For $\lambda=0$, $\beta=1$ and $\alpha=0$ our result coincides to that of Janteng et. al.(see[19]).*

Remark 11. *Letting $\beta=0$ in Theorem 8 we get the result due to Mishra and Gochhayat [28].*

Theorem 12. *If the function $h(z)$ defined by (1.1) belongs to the class $R_{\lambda}^{\beta}(\alpha)$, then for $\left(0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}}\right)$, we have*

$$|a_2a_3 - a_4| \leq \frac{2(1-\alpha)}{3\sqrt{3}4^{\beta}(3+\lambda)^{\beta}} \frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}}. \tag{3.16}$$

Proof. Let the function $h(z)$ given by (1.1) be in the class $R_{\lambda}^{\beta}(\alpha)$. Proceeding as in Theorem 6 and putting the values of a_2, a_3 and a_4 in the functional $|a_2a_3 - a_4|$ we get

$$|a_2a_3 - a_4| = \frac{(1-\alpha)}{4^{\beta}(3+\lambda)^{\beta}}\Big|k_1d_1d_2 + k_2d_3\Big|, \tag{3.17}$$

where

$$k_1 = 2^{\beta}(1-\alpha), \quad k_2 = -(3+\lambda)^{\beta}. \tag{3.18}$$

Substituting the values of d_2 and d_3 from (2.2) and (2.3) of Lemma 5 on the right hand side of (3.17) we have

$$\begin{aligned}|k_1d_1d_2 + k_2d_3| &= \Bigg|\frac{k_1d_1}{2}\left\{d_1^2 + x(4-d_1^2)\right\}\\ &+ \frac{k_2}{4}\left\{d_1^3 + 2d_1x(4-d_1^2) - d_1x^2(4-d_1^2) + 2(4-d_1^2)(1-|x|^2)z\right\}\Bigg|\\ &= \Bigg|\frac{1}{2}\left(k_1d_1^3 + k_1d_1x(4-d_1^2)\right) + \frac{1}{4}\left(k_2d_1^3 + 2k_2d_1(4-d_1^2)x\right)\\ &\quad -k_2d_1(4-d_1^2)x + 2k_2(4-d_1^2)(1-|x|^2)z\Bigg|,\end{aligned}$$

which implies

$$\begin{aligned}4|k_1d_1d_2 + k_2d_3| = \Big|&2k_1d_1^3 + 2k_1d_1(4-d_1^2)x + k_2d_1^3 + 2k_2d_1(4-d_1^2)x\\ &-k_2d_1(4-d_1^2)x^2 + 2k_2(4-d_1^2)z - 2k_2(4-d_1^2)|x|^2z\Big|.\end{aligned}$$

Using the triangle inequality and the fact that $|z| \leq 1$ in the above equation, we have

$$\begin{aligned}4|k_1d_1d_2 + k_2d_3| &\leq (2k_1+k_2)d_1^3 + 2(k_1+k_2)d_1|x|(4-d_1^2) + 2k_2(4-d_1^2) + k_2(d_1+2)(4-d_1^2)|x|^2\\ &= \Big|\left[2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta}\right]d_1^3 + 2\left(2^{\beta}(1-\alpha) - (3+\lambda)^{\beta}\right)d_1|x|(4-d_1^2)\\ &- 2(3+\lambda)^{\beta}(4-d_1^2) + (3+\lambda)^{\beta}(d_1+2)(4-d_1^2)|x|^2\Big|.\end{aligned} \tag{3.19}$$

Since $|d_1| < 2$ by Lemma 4 we may assume without any restriction $d_1 = d \in [0, 2]$. Using the well-known result that $d_1 + a > d_1 - a$ for $a \geq 0$, replacing $|x|$ by ρ $(0 \leq$

$\rho \leq 1)$ and assuming $\alpha \leq \frac{[2^{\beta+1}-(3+\lambda)^{\beta}]}{2^{\beta+1}}$ on the right hand side of the above inequality (3.19) we get

$$4|k_1d_1d_2 + k_2d_3| \leq \big[2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta}\big]d^3 + 2\big[-2^{\beta}(1+\alpha) + (3+\lambda)^{\beta}\big]d(4-d^2)\rho + 2(3+\lambda)^{\beta}(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2)\rho^2 = K(d,\rho)(say), \tag{3.20}$$

where

$$K(d,\rho) = \big[2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta}\big]d^3 + 2\big[-2^{\beta}(1+\alpha) + (3+\lambda)^{\beta}\big]d(4-d^2)\rho + 2(3+\lambda)^{\beta}(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2)\rho^2. \tag{3.21}$$

Now, we have to maximize the function $K(d,\rho)$ over closed region $[0,2]\times[0,1]$. Differentiating K partially with respect to ρ we get

$$\frac{\partial K}{\partial \rho} = 2[(3+\lambda)^{\beta} - 2^{\beta}(1+\alpha)]d(4-d^2) + 2(3+\lambda)^{\beta}(d-2)(4-d^2)\rho. \tag{3.22}$$

For $0<\rho<1$, for fixed d with $0<d<2$ and for α with $0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}}$, we observe from (3.22) that $\frac{\partial K}{\partial \rho} \geq 0$ which implies the function $K(d,\rho)$ is an increasing function of ρ and hence it cannot have a maximum at any point in the interior of the closed region $[0,2]\times[0,1]$. Thus for fixed d, $0 \leq d \leq 2$, we have

$$max_{0\leq\rho\leq 1}K(d,\rho) = K(d,1) = L(d)\ (say)$$

where

$$\begin{aligned} L(d) &= \big[2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta}\big]d^3 + 2(3+\lambda)^{\beta}(4-d^2) \\ &\quad + 2\big[(3+\lambda)^{\beta} - 2^{\beta}(1+\alpha)\big]d(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2) \\ &= \big[2^{\beta+2} - 4(3+\lambda)^{\beta}\big]d^3 + 4\big[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)\big]d. \end{aligned} \tag{3.23}$$

A function $L(d)$ to be maximum or minimum on the interval $[0,2]$, we have

$$L'(d) = 3\big[2^{\beta+2} - 4(3+\lambda)^{\beta}\big]d^2 + 4\big[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)\big] = 0,$$

which implies

$$d = \sqrt{\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]}{3[(3+\lambda)^{\beta} - 2^{\beta}]}} \in [0,2], \quad \left(0 \leq \alpha \leq \frac{2^{\beta+1} - (3+\lambda)^{\beta}}{2^{\beta+1}}\right). \tag{3.24}$$

Also,

$$L''(d) = 6\big[2^{\beta+1} - 4(3+\lambda)^{\beta}\big]d. \tag{3.25}$$

Substitute the value of d from (3.24) in (3.25) we get

$$L''(d) = \frac{12}{\sqrt{3}}\left[2^{\beta+1} - 4(3+\lambda)^{\beta}\right]\sqrt{\frac{[2^{\beta+1}(1+\alpha) - 3(3+\lambda)^{\beta}]}{[2^{\beta+2} - 4(3+\lambda)^{\beta}]}} < 0,$$

for $0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}}$. Hence it follows from differential calculus, the function $L(d)$ is maximum at the point d given by (3.24). Putting the value of d from (3.24) in (3.23) and simplifying we get

$$\begin{aligned} L_{max} &= \left[2^{\beta+2} - 4(3+\lambda)^{\beta}\right]\left[\frac{[2^{\beta+1}(1+\alpha) - 3(3+\lambda)^{\beta}]}{3[2^{\beta} - (3+\lambda)^{\beta}]}\right]^{\frac{3}{2}} \\ &+ 4[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]\left[\frac{[2^{\beta+1}(1+\alpha) - 3(3+\lambda)^{\beta}]}{3[2^{\beta} - (3+\lambda)^{\beta}]}\right]^{\frac{1}{2}} \\ &= \frac{-4[-2^{\beta+1}(1+\alpha) + 3(3+\lambda)^{\beta}]^{\frac{3}{2}}}{3^{\frac{3}{2}}[-2^{\beta} + (3+\lambda)^{\beta}]^{\frac{1}{2}}} + \frac{4[-2^{\beta+1}(1+\alpha) + 3(3+\lambda)^{\beta}]^{\frac{3}{2}}}{\sqrt{3}[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}} \\ &= \frac{-4[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}} + 12[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{3\sqrt{3}[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}} \\ &= \frac{8}{3\sqrt{3}}\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{\sqrt{(3+\lambda)^{\beta} - 2^{\beta}}}. \end{aligned} \tag{3.26}$$

From (3.20) and (3.26) we have

$$|k_1d_1d_2 + k_2d_3| \leq= \frac{2}{3\sqrt{3}}\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{\sqrt{(3+\lambda)^{\beta} - 2^{\beta}}}. \tag{3.27}$$

The relations (3.17) and (3.27) give

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1-\alpha}{4^{\beta}(3+\lambda)^{\beta}}\frac{2}{3\sqrt{3}}\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}} \\ &= \frac{2(1-\alpha)}{3\sqrt{3}4^{\beta}(3+\lambda)^{\beta}}\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}}. \end{aligned}$$

This completes the proof of the Theorem 12. □

Remark 13. *Putting $\lambda = 0$, $\beta = 1$ in the above theorem we get the following results due to Vamshree Krishna et al.[48].*

Corollary 14. *(see[48]) Let $f \in RT(\alpha)$ $(0 \leq \alpha \leq \frac{1}{4})$. Then*

$$|a_2a_3 - a_4| \leq \left(\frac{1-\alpha}{6}\right)\left(\frac{5-4\alpha}{3}\right)^{\frac{3}{2}}.$$

Putting $\alpha = 0$ in Corollary 14 we get $|a_2a_3 - a_4| \leq \frac{5}{18}\sqrt{\frac{5}{3}}$. This result is coincide with Babalola (see[1]).

Theorem 15. *Let $h(z) \in R_{\lambda}^{\beta}(\alpha)$ $\left(0 \leq \alpha \leq \frac{2(3+\lambda)^{\beta}-2^{2\beta}}{2(3+\lambda)^{\beta}}\right)$. Then*

$$|a_3 - a_2^2| \leq \frac{2(1-\alpha)}{(3+\lambda)^{\beta}}.$$

Proof. Substituting the values of a_2 and a_3 from (3.3) in coefficient functional $|a_3 - a_2^2|$ we obtain

$$\begin{aligned} |a_3 - a_2^2| &= \left|\frac{(1-\alpha)d_2}{(3+\lambda)^{\beta}} - \frac{(1-\alpha)^2 d_1^2}{2^{2\beta}}\right| \\ &= \frac{(1-\alpha)}{(3+\lambda)^{\beta}2^{2\beta}}\left|2^{2\beta}d_2 - (3+\lambda)^{\beta}(1-\alpha)d_1^2\right| \\ &= \frac{1-\alpha}{(3+\lambda)^{\beta}2^{2\beta}}\left|l_1 d_2 + l_2 d_1^2\right|, \end{aligned} \tag{3.28}$$

where

$$l_1 = 2^{2\beta}, \quad l_2 = -(3+\lambda)^{\beta}(1-\alpha).$$

Putting the value of d_2 from (2.2) of Lemma 5 in the right hand side of (3.28) we obtain

$$|l_1 d_2 + l_2 d_1^2| = \left|\frac{l_1}{2}\left\{d_1^2 + x(4-d_1^2)\right\} + l_2 d_1^2\right| = \left|\frac{l_1 d_1^2 + l_1 x(4-d_1^2) + 2l_2 d_1^2}{2}\right|,$$

which implies

$$\begin{aligned} 2|l_1 d_2 + l_2 d_1^2| &= \left|(l_1 + 2l_2)d_1^2 + l_1 x(4-d_1^2)\right| \\ &= \left|[2^{2\beta} - 2(3+\lambda)^{\beta}(1-\alpha)]d_1^2 + 2^{2\beta}x(4-d_1^2)\right|. \end{aligned} \tag{3.29}$$

Choosing $d_1 = d \in [0,2]$, applying triangle inequality, replacing $|x|$ by ρ on the right hand side of (3.29) and assume that $\alpha \leq \frac{2(3+\lambda)^{\beta}-2^{2\beta}}{2(3+\lambda)^{\beta}}$, we have

$$\begin{aligned} 2|l_1 d_2 + l_2 d_1^2| &\leq \left[2(3+\lambda)^{\beta}(1-\alpha) - 2^{2\beta}\right]d^2 + 2^{2\beta}(4-d^2)\rho \\ &= M(d,\rho)(say) \quad (0 \leq \rho = |x| \leq 1), \end{aligned} \tag{3.30}$$

where

$$M(d,\rho) = \left[2(3+\lambda)^{\beta}(1-\alpha) - 2^{2\beta}\right]d^2 + 2^{2\beta}(4-d^2)\rho.$$

In order to determine the maximum value of the function $M(d,\rho)$ differentiating $M(d,\rho)$, partially with respect to ρ, we get

$$\frac{\partial M}{\partial \rho} = 2^{2\beta}(4-d^2) > 0 \quad for \quad d \in [0,2]$$

For $0 < d < 2$, $\frac{\partial M}{\partial \rho} > 0$. Hence the function $M(d, \rho)$ is an increasing function of ρ. The maximum value of M occurs at $\rho = 1$ and given by

$$max_{0 \leq \rho \leq 1} M(d, \rho) = M(d, 1) = N(d)$$

where

$$\begin{aligned} N(d) &= \left[2(3+\lambda)^{\beta}(1-\alpha) - 2^{2\beta}\right]d^2 + 2^{2\beta}(4-d^2) \\ &= \left[2(3+\lambda)^{\beta}(1-\alpha) - 2^{2\beta+1}\right]d^2 + 2^{2\beta+2}. \end{aligned} \tag{3.31}$$

Now

$$N'(d) = 4\left[(3+\lambda)^{\beta}(1-\alpha) - 2^{2\beta}\right]d \leq 0, \quad \forall \quad d \in [0,2], \quad \alpha \in \left[0, \frac{2(3+\lambda)^{\beta} - 2^{2\beta}}{2(3+\lambda)^{\beta}}\right].$$

Therefore $N(d)$ becomes a decreasing function of d whose maximum value occur at $d = 0$. From (3.31) we get

$$max_{0 \leq d \leq 2} N(d) = N(0) = 2^{2\beta+2}. \tag{3.32}$$

It follows from (3.30) and (3.32) that

$$|l_1 d_2 + l_2 d_1^2| \leq 2^{2\beta+1}. \tag{3.33}$$

Using (3.33) in (3.28) gives

$$|a_3 - a_2^2| \leq \frac{1-\alpha}{(3+\lambda)^{\beta} 2^{2\beta}} 2^{2\beta+1} = \frac{2(1-\alpha)}{(3+\lambda)^{\beta}}$$

The proof of Theorem 15 is thus completed. □

Remark 16. *Taking $\lambda = 0$, $\beta = 1$ in Theorem 15 we get the estimate of $|a_3 - a_2^2| \leq \frac{2}{3}(1-\alpha)$, studied by Vamshree Krishna et. al [48].*

Remark 17. *Letting $\lambda = 0$, $\beta = 1$ and $\alpha = 0$ we get the results $|a_3 - a_2^2| \leq \frac{2}{3}$ due to Babalola [1]*

Theorem 18. *Let $f \in R_{\lambda}^{\beta}(\alpha)$. Then*

$$|h_3(1)| \leq \frac{4(1-\alpha)^2}{(3+\lambda)^{\beta}}\left[\frac{2(1-\alpha)}{(3+\lambda)^{2\beta}} + \frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{3\sqrt{3}4^{2\beta}[(3+\lambda)^{\beta} - 2^{\beta}]^{\frac{1}{2}}} + \frac{1}{(5+\lambda)^{\beta}}\right]$$

Proof. Using the results from Theorem 6, 8, 12 and 15 in (1.3) we obtain

$$H_3(1) \leq \frac{2(1-\alpha)}{(3+\lambda)^{\beta}}\frac{4(1-\alpha)^2}{(3+\lambda)^{2\beta}} + \frac{2(1-\alpha)}{4^{\beta}}\frac{2(1-\alpha)[3(3+\lambda)^{\beta}-2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{3\sqrt{3}4^{\beta}(3+\lambda)^{\beta}[(3+\lambda)^{\beta}-2^{\beta}]^{\frac{1}{2}}} + \frac{2(1-\alpha)}{(5+\lambda)^{\beta}}\frac{2(1-\alpha)}{(3+\lambda)^{\beta}}$$

$$= \frac{8(1-\alpha)^3}{(3+\lambda)^{3\beta}} + \frac{4(1-\alpha)^2[3(3+\lambda)^{\beta}-2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{4^{2\beta}3\sqrt{3}(3+\lambda)^{\beta}[(3+\lambda)^{\beta}-2^{\beta}]^{\frac{1}{2}}} + \frac{4(1-\alpha)^2}{(3+\lambda)^{\beta}(5+\lambda)^{\beta}}$$

$$= \frac{4(1-\alpha)^2}{(3+\lambda)^{\beta}}\left[\frac{2(1-\alpha)}{(3+\lambda)^{2\beta}} + \frac{[3(3+\lambda)^{\beta}-2^{\beta+1}(1+\alpha)]^{\frac{3}{2}}}{3\sqrt{3}4^{2\beta}[(3+\lambda)^{\beta}-2^{\beta}]^{\frac{1}{2}}} + \frac{1}{(5+\lambda)^{\beta}}\right].$$

□

This complete the proof of Theorem 18.

Remark 19. *Putting $\lambda = 0$, $\beta = 1$ in above theorem we get the results due to Vamshree Krishan (see[48]).*

Remark 20. *Putting $\lambda = 0$, $\beta = 1$, $\alpha = 0$ in Theorem 18 we get the results of Babalola (see[1]).*

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