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CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATION WITH NONLOCAL ERDÉLYI-KOBER TYPE INTEGRAL BOUNDARY CONDITIONS IN BANACH SPACES

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Abstract. In this paper, we study nonlocal boundary value problems of nonlinear Caputo fractional differential equations supplemented with Erdélyi-Kober type fractional integral boundary conditions. Existence results are obtained by applying the Mönch's fixed point theorem and the technique of measures of noncompactness. An example illustrating the main result is also constructed.

1 Introduction

Fractional calculus is an extension of the ordinary calculus, its main area of interest is the definition of real or complex number powers of the classical differentiation operator, so the fractional calculus refers to integration and differentiation to an arbitrary order. In the last few decades, fractional differential equations have been of great interest. It was caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For more information on the subject, the interested reader is referred to texts such as [18, 22, 26, 28, 25], some recent developments of fractional differential and integral equations are given in [1, 4, 11, 29].

In particular, many authors have been studying the existence of solutions of fractional differential problems under various boundary conditions, by different ways such as integral boundary conditions involving Riemann-Liouville or Hadamard type integral boundary conditions, fractional derivative boundary conditions, multipoint boundary conditions and nonlocal conditions, see [4, 11, 13, 16, 20, 21, 30].

Especially, the existence of a solution for abstract Cauchy differential equations with nonlocal conditions in a Banach space has been considered first by Byszewski

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[12]. In physical science, the nonlocal condition may be connected with better effect in applications than the classical initial condition since nonlocal conditions are normally more exact for physical estimations than the classical initial conditions. We refer to [1, 13, 14, 16, 20, 21, 30] and references given therein, for some examples of nonlocal problems.

In [29] authors established the existence of solutions for the following nonlinear Riemann-Liouville fractional differential equation subject to nonlocal Erdelyi-Kober fractional integral conditions:

$$D^q x(t) = f(t, x(t)), \qquad t \in (0, T).$$

$$x(0) = 0, \qquad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i),$$

where $1 < q \leq 2, D^q$, is the standard Riemann-Liouville fractional derivative of order q. $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}, i = 1, 2, ..., m$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha, \beta_i \in \mathbb{R}, \xi_i \in (0, T), i = 1, 2, ..., m$, are given constants.

In [4], authors studied a new class of boundary value problems of Caputo fractional differential equations:

$$^c D^q x(t) = f(t, x(t)), \qquad t \in [0, T].$$

supplemented with Riemann-Liouville and Erdelyi-Kober fractional integral boundary conditions

$$x(0) = \alpha \frac{1}{\Gamma(p)} \int_0^{\zeta} (\zeta - s)^{p-1} ds := \alpha J^p x(\zeta),$$

$$x(T) = \beta \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^{\xi} \frac{s^{\eta\gamma+\eta-1} x(s)}{(\xi^\eta - s^\eta)^{1-\delta}} ds := \beta I_\eta^{\gamma,\delta} x(\xi), 0 < \xi, \zeta < T,$$

where $^c D^q$ is the Caputo fractional derivative of order $1 < q \leq 2$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, J^p denote Riemann-Liouville fractional integral of order $p > 0$ and $I_\eta^{\gamma,\delta}$ denote Erdélyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, $\gamma \in \mathbb{R}$.

Motivated by the studies above among others, in this paper, we concentrate on the following boundary value problem of nonlinear Caputo fractional differential equation

$$D^\alpha x(t) = f(t, x(t)), \quad t \in J := [0, T], \tag{1.1}$$

associated with the following Erdélyi-Kober fractional integral boundary conditions:

$$\begin{aligned} x(T) &= \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} x(\beta_i), \quad 0 < \beta_i < T, \\ x'(T) &= \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} x'(\sigma_i), \quad 0 < \sigma_i < T, \\ x''(T) &= \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} x''(\varepsilon_i), \quad 0 < \varepsilon_i < T, \end{aligned} \tag{1.2}$$

where D^α is the Caputo fractional derivative of order $2 < \alpha \leq 3$ and $I_{\eta_i}^{\gamma_i,\delta_i}$ denote Erdélyi-Kober fractional integral of order $\delta_i > 0$, $\eta_i > 0$, $\gamma_i \in \mathbb{R}$. Let $E = C(J,\mathbb{R})$ be the Banach space of all continuous functions from $x : J \to \mathbb{R}$ with $\|x\|_\infty = \sup\{\|x(t)\| : \ t \in J\}$. Suppose that $f : J \times E \to E$ is a continuous function, $a_i, b_i, d_i, i = 1, 2, ..., m$ are real constants. Recall that Erdélyi-Kober fractional integral operators play an important role especially in engineering, for more details on the Erdélyi-Kober fractional integrals, see [4, 29].

In the present paper, we initiate the study of boundary value problems like (1.1)-(1.2), in which Caputo fractional differential equations are matched to Erdélyi-Kober fractional integral boundary conditions. We will present the existence results for the problem (1.1)-(1.2) which rely on Mönchs fixed point theorem combined with the technique of Kuratowski measure of noncompactness. that technique turns out to be a very useful tool in existence for several kinds of integral equations and subsequently developed and used in many papers, see, for instance. The strong measure of noncompactness was considered first by Banaŝ et al. [7, 8], for more details see, [3, 5, 6, 9, 10, 17, 23, 27].

The rest of the paper is organized as follows. In Section 2, we recall some notations, definitions, and lemmas that we need in the sequel. Section 3 treats the existence of solutions in Banach spaces. In Section 4, an example is treated.

2 Preliminaries

At firstly, we introduce some concepts on fractional calculus and some properties that will be used later. For more details, we refer to [2, 5, 7, 18, 28, 27].

Let $L^1(J,E)$ be the Banach space of measurable functions $x : J \to E$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(t)|\, dt.$$

Definition 1. *[18] Let $f \in C(J,\mathbb{R})$, then the Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, t > 0, \tag{2.1}$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definition 2. *[18] Let $n-1 < \alpha < n$, the Caputo derivative of order α of a function $f \in C^n(J,\mathbb{R})$, is given by*

$$\begin{aligned} {}^C D^\alpha f(t) &= I^{n-p} f^{(n)}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, t > 0. \end{aligned} \tag{2.2}$$

Definition 3. *The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} ds, \tag{2.3}$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Remark 4. *For $\eta = 1$ the above operator is reduced to the Kober operator*

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}} ds, \gamma, \delta > 0, \tag{2.4}$$

that was introduced for the first time by Kober in [19]. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} ds, \delta > 0, \tag{2.5}$$

Lemma 5. *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \tag{2.6}$$

Lemma 6. *[18] Let $q, r > 0$, and $n = [q] + 1$, then*

$$\begin{aligned} I^q t^{r-1}(t) &= \frac{\Gamma(r)}{\Gamma(q+r)} t^{r+q-1} \\ {}^c D^q t^{r-1}(t) &= \frac{\Gamma(r)}{\Gamma(r-q)} t^{r-q-1}, \end{aligned} \tag{2.7}$$

and

$${}^c D^q t^k = 0, k = 0, 1, ..., n-1. \tag{2.8}$$

Lemma 7. *[18] For $q > 0$ and $x \in C([0,T],\mathbb{R})$. Then the fractional differential equation*

$$^cD^q x(t) = 0$$

has a unique solution

$$x(t) = c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

then

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

where $n{-}1 \leq q < n$ and $c_i \in \mathbb{R}$, $i = 0, 1, ..., n{-}1$.

Lemma 8. *[18] Let $q, r \geq 0$, $f \in L^1([0,T],\mathbb{R})$. Then*

$$I^q I^r f(t) = I^{q+r} f(t) = I^q I^r f(t),$$
$$^cD^q I^r f(t) = I^{r-q} f(t), \quad r > q,$$

and

$$^cD^q I^q f(t) = f(t), \quad t \in [0,T].$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 9. *([5, 7]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty]$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\};\ \text{here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies some important properties [5, 7]:

(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\mu(B) = \mu(\overline{B})$.
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$
(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.
(f) $\mu(convB) = \mu(B)$.

Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in ([5, 7]).

Definition 10. *A map $f : J \times E \to E$ is said to be Caratheodory if*
(i) $t \mapsto f(t,u)$ is measurable for each $u \in E$;
(ii) $u \mapsto F(t,u)$ is continuous for almost all $t \in J$.

Notation 11. *for a given set V of functions $v : J \to E$, let us denote by*

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 12. *([23, 2, 27]) Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup 0 \Rightarrow \mu(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 13. *([27]) Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\mu(B); \text{ here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t) f(s, y(s)) ds : y \in V\right\}\right) \leq \int_J \|G(t,s)\| p(s) \mu(V(s)) ds.$$

3 Main results

For the existence of solutions for the problem (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 14. *Let $h : [1, T) \to E$ be a continuous function. Then,for any $x \in C([0, T], \mathbb{R})$, x is a solution of the following nonlinear fractional differential equation with Erdélyi-Kober fractional integral conditions:*

$$\begin{cases} D^{\alpha} x(t) = f(t, x(t)), t \in [0, T] \\ x(T) = \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i, \delta_i} x(\beta_i) \\ x'(T) = \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i, \delta_i} x'(\sigma_i) \\ x''(T) = \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i, \delta_i} x''(\varepsilon_i). \end{cases} \tag{3.1}$$

if and only if

$$x(t) = I^{\alpha} f(t, x(t)) + \frac{1}{v_0(a_i, \beta_i)} \left\{ \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha} f(\beta_i, x(\beta_i)) - I^{\alpha} f(T, x(T)) \right\}$$
$$+ \frac{1}{v_0(b_i, \sigma_i)} \left(t - \frac{v_1(a_i, \beta_i)}{v_0(a_i, \beta_i)} \right) \left\{ \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-1} f(\sigma_i, x(\sigma_i)) - I^{\alpha-1} f(T, x(T)) \right\}$$
$$+ \frac{1}{v_0(d_i, \varepsilon_i)} \left(\frac{v_2(a_i, \beta_i)}{2v_0(a_i, \beta_i)} \right) + \frac{v_1(a_i, \beta_i) v_1(b_i, \sigma_i)}{v_0(b_i, \sigma_i) v_0(a_i, \beta_i)} - \frac{v_1(b_i, \sigma_i) t}{v_0(b_i, \sigma_i)} + \frac{t^2}{2} \Bigg)$$
$$\times \left\{ \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-2} f(\varepsilon_i, x(\varepsilon_i)) - I^{\alpha-2} f(T, x(T)) \right\} \tag{3.2}$$

where

$$v_0(a_i, \beta_i) = \left(1 - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + 1)}{\Gamma(\gamma_i + \delta_i + 1)} \right) \tag{3.3}$$

$$v_1(a_i, \beta_i) = \left(T - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{1}{\eta_i} + 1)\beta_i}{\Gamma(\gamma_i + \frac{1}{\eta_i} + \delta_i + 1)} \right) \tag{3.4}$$

$$v_2(a_i, \beta_i) = \left(T^2 - \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{2}{\eta_i} + 1)\beta_i^2}{\Gamma(\gamma_i + \frac{2}{\eta_i} + \delta_i + 1)} \right) \tag{3.5}$$

Proof. Using Lemma (7), the general solution of the nonlinear fractional differential equation in (3.1) can be represented as

$$x(t) = c_0 + c_1 t + c_2 t^2 + I^{\alpha} h(t), c_0, c_1, c_2 \in \mathbb{R}. \tag{3.6}$$

By using the first integral condition of problem (3.1) and applying the Erdélyi-Kober integral on (3.6), we get

$$c_0 + c_1 T + c_2 T^2 + I^{\alpha} h(T) = \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha} h(\beta_i) + c_0 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + 1)}{\Gamma(\gamma_i + \delta_i + 1)}$$
$$+ c_1 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{1}{\eta_i} + 1)\beta_i}{\Gamma(\gamma_i + \frac{1}{\eta_i} + \delta_i + 1)} + c_2 \sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{2}{\eta_i} + 1)\beta_i^2}{\Gamma(\gamma_i + \frac{2}{\eta_i} + \delta_i + 1)}.$$

After collecting the similar terms in one part, we get the following equation:

$$c_0\left(1-\sum_{i=1}^{m} a_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)+c_1\left(T-\sum_{i=1}^{m} a_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\beta_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}\right)$$
$$+c_2\left(T^2-\sum_{i=1}^{m} a_i\frac{\Gamma(\gamma_i+\frac{2}{\eta_i}+1)\beta_i^2}{\Gamma(\gamma_i+\frac{2}{\eta_i}+\delta_i+1)}\right) \tag{3.7}$$
$$=\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}h(\beta_i)-I^{\alpha}h(T).$$

Rewriting equation (3.7) by using (3.3), (3.4), and (3.5), we obtain

$$c_0v_0(a_i,\beta_i)+c_1v_1(a_i,\beta_i)+c_2v_2(a_i,\beta_i)=\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}h(\beta_i)-I^{\alpha}h(T). \tag{3.8}$$

Then, taking the derivative of (3.6) and using the second integral condition of (3.1), one has

$$x'(T)=c_1+c_2T+I^{\alpha-1}h(T). \tag{3.9}$$

Now, applying the Erdélyi-Kober integral on (3.9), we have

$$c_1+2c_2T+I^{\alpha-1}h(T)=\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}h(\sigma_i)+c_1\sum_{i=1}^{m} b_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}$$
$$+2c_2\sum_{i=1}^{m} b_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\sigma_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}. \tag{3.10}$$

The above equation (3.10) implies that

$$c_1\left(1-\sum_{i=1}^{m} b_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)+2c_2\left(T-\sum_{i=1}^{m} b_i\frac{\Gamma(\gamma_i+\frac{1}{\eta_i}+1)\sigma_i}{\Gamma(\gamma_i+\frac{1}{\eta_i}+\delta_i+1)}\right)$$
$$=\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}h(\sigma_i)-I^{\alpha-1}h(T), \tag{3.11}$$

also, by using (3.3) and (3.4), equation (3.11) can be written as

$$c_1v_0(b_i,\sigma_i)+c_2v_1(b_i,\sigma_i)=\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}h(\sigma_i)-I^{\alpha-1}h(T). \tag{3.12}$$

By using the last integral condition of (3.1) and applying Erdélyi-Kober integral operator on the second derivative of (3.9), we have

$$2c_2+I^{\alpha-2}h(T)=\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}h(\varepsilon_i)+2c_2\sum_{i=1}^{m} d_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}. \tag{3.13}$$

Hence, we obtain the following equation:

$$2c_2\left(1-\sum_{i=1}^{m}d_i\frac{\Gamma(\gamma_i+1)}{\Gamma(\gamma_i+\delta_i+1)}\right)=\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\sigma_i)-I^{\alpha-2}h(T), \tag{3.14}$$

by using (3.3), equation (3.14) can be written as

$$2c_2v_0(d_i,\varepsilon_i)=\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\varepsilon_i)-I^{\alpha-2}h(T). \tag{3.15}$$

Moreover, equation (3.15) implies that

$$c_2=\frac{1}{2v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\varepsilon_i)-I^{\alpha-2}h(T),\right\} \tag{3.16}$$

substituting the values of (3.16) in (3.12), we obtain

$$\begin{aligned}c_1&=\frac{1}{v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^{m}b_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-1}h(\sigma_i)-I^{\alpha-1}h(T)\right\}\\&-\frac{v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\varepsilon_i)-I^{\alpha-2}h(T)\right\}.\end{aligned} \tag{3.17}$$

Now, substituting the values of (3.16) and (3.16) in (3.12), we have

$$\begin{aligned}c_0&=\frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m}a_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha}h(\beta_i)-I^{\alpha}h(T)\right\}\\&-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^{m}b_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-1}h(\sigma_i)-I^{\alpha-1}h(T)\right\}\\&+\frac{v_1(b_i,\sigma_i)v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)v_0(b_i,\sigma_i)v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\varepsilon_i)-I^{\alpha-2}h(T)\right\}\\&-\frac{v_2(a_i,\beta_i)}{2v_0(d_i,\varepsilon_i)v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m}d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}h(\varepsilon_i)-I^{\alpha-2}h(T)\right\}.\end{aligned} \tag{3.18}$$

Finally, substituting the values of (3.18), (3.17), and (3.16) in equation (3.6), we obtain the general solution of problem (3.1) which is (3.2). Converse is also true by using the direct computation. □

In the following,we prove existence results, for the boundary value problem (1.1)-(1.2) by using a Mönch fixed point theorem.
(H1) $f : J\times E\to E$ satisfies the Caratheodory conditions;
(H2) There exists $P\in L^1(J,\mathbb{R}^+)\cap C(J,\mathbb{R}^+)$, such that,

$$\|f(t,x)\| \leq P(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E;$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+} \mu(f(J_{t,h} \times B)) \leq P(t)\mu(B); \quad here \quad J_{t,h} = [t-h,t] \cap J.$$

Theorem 15. *Assume that conditions (H1)-(H3) hold. Let* $P^* = \sup_{t\in J} P(t)$. *If*

$$p^* M < 1 \tag{3.19}$$

With

$$\begin{aligned} M := &\Bigg\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i \frac{\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + 1)\beta_i^\alpha}{\Gamma(\alpha+1)\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + \delta_i + 1)} - \frac{T^\alpha}{\Gamma(\alpha+1)} \right\} \\ &+ \frac{1}{v_0(b_i,\sigma_i)} \left(T - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i \frac{\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + 1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)} \right\} \\ &+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2} \right) \\ &\times \left\{ \sum_{i=1}^m d_i \frac{\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + 1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \right\} \Bigg\}, \end{aligned}$$

then the BVP (1.1)-(1.2) *has at least one solution.*

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $\mathfrak{F} : C(J,E) \to C(J,E)$ defined by

$$\begin{aligned} \mathfrak{F}x(t) = &\, I^\alpha f(t,x(t)) + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i,\delta_i} I^\alpha f(\beta_i, x(\beta_i)) - I^\alpha f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(b_i,\sigma_i)} \left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1} f(\sigma_i, x(\sigma_i)) - I^{\alpha-1} f(T,x(T)) \right\} \\ &+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2} \right) \\ &\times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2} f(\varepsilon_i, x(\varepsilon_i)) - I^{\alpha-2} f(T,x(T)) \right\}. \end{aligned} \tag{3.20}$$

Clearly, the fixed points of the operator $\mathfrak{F}$ are solutions of the problem (1.1)-(1.2). We consider

$$D = \{x \in C(J,E) : \|x\| \leq R\}.$$

where R satisfies inequality (3.19), Clearly, the subset D is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps. □

Step 1: First we show that $\mathfrak{F}$ is continuous:
Let x_n be a sequence such that $x_n \to x$ in $C(J,E)$. Then for each $t \in J$,

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| \le\ & I^{\alpha}\|f(s,x_n(s)) - f(s,x(s))\|(t) + \frac{1}{v_0(a_i,\beta_i)} \\
& \times \left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}(1)(\beta_i) - I^{\alpha}(1)(T)\right\} \|f(s,x_n(s)) - f(s,x(s))\| \\
& + \frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right) \\
& \times \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}(1)(\sigma_i) - I^{\alpha-1}(1)(T)\right\} \|f(s,x_n(s)) - f(s,x(s))\| \\
& + \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right) \\
& \times \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}(1)(\varepsilon_i) - I^{\alpha-2}(1)(T)\right\} \|f(s,x_n(s)) - f(s,x(s))\| \\
\le\ & \left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)}\right. \\
& \times \left\{\sum_{i=1}^{m} a_i \frac{\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + 1)\beta_i^{\alpha}}{\Gamma(\alpha+1)\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right\} + \frac{1}{v_0(b_i,\sigma_i)} \\
& \times \left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i \frac{\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + 1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right\} \\
& + \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2}\right) \\
& \left.\times \left\{\sum_{i=1}^{m} d_i \frac{\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + 1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\right\}\right\} \\
& \times \|f(s,x_n(s)) - f(s,x(s))\|
\end{aligned}$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathfrak{F}(x_n) - \mathfrak{F}(x)\|_{\infty} \to 0 \text{ as } n \to \infty.$$

Step 2: Second we show that $\mathfrak{F}$ maps D into itself :

Take $x \in D$, by $(H2)$, we have, for each $t \in J$ and assume that $\mathfrak{F}x(t) \neq 0$.

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| &\leq I^{\alpha}\|f(s,x(s))\|(t) \\
&+ \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}\|f(s,x(s))\|(\beta_i) - I^{\alpha}\|f(s,x(s))\|(T)\right\} \\
&+ \frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right) \\
&\times \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}\|f(s,x(s))\|(\sigma_i) - I^{\alpha-1}\|f(s,x(s))\|(T)\right\} \\
&+ \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right) \\
&\times \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}\|f(s,x(s))\|(\varepsilon_i) - I^{\alpha-2}\|f(s,x(s))\|(T)\right\} \\
&\leq I^{\alpha}p(s)\|x(s)\|(t) \\
&+ \frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}p(s)\|x(s)\|(\beta_i) - I^{\alpha}p(s)\|x(s)\|(T)\right\} \\
&+ \frac{1}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right) \\
&\times \left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}p(s)\|x(s)\|(\sigma_i) - I^{\alpha-1}p(s)\|x(s)\|(T)\right\} \\
&+ \frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)t}{v_0(b_i,\sigma_i)} + \frac{t^2}{2}\right) \\
&\times \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}p(s)\|x(s)\|(\varepsilon_i) - I^{\alpha-2}p(s)\|x(s)\|(T)\right\} \\
&\leq p^*RI^{\alpha}(1)(T) + \frac{p^*R}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha}(1)(\beta_i) - I^{\alpha}(1)(T)\right\} \\
&+ \frac{p^*R}{v_0(b_i,\sigma_i)}\left(t - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}(1)(\sigma_i) - I^{\alpha-1}(1)(T)\right\} \\
&+ \frac{p^*R}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2}\right) \\
&\times \left\{\sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}(1)(\varepsilon_i) - I^{\alpha-2}(1)(T)\right\}
\end{aligned}$$

Consequently

$$\leq P^*R\left\{\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{v_0(a_i,\beta_i)}\left\{\sum_{i=1}^m a_i\frac{\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+1)\beta_i^\alpha}{\Gamma(\alpha+1)\Gamma(\gamma_i+\frac{\alpha}{\eta_i}+\delta_i+1)}-\frac{T^\alpha}{\Gamma(\alpha+1)}\right\}\right.$$
$$+\frac{1}{v_0(b_i,\sigma_i)}\left(t-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)\left\{\sum_{i=1}^m b_i\frac{\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i+\frac{\alpha-1}{\eta_i}+\delta_i+1)}-\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right\}$$
$$+\frac{1}{v_0(d_i,\varepsilon_i)}\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)}+\frac{T^2}{2}\right)$$
$$\left.\times\left\{\sum_{i=1}^m d_i\frac{\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i+\frac{\alpha-2}{\eta_i}+\delta_i+1)}-\frac{T^{\alpha-2}}{\Gamma(\alpha-1)}\right\}\right\}$$
$$\leq P^*RM$$
$$\leq R.$$

Step 3: we show that $\mathfrak{F}(D)$ is equicontinuous :
By Step 2, it is obvious that $\mathfrak{F}(D)\subset C(J,E)$ is bounded. For the equicontinuity of $\mathfrak{F}(D)$, let $t_1,t_2\in J$, $t_1<t_2$ and $x\in D$, so $\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\neq 0$. Then

$$\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|\leq|I^\alpha f(s,x(s))(t_2)-I^\alpha f(s,x(s))(t_1)|+\left((t_2-t_1)-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)$$
$$\times\frac{1}{v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^m b_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-1}|f(s,x(s))|(\sigma_i)-I^{\alpha-1}|f(s,x(s))|(T)\right\}$$
$$+\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)(t_2-t_1)}{v_0(b_i,\sigma_i)}+\frac{(t_2^2-t_1^2)}{2}\right)$$
$$\times\frac{1}{v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^m d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}|f(s,x(s))|(\varepsilon_i)-I^{\alpha-2}|f(s,x(s))|(T)\right\}$$
$$\leq\frac{Rp^*}{\Gamma(\alpha+1)}\{(t_2^\alpha-t_1^\alpha)+2(t_2-t_1)^\alpha\}+\left((t_2-t_1)-\frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)}\right)$$
$$\times\frac{1}{v_0(b_i,\sigma_i)}\left\{\sum_{i=1}^m b_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-1}|f(s,x(s))|(\sigma_i)-I^{\alpha-1}|f(s,x(s))|(T)\right\}$$
$$+\left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)})+\frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)}-\frac{v_1(b_i,\sigma_i)(t_2-t_1)}{v_0(b_i,\sigma_i)}+\frac{(t_2^2-t_1^2)}{2}\right)$$
$$\times\frac{1}{v_0(d_i,\varepsilon_i)}\left\{\sum_{i=1}^m d_iJ_{\eta_i}^{\gamma_i,\delta_i}I^{\alpha-2}|f(s,x(s))|(\varepsilon_i)-I^{\alpha-2}|f(s,x(s))|(T)\right\}.$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero. Hence $N(D)\subset D$.

Finally we show that the implication holds:
Let $V \subset D$ such that $V = \overline{conv}(\mathfrak{F}(V) \cup \{0\})$. Since V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu(V(t))$ is continuous on J.
By assumption $(H2)$ and the properties of the measure μ we have for each $t \in J$.

$$
\begin{aligned}
v(t) &\leq \mu(\mathfrak{F}(V)(t) \cup \{0\})) \leq \mu((\mathfrak{F}V)(t)) \\
&\leq \mu \Bigg\{ I^\alpha f(s, x(s))(t) + \frac{1}{v_0(a_i, \beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i, \delta_i} I^\alpha f(s, V(s))(\beta_i) - I^\alpha f(s, V(s))(T) \right\} \\
&+ \frac{1}{v_0(b_i, \sigma_i)} \left(t - \frac{v_1(a_i, \beta_i)}{v_0(a_i, \beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-1} f(s, V(s))(\sigma_i) - I^{\alpha-1} f(s, V(s))(T) \right\} \\
&+ \frac{1}{v_0(d_i, \varepsilon_i)} \left(\frac{v_2(a_i, \beta_i)}{2v_0(a_i, \beta_i)}) + \frac{v_1(a_i, \beta_i) v_1(b_i, \sigma_i)}{v_0(b_i, \sigma_i) v_0(a_i, \beta_i)} - \frac{v_1(b_i, \sigma_i) t}{v_0(b_i, \sigma_i)} + \frac{t^2}{2} \right) \\
&\times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-2} f(s, V(s))(\varepsilon_i) - I^{\alpha-2} f(s, V(s))(T) \right\} \Bigg\} \\
&\leq I^\alpha p(s) \mu(V(s))(t) + \frac{1}{v_0(a_i, \beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i, \delta_i} I^\alpha p(s) \mu(V(s))(\beta_i) - I^\alpha p(s) \mu(V(s))(T) \right\} \\
&+ \frac{1}{v_0(b_i, \sigma_i)} \left(t - \frac{v_1(a_i, \beta_i)}{v_0(a_i, \beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-1} p(s) \mu(V(s))(\sigma_i) - I^{\alpha-1} p(s) \mu(V(s))(T) \right\} \\
&+ \frac{1}{v_0(d_i, \varepsilon_i)} \left(\frac{v_2(a_i, \beta_i)}{2v_0(a_i, \beta_i)}) + \frac{v_1(a_i, \beta_i) v_1(b_i, \sigma_i)}{v_0(b_i, \sigma_i) v_0(a_i, \beta_i)} - \frac{v_1(b_i, \sigma_i) t}{v_0(b_i, \sigma_i)} + \frac{t^2}{2} \right) \\
&\times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-2} p(s) \mu(V(s))(\varepsilon_i) - I^{\alpha-2} p(s) \mu(V(s))(T) \right\} \\
&\leq I^\alpha p(s) v(s)(t) + \frac{1}{v_0(a_i, \beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i, \delta_i} I^\alpha p(s) v(s)(\beta_i) - I^\alpha p(s) v(s)(T) \right\} \\
&+ \frac{1}{v_0(b_i, \sigma_i)} \left(t - \frac{v_1(a_i, \beta_i)}{v_0(a_i, \beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-1} p(s) v(s)(\sigma_i) - I^{\alpha-1} p(s) v(s)(T) \right\} \\
&+ \frac{1}{v_0(d_i, \varepsilon_i)} \left(\frac{v_2(a_i, \beta_i)}{2v_0(a_i, \beta_i)}) + \frac{v_1(a_i, \beta_i) v_1(b_i, \sigma_i)}{v_0(b_i, \sigma_i) v_0(a_i, \beta_i)} - \frac{v_1(b_i, \sigma_i) t}{v_0(b_i, \sigma_i)} + \frac{t^2}{2} \right) \\
&\times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i, \delta_i} I^{\alpha-2} p(s) v(s)(\varepsilon_i) - I^{\alpha-2} p(s) v(s)(T) \right\}
\end{aligned}
$$

$$\leq P^*\|v\| \left\{ I^\alpha(1)(T) + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i J_{\eta_i}^{\gamma_i,\delta_i} I^\alpha(1)(\beta_i) - I^\alpha(1)(T) \right\} \right.$$
$$+ \frac{1}{v_0(b_i,\sigma_i)} \left(T - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-1}(1)(\sigma_i) - I^{\alpha-1}(1)(T) \right\}$$
$$+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2} \right)$$
$$\left. \times \left\{ \sum_{i=1}^m d_i J_{\eta_i}^{\gamma_i,\delta_i} I^{\alpha-2}(1)(\varepsilon_i) - I^{\alpha-2}(1)(T) \right\} \right\}$$
$$\leq P^*\|v\| \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{v_0(a_i,\beta_i)} \left\{ \sum_{i=1}^m a_i \frac{\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + 1)\beta_i^\alpha}{\Gamma(\alpha+1)\Gamma(\gamma_i + \frac{\alpha}{\eta_i} + \delta_i + 1)} - \frac{T^\alpha}{\Gamma(\alpha+1)} \right\} \right.$$
$$+ \frac{1}{v_0(b_i,\sigma_i)} \left(T - \frac{v_1(a_i,\beta_i)}{v_0(a_i,\beta_i)} \right) \left\{ \sum_{i=1}^m b_i \frac{\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + 1)\sigma_i^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma_i + \frac{\alpha-1}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-1}}{\Gamma(\alpha)} \right\}$$
$$+ \frac{1}{v_0(d_i,\varepsilon_i)} \left(\frac{v_2(a_i,\beta_i)}{2v_0(a_i,\beta_i)}) + \frac{v_1(a_i,\beta_i)v_1(b_i,\sigma_i)}{v_0(b_i,\sigma_i)v_0(a_i,\beta_i)} - \frac{v_1(b_i,\sigma_i)T}{v_0(b_i,\sigma_i)} + \frac{T^2}{2} \right)$$
$$\left. \times \left\{ \sum_{i=1}^m d_i \frac{\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + 1)\varepsilon_i^{\alpha-2}}{\Gamma(\alpha-1)\Gamma(\gamma_i + \frac{\alpha-2}{\eta_i} + \delta_i + 1)} - \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} \right\} \right\}$$
$$:= p^*\|v\|M.$$

This means that

$$\|v\|(1 - p^*M) \leq 0.$$

By (3.19) it follows that $\|v\| = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem (13), we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (1.1)-(1.2).

4 Example

Let

$$E = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^\infty |x_n| < \infty\}$$

with the norm

$$\|x\|_E = \sum_{n=1}^\infty |x_n|$$

Let us consider problem (1.1)-(1.2) with specific data:

$$\begin{array}{llll} T=1, & m=1, & \alpha=5/2, & \beta_1=1/2 \\ \sigma_1=3/2, & \varepsilon_1=5/7, & \eta_1=7/5, & \gamma_1=2/3 \\ \delta_1=3/2, & a_1=3/2, & b_1=1/2, & d_1=3/4. \end{array} \tag{4.1}$$

Using the given values of the parameters in (3.3)-(3.4) and (3.5), we find that

$$\begin{array}{lll} v_0(a_1,\beta_1)=0.4226, & v_0(b_1,\sigma_1)=0.8075, & v_0(d_1,\varepsilon_1)=0.7113 \\ v_1(a_1,\beta_1)=0.6445, & v_1(b_1,\sigma_1)=0.8815 & \\ v_2(a_1,\beta_1)=0.7531 & & \end{array} \tag{4.2}$$

In order to illustrate Theorem (15),we take

$$f(t,x(t))=\frac{t\sqrt{\pi}-1}{7^3}x(t), t\in[0,1]$$

Clearly, the function f is continuous, we have

$$|f(t,x(t))|\leq\frac{\sqrt{\pi}}{7^3}|x|$$

Hence, the hypothesis (H2) is satisfied with $p^*=\frac{\sqrt{\pi}}{7^3}$. We shall show that condition (3.19) holds with $T=1$. Indeed,

$$p^*M\simeq 0.3817<1$$

Simple computations show that all conditions of Theorem (15) are satisfied. It follows that the problem (1.1)-(1.2) with data (4.1) and (4.2) has at least solution defined on $[0,1]$.

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