Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 15 (2020), 459 – 470

ON SOME SPECIAL CURVES IN LORENTZ-MINKOWSKI PLANE

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Abstract. Here we consider the plane curves whose curvature κ depends on the distance from the origin in Lorentz-Minkowski plane $\mathbb{E}_1^2$. We obtain their explicit parameterizations in the cases when the plane curves have curvatures which are linear or quadratic functions of the distance of their points from the origin in $\mathbb{E}_1^2$ up to a real positive multiplier $\sigma \in \mathbb{R}^+$.
We have derived also the algebraic equations which are uniformized by these parameterziations.

1 Introduction

The fundamental existence and uniqueness theorem in the theory of plane curves states that such curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length [1]. However, the simplicity of the situation is quite elusive because in many cases it is impossible to find the explicit parameterization.
Singer [13] has considered a different sort of the problem. He asked the question "whether it is possible to determine the plane curve whose curvature is given in terms of its position". In the same paper, he has solved this problem by embedding it in some dynamical system (which still has to be resolved!).
Alternatively, by making use of the notion of geometric angular momentum (with respect to the origin) and geometric linear momentum (with respect to the fixed lightlike geodesic), Castro *et all* [2, 3] have developed two effective integrability methods to determine such curves through quadratures. In [4] and [10] have been studied some concrete plane curves whose curvature depends on the distance from the origin in the Euclidean plane like Bernoulli's lemniscate and co-lemniscate, and Sturmian spirals [11].
The present paper should be considered as a part of a longstanding Programme of studies of non-Euclidean analogues of the most interesting classical curves like elasticas, lemniscates, spirals, etc. [5, 14]. Concretely, in this paper we consider

2020 Mathematics Subject Classification: 53C40; 53C50
Keywords: Curvature, plane curves, Lorentz-Minkowski plane

the plane curves whose curvature κ depends on the distance from the origin in the Lorentz-Minkowski plane $\mathbb{E}_1^2$. Here we derive the parametric equations of the plane curves with $\kappa(s) = \sigma r(s)$ and $\kappa(s) = \sigma r^2(s)$ where $r(s)$ is the distance from the origin in $\mathbb{E}_1^2$ and $\sigma \in \mathbb{R}^+$.

2 Preliminaries

The Lorentz-Minkowski plane $\mathbb{E}_1^2$ is the Euclidean plane $\mathbb{E}^2$ equipped with indefinite flat metric given by

$$g = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2$$

where (x_1, x_2) is a rectangular coordinate system of $\mathbb{E}_1^2$. Recall that a vector $v \in \mathbb{E}_1^2 \backslash \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$. The null curves in the Lorentz plane are lines, which curvature is identically zero. This means also, that the null curves can not be considered as curves whose curvatures are given in terms of their positions. Our interest here is in the other two classes, i.e., the spacelike and the timelike curves.
The norm of any vector v in them is given by the formula $||v|| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^2$, can locally be spacelike or timelike, if all of its velocity vectors $\dot{\alpha}(s)$ are spacelike, respectively timelike. A spacelike or timelike curve α is parameterized by the arc-length parameter s if $g(\dot{\alpha}(s), \dot{\alpha}(s)) = \pm 1$ ([12]). A spacelike curve in Lorentz-Minkowski plane can be parametrized [6, 7, 8] as

$$\mathbf{x}(s) = \left(\int_0^s \sinh \varphi(s) \mathrm{d}s, \int_0^s \cosh \varphi(s) \mathrm{d}s \right) \tag{2.1}$$

with the Frenet vector fields and curvature function given below

$$\begin{aligned} \mathbf{T}(s) &= (\sinh \varphi(s), \cosh \varphi(s)) \\ \mathbf{N}(s) &= (\cosh \varphi(s), \sinh \varphi(s)) \\ \kappa(s) &= -g\left(\dot{\mathbf{T}}(s), \mathbf{N}(s)\right) = \dot{\varphi}(s). \end{aligned} \tag{2.2}$$

The Frenet vectors fields satisfy the following relations

$$\dot{\mathbf{x}} = \mathbf{T}, \qquad \dot{\mathbf{T}} = \kappa \mathbf{N}, \qquad \dot{\mathbf{N}} = \kappa \mathbf{T}. \tag{2.3}$$

Respectively, a timelike curve in Lorentz-Minkowski plane can be parameterized [6, 7, 8] as

$$\mathbf{x}(s) = \left(\int_0^s \cosh \varphi(s) \mathrm{d}s, \int_0^s \sinh \varphi(s) \mathrm{d}s \right) \tag{2.4}$$

with the Frenet vector fields and the curvature function of the form

$$\begin{aligned}\mathbf{T}(s) &= (\cosh\varphi(s), \sinh\varphi(s))\\ \mathbf{N}(s) &= (\sinh\varphi(s), \cosh\varphi(s))\\ \kappa(s) &= g\left(\dot{\mathbf{T}}(s), \mathbf{N}(s)\right) = \dot{\varphi}(s)\end{aligned} \tag{2.5}$$

which obey to the following relations

$$\dot{\mathbf{x}} = \mathbf{T}, \qquad \dot{\mathbf{T}} = \kappa\mathbf{N}, \qquad \dot{\mathbf{N}} = \kappa\mathbf{T}. \tag{2.6}$$

3 Spacelike Curves in Lorentz-Minkowski Plane

In this and next section, we derive the general parameterizations of spacelike (and respectively timelike) curves whose curvature functions depend on the distance from origin in Lorentz-Minkowski plane and then apply them to the specified above cases.

Let $\mathbf{x}(s) = (x(s), z(s))$ be a spacelike curve in Lorentz-Minkowski plane. Then $\mathbf{x}(s)$ can be written as follows

$$\mathbf{x}(s) = \xi(s)\mathbf{T}(s) + \eta(s)\mathbf{N}(s). \tag{3.1}$$

Differentiating (3.1) with respect to s and using (2.3), we get

$$\mathbf{T} = (\dot{\xi} + \kappa\eta)\mathbf{T} + (\dot{\eta} + \kappa\xi)\,\mathbf{N}. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\dot{\xi} = 1 - \kappa\eta, \qquad \dot{\eta} = -\kappa\xi, \qquad \kappa = \sigma r = \sigma\sqrt{\varepsilon\left(\xi^2 - \eta^2\right)} \tag{3.3}$$

where $\varepsilon = \pm 1$ such that $\varepsilon\left(\xi^2 - \eta^2\right) \geq 0$. Multiplying the first equation by ξ, the second by η and subtracting the so obtained expressions, we find

$$\xi = \varepsilon r\dot{r}. \tag{3.4}$$

Substituting (3.4) back in (3.3) and integrating we end up with the formula

$$\eta = -\varepsilon\int r\kappa(r)\,\mathrm{d}r. \tag{3.5}$$

3.1 Spacelike curves with $\kappa(s) = \sigma r(s)$

Let us consider the spacelike curves whose curvature can be written as $\kappa(s) = \sigma r(s)$, where $r(s)$ is a distance function from the origin and $\sigma \in \mathbb{R}^+$.

Since $\kappa = \sigma r$, from (3.4) and (3.5), we get

$$\eta = -\frac{\varepsilon\sigma r^3}{3}, \qquad \xi = r\sqrt{\varepsilon + \frac{\sigma^2 r^4}{9}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \varepsilon\sqrt{\varepsilon + \frac{\sigma^2 r^4}{9}}\cdot \tag{3.6}$$

i) Let $\varepsilon = 1$. From the general expressions (3.6) we have

$$\eta = -\frac{\sigma r^3}{3}, \qquad \xi = r\sqrt{1 + \frac{\sigma^2 r^4}{9}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 + \frac{\sigma^2 r^4}{9}} \tag{3.7}$$

while the last equation of (3.7) means

$$\mathrm{d}s = \frac{\mathrm{d}r}{\sqrt{1 + \frac{\sigma^2 r^4}{9}}} \tag{3.8}$$

so that finally we have

$$\varphi = \int \kappa(s)\mathrm{d}s = \int \frac{\kappa(r)\mathrm{d}r}{\sqrt{1 + \frac{\sigma^2 r^4}{9}}} = \sigma\int \frac{r\mathrm{d}r}{\sqrt{1 + \frac{\sigma^2 r^4}{9}}}\cdot \tag{3.9}$$

The integration gives us

$$\varphi = \frac{3}{2}\,\mathrm{arcsinh}\,\frac{\sigma}{3}r^2 \quad \Longrightarrow \quad r = \sqrt{\frac{3}{\sigma}\sinh\frac{2\varphi}{3}}, \qquad \text{where} \quad \varphi \geq 0 \tag{3.10}$$

and therefore

$$\xi = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{2\varphi}{3}, \qquad \eta = -\left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{2\varphi}{3}\cdot$$

By rewriting (3.1) in its components, we obtain the relations

$$x = \xi\sinh\varphi + \eta\cosh\varphi, \qquad z = \xi\cosh\varphi + \eta\sinh\varphi \tag{3.11}$$

which combined with the above findings provides the sought parametrization of the curves in the form

$$x = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{\varphi}{3}, \qquad z = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{\varphi}{3}\cdot \tag{3.12}$$

This parameterization suggests also that we are dealing with the algebraic curves, and indeed - a little manipulation with hyperbolic functions makes evident the fact that the curves in (3.12) actually uniformize the algebraic curves (see also the Remark at the end of Section 5.1 in [3])

$$\boxed{\sigma\left(z^2 - x^2\right)^2 - 6xz = 0.} \tag{3.13}$$

Just for a completeness we will mention here that the Bernoullian lemniscates (which share the same curvature function in the Euclidean plane) uniformize the quartics in the form (for more details cf. [9, page 62])

$$\boxed{\left(x^2+z^2\right)^2-\sigma\left(x^2-z^2\right)=0.} \tag{3.14}$$

ii) Let $\varepsilon=-1$. From (3.6), we get

$$\eta=\frac{\sigma r^3}{3}, \qquad \xi=r\sqrt{\frac{\sigma^2 r^4}{9}-1}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s}=-\sqrt{\frac{\sigma^2 r^4}{9}-1} \tag{3.15}$$

while from the last equation in (3.15) we have

$$\mathrm{d}s=-\frac{\mathrm{d}r}{\sqrt{\frac{\sigma^2 r^4}{9}-1}} \tag{3.16}$$

and finally

$$\varphi=\int\kappa(s)\mathrm{d}s=-\int\frac{\kappa(r)\mathrm{d}r}{\sqrt{\frac{\sigma^2 r^4}{9}-1}}=-\sigma\int\frac{r\mathrm{d}r}{\sqrt{\frac{\sigma^2 r^4}{9}-1}}\cdot \tag{3.17}$$

This time as a result of the integration we have

$$\varphi=-\frac{3}{2}\operatorname{arccosh}\frac{\sigma r^2}{3} \quad\Longrightarrow\quad r=\sqrt{\frac{3}{\sigma}\cosh\frac{2\varphi}{3}}\cdot \tag{3.18}$$

The last relation and the first two formulas in (3.15) produce respectively

$$\xi=\left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{2\varphi}{3}, \qquad \eta=\left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{2\varphi}{3}$$

which combined with (3.11) allows us to express the curve in explicit form

$$x=\left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{5\varphi}{3}, \qquad z=\left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{5\varphi}{3}\cdot \tag{3.19}$$

Curves of the above type uniformize the polynomial equation

$$\boxed{r^2\left(\sigma r^2+3\right)\left(4\sigma^2 r^4-6\sigma r^2-9\right)^2-486x^2=0} \tag{3.20}$$

in which $r^2=x^2-z^2$.

3.2 Spacelike curves with $\kappa(s) = \sigma r^2(s)$

From (3.4) and (3.5) for the spacelike curves with curvature $\kappa = \sigma r^2$ we have

$$\eta = -\frac{\varepsilon \sigma r^4}{4}, \qquad \xi = r\sqrt{\varepsilon + \frac{\sigma^2 r^6}{16}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \varepsilon\sqrt{\varepsilon + \frac{\sigma^2 r^6}{16}}\cdot \tag{3.21}$$

i) Let $\varepsilon = 1$. Then from (3.21), we get

$$\eta = -\frac{\sigma r^4}{4}, \qquad \xi = r\sqrt{1 + \frac{\sigma^2 r^6}{16}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 + \frac{\sigma^2 r^6}{16}}\cdot \tag{3.22}$$

Then similarly, we find

$$x = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{\varphi}{4}, \qquad z = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{\varphi}{4}\cdot \tag{3.23}$$

and

$$\boxed{\sigma\left(x^2 - z^2\right)^3 + 4x\left(x^2 + 3z^2\right) = 0.} \tag{3.24}$$

ii) Let $\varepsilon = -1$. from (3.21), we get

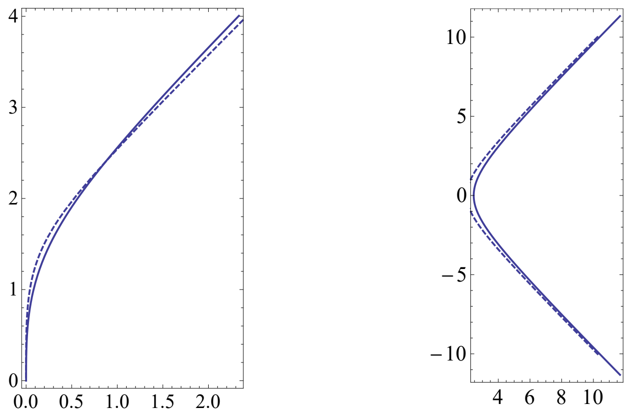

Figure 1: The left figure is obtained by (3.12) and (3.23) and the right one by (3.19) and (3.26). In all cases $\sigma = 0.5$. The dashed lines represent the curves with $\kappa(s) = \sigma r^2(s)$.

$$\eta = \frac{\sigma r^4}{4}, \qquad \xi = r\sqrt{\frac{\sigma^2 r^6}{16} - 1}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = -\sqrt{\frac{\sigma^2 r^6}{16} - 1}. \tag{3.25}$$

Then, using the same methods as above we find

$$x = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{7\varphi}{4}, \qquad z = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{7\varphi}{4}\cdot \tag{3.26}$$

4 Timelike Curves in Lorentz-Minkowski Plane

Let $\mathbf{x}(s) = (x(s), z(s))$ be a timelike curve in Lorentz-Minkowski plane. Then $\mathbf{x}(s)$ can be written as follows

$$\mathbf{x}(s) = \xi(s)\mathbf{T}(s) + \eta(s)\mathbf{N}(s). \tag{4.1}$$

Differentiating (4.1) with respect to s and using (2.6), we get

$$\mathbf{T} = (\dot{\xi} + \kappa\eta)\mathbf{T} + (\dot{\eta} + \kappa\xi)\,\mathbf{N}. \tag{4.2}$$

From (4.1) and (4.2), we have

$$\dot{\xi} = 1 - \kappa\eta, \qquad \dot{\eta} = -\kappa\xi, \qquad \kappa = \sigma r = \sigma\sqrt{\varepsilon\left(\eta^2 - \xi^2\right)} \tag{4.3}$$

where $\varepsilon = \pm 1$ such that $\varepsilon\left(\xi^2 - \eta^2\right) \geq 0$. Multiplying the second equation above by η, the first by ξ, and subtracting the so obtained expressions, we find

$$\xi = -\varepsilon r\dot{r}. \tag{4.4}$$

Substituting (4.4) in (4.3) and then integrating the obtained expression, we get

$$\eta = \varepsilon\int r\kappa(r)\,\mathrm{d}r. \tag{4.5}$$

Rewriting (4.1) in components one ends with the useful formulas

$$x = \xi\sinh\varphi + \eta\cosh\varphi, \qquad z = \xi\cosh\varphi + \eta\sinh\varphi. \tag{4.6}$$

4.1 Timelike curves with $\kappa(s) = \sigma r(s)$

In this subsection we consider the timelike curves whose curvature can be written as $\kappa(s) = \sigma r(s)$, where $r(s)$ is a distance function from the origin and $\sigma \in \mathbb{R}^+$.

Since $\kappa = \sigma r$, from (4.4) and (4.5), we get

$$\eta = \frac{\varepsilon\sigma r^3}{3}, \qquad \xi = r\sqrt{\frac{\sigma^2 r^4}{9} - \varepsilon}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = -\varepsilon\sqrt{\frac{\sigma^2 r^4}{9} - \varepsilon}. \tag{4.7}$$

$i)$ Let $\varepsilon = 1$. Then from (4.7), we get

$$\eta = \frac{\sigma r^3}{3}, \qquad \xi = r\sqrt{\frac{\sigma^2 r^4}{9} - 1}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = -\sqrt{\frac{\sigma^2 r^4}{9} - 1}.$$

Following similar methods to that ones in the spacelike case, we obtain

$$x = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{5\varphi}{3}, \qquad z = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{5\varphi}{3}. \tag{4.8}$$

These curves uniformize the 16-th degree polynomial equation

$$\boxed{r^2\left(\sigma r^2+3\right)\left(4\sigma^2 r^4-6\sigma r^2-9\right)^2-486z^2=0} \tag{4.9}$$

in which $r^2=z^2-x^2$.

ii) Let $\varepsilon=-1$. Then from (4.7), we get

$$\eta=-\frac{\sigma r^3}{3},\qquad \xi=r\sqrt{\frac{\sigma^2 r^4}{9}+1},\qquad \frac{\mathrm{d}r}{\mathrm{d}s}=\sqrt{\frac{\sigma^2 r^4}{9}+1}.$$

The straightforward application of the scheme produces

$$x=\left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{\varphi}{3},\qquad z=\left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{\varphi}{3} \tag{4.10}$$

but one should keep in mind that the parameter φ should take only positive values, i.e., $\varphi\geq 0$.
The algebraic curves which are uniformized by (4.10) are exactly those presented in (3.13) as can be checked directly.

4.2 Timelike curves with $\kappa(s)=\sigma r^2(s)$

Let us consider finally the timelike curves whose curvature can be written as $\kappa(s)=\sigma r^2(s)$, where $r(s)$ is a distance function from the origin and $\sigma\in\mathbb{R}^+$.

Since $\kappa=\sigma r^2$, (4.4) and (4.5) tell us that we actually have

$$\eta=\frac{\varepsilon\sigma r^4}{4},\qquad \xi=r\sqrt{\frac{\sigma^2 r^6}{16}-\varepsilon},\qquad \frac{\mathrm{d}r}{\mathrm{d}s}=-\varepsilon\sqrt{\frac{\sigma^2 r^6}{16}-\varepsilon}. \tag{4.11}$$

i) Let $\varepsilon=1$. Then from (4.11) we can write

$$\eta=\frac{\sigma r^4}{4},\qquad \xi=r\sqrt{\frac{\sigma^2 r^6}{16}-1},\qquad \frac{\mathrm{d}r}{\mathrm{d}s}=-\sqrt{\frac{\sigma^2 r^6}{16}-1} \tag{4.12}$$

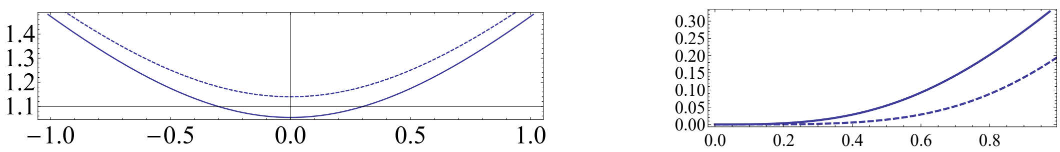

Figure 2: The left figure is obtained by (4.8) and (4.13). The right one by (4.10) and (4.14). Dashed lines represent the curves with $\kappa(s)=\sigma r^2(s)$. In all cases $\sigma\equiv 2.7$.

which leads (after some calculations) to the expressions

$$x=\left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{7\varphi}{4},\qquad z=\left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{7\varphi}{4}\cdot \tag{4.13}$$

ii) Let $\varepsilon=-1$. In this case from (4.11) we have

$$\eta=-\frac{\sigma r^4}{4},\qquad \xi=r\sqrt{\frac{\sigma^2r^6}{16}+1},\qquad \frac{\mathrm{d}r}{\mathrm{d}s}=\sqrt{\frac{\sigma^2r^6}{16}+1}.$$

Proceeding in a similar way as explained above we obtain finally the desired formula

$$x=\left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{\varphi}{4},\qquad z=\left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{\varphi}{4} \tag{4.14}$$

and respectively the algebraic curve

$$\boxed{\sigma\left(x^2-z^2\right)^3-4z\left(3x^2+z^2\right)=0.} \tag{4.15}$$

5 Concluding Remarks

In this study we have treated in detail two examples of curves whose curvatures can be expressed as functions of their position vectors. In fact, the present work is a continuation of the analogous studies of the Sturmian spirals [5] and elastic Sturmian spirals [14] in the Lorentz-Minkowski plane. We have obtained the parametric and the implicit equations of the curves for which $\kappa(s)=\sigma r(s)$ and $\kappa(s)=\sigma r^2(s)$ where $r(s)$ is the distance from the origin in $\mathbb{E}_1^2$ and $\sigma\in\mathbb{R}^+$.

Let us mention that the straightforward approach for the reconstruction of these curves is to integrate directly the equations of the type of the last equation in (3.7) and this produces

$$r(s)=a\,\frac{\lambda\mathrm{cn}(\frac{\lambda\sqrt{\sigma}}{\sqrt{6}}s,k)+\mathrm{sn}(\frac{\lambda\sqrt{\sigma}}{\sqrt{6}}s,k)}{\lambda\mathrm{cn}(\frac{\lambda\sqrt{\sigma}}{\sqrt{6}}s,k)-\mathrm{sn}(\frac{\lambda\sqrt{\sigma}}{\sqrt{6}}s,k)} \tag{5.1}$$

in which $\mathrm{sn}(\cdot,\cdot)$ and $\mathrm{cn}(\cdot,\cdot)$ are the Jacobian elliptic functions

$$a=\sqrt{\frac{3}{\sigma}},\qquad \lambda=\sqrt{3+2\sqrt{2}} \tag{5.2}$$

and $k=2\sqrt{3\sqrt{2}-4}$ is the so called elliptic module.

In principle, the next step will require the integration of the last equation in (2.2) which is possible but the result will be spelled out in terms of all three kinds of elliptic integrals which is a faraway from being transparent.

That is why we have taken another more geometrical route and the so obtained

Table 1: Parameterizations and implicit equations.

Spacelike $\varepsilon = 1$ and $\kappa = \sigma r$ $\sigma\left(z^2 - x^2\right)^2 - 6xz = 0$	$x = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{\varphi}{3}, \qquad \varphi \geq 0$ $z = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{\varphi}{3}$
Spacelike $\varepsilon = -1$ and $\kappa = \sigma r$ $r^2\left(\sigma r^2 + 3\right)\left(4\sigma^2 r^4 - 6\sigma r^2 - 9\right)^2 - 486x^2 = 0$	$x = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{5\varphi}{3}$ $z = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{5\varphi}{3}$
Spacelike $\varepsilon = 1$ and $\kappa = \sigma r^2$ $\sigma\left(x^2 - z^2\right)^3 + 4x\left(x^2 + 3z^2\right) = 0$	$x = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{\varphi}{4}$ $z = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{\varphi}{4}$
Spacelike $\varepsilon = -1$ and $\kappa = \sigma r^2$ *N/A*	$x = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{7\varphi}{4}$ $z = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{7\varphi}{4}$
Timelike $\varepsilon = 1$ and $\kappa = \sigma r$ $r^2\left(\sigma r^2 + 3\right)\left(4\sigma^2 r^4 - 6\sigma r^2 - 9\right)^2 - 486z^2 = 0$	$x = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{5\varphi}{3}$ $z = \left(\frac{3}{\sigma}\cosh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{5\varphi}{3}$
Timelike $\varepsilon = -1$ and $\kappa = \sigma r$ $\sigma\left(z^2 - x^2\right)^2 - 6xz = 0$	$x = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\cosh\frac{\varphi}{3}, \qquad \varphi \geq 0$ $z = \left(\frac{3}{\sigma}\sinh\frac{2\varphi}{3}\right)^{\frac{1}{2}}\sinh\frac{\varphi}{3}$
Timelike $\varepsilon = 1$ and $\kappa = \sigma r^2$ *N/A*	$x = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{7\varphi}{4}$ $z = \left(\frac{4}{\sigma}\cosh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{7\varphi}{4}$
Timelike $\varepsilon = -1$ and $\kappa = \sigma r^2$ $\sigma\left(x^2 - z^2\right)^3 - 4z\left(3x^2 + z^2\right) = 0$	$x = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\cosh\frac{\varphi}{4}$ $z = \left(\frac{4}{\sigma}\sinh\frac{3\varphi}{4}\right)^{\frac{1}{3}}\sinh\frac{\varphi}{4}$

results are summarized for clarity and future use in Table 1. In most of the cases considered in the paper we were able to find also the implicit equations which are uniformized by the explicit parameterizations and those for which they are available are listed in the table as well. The representative illustrations of curves are drawn using the derived parameterizations via the computer algebra system `Mathematica`®. Computer experiments show (cf. Fig. 1 and Fig. 2) that the geometry of the curves is more sensitive upon the numerical value of the parameter σ and not so much on the functions of the distance which enter in the expressions for the curvatures.

Acknowledgments

The authors are grateful to Professor Ildelfonso Castro (Universidad de Jaén, Spain) for providing copies of [2, 3] and the professional comments about the present paper.

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