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SEMI-LOCAL CONVERGENCE OF A SEVENTH-ORDER METHOD IN BANACH SPACES UNDER ω-CONTINUITY CONDITION

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Abstract. The article is about the analysis of semi-local convergence of a seventh-order iterative method used for finding the roots of a nonlinear equation in Banach spaces. In this article, the imposed hypotheses is amiable than the well-known Lipschitz and Hölder continuity conditions. The R-order convergence of the considered scheme is proved to be at least $4+3q$. An approximate apriori error bound for this method is also elaborated and the domain of existence and uniqueness of the solution in the convergence theorem. Two numerical illustrations have been worked out to exhibit the virtue of the developed theory.

1 Introduction

To solve the nonlinear problems in some areas, like scientific and engineering computing, we need to locate an approximate root x^* of a nonlinear equation which is defined as

$$\Delta(x) = 0, \tag{1.1}$$

where Δ is continuously second-order Fréchet differentiable operator such that $\Delta : \Omega \subseteq X_1 \to X_2$, where X_1 and X_2 are two Banach spaces and Ω is a non-empty open convex subset of X_1. The nonlinear equation (1.1) can be solved by well-known Newton's method having order two. The semi-local convergence of this method initially studied by Kantorovich in [7] and then by Rall in [13]. But, Rall used different method to show the convergence of Newton's method by applying the recurrence relation techniques. It is well known that the application of the higher order algorithms are computationally ampliative but have some important applications when calculating the stiff system of equations which requires rapid convergence.

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Semi-local convergence of various higher order iterative methods have been dis-serted by many researchers such as third-order in [4-12], fourth-order in [3, 17], fifth-order in [1, 2], sixth-order in [14, 18], seventh-order in [15] etc. and references therein. Keeping in mind, the necessity of the higher order convergence, Jaiswal in [6], has discussed the semi-local convergence of seventh-order iterative scheme accustomed by Xiao and Yin [16], when second order Fréchet derivative satisfied Lipschitz continuity. The author has also discussed the computational efficiency in the revisited form. In this paper, we relax one of the conditions assumed in [6] and discussed the semi-local convergence analysis under ω-continuity condition, which is the generalization of Lipschitz and Hölder continuity conditions. The differential equations, boundary value problems, integral equations, etc. may be reduced into the form of the equation (1.1). To solve the equation (1.1), we use an iterative method with an initial guess x_0, that constructed the sequence $\{x_n\}$ and also confirms $lim_{n\to\infty}x_n = x^*$ such that $\Delta(x^*) = 0$. The convergence of the sequence $\{x_n\}$ to an approximate solution x^* and the convergence speed are the two main aspects of any iterative methods. Consider the algorithm given in [16]

$$\begin{aligned}
y_k &= x_k - \frac{1}{2}\Gamma_k\Delta(x_k),\\
z_k^{(1)} &= x_k - [\Delta'(y_k)]^{-1}\Delta(x_k),\\
z_k^{(2)} &= z_k^{(1)} - [2[\Delta'(y_k)]^{-1} - \Gamma_k]\Delta(z_k^{(1)}),\\
x_{k+1} &= z_k^{(2)} - [2[\Delta'(y_k)]^{-1} - \Gamma_k]\Delta(z_k^{(2)}),
\end{aligned} \tag{1.2}$$

where, $\Gamma_k = [\Delta'(x_k)]^{-1}$. Furthermore, the convergence result is said to be global if the condition imposed only on Δ, whereas the convergence result is said to be local if some conditions satisfy by the solution x^*.

2 Fundamental results

Here, we assume the following symbols throughout the paper:
Ω $\equiv$ non-empty open subset of X_1;
X_1, X_2 $\equiv$ Banach spaces; $D(x,r) = \{y \in X_1 : \|y-x\| < r\}$;
$\overline{D(x,r)} = \{y \in X_1 : \|y-x\| \le r\}$; $u_k = x_k - \Gamma_k\Delta(x_k)$;
$H(x_0) = \Gamma_0[\Delta'(y_0) - \Delta'(x_0)]$ and $J_\Delta(x_0) = [2[\Delta'(y_0)]^{-1} - \Gamma_0]$.

Let $x_0 \in \Omega$ and $\Delta : \Omega \subseteq X_1 \to X_2$ is assumed to be a nonlinear operator. Suppose the following conditions are true:
$(A1)\|\Gamma_0\Delta(x_0)\| \le \sigma$,
$(A2)\|\Gamma_0\| \le \tau$,
$(A3)\|\Delta''(x)\| \le \Theta$, $x \in \Omega$,
$(A4)\|\Delta''(x) - \Delta''(y)\| \le \omega(\|x-y\|), \forall\ x, y \in \Omega$,

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, is a continuous and non-decreasing function satisfying $\omega(\epsilon z) \leq \varphi(\epsilon)\omega(z), \epsilon \in [0,1]$, and $z \in [0,+\infty)$, with $\varphi : [0,1] \to \mathbb{R}_+$, is also continuous and non-decreasing function. Let us consider the subsequent lemmas, which will be used later on.

Lemma 1. *[1] Let $\Delta : \Omega \subseteq X_1 \to X_2$ be a nonlinear operator which is continuously twice Fréchet differentiable in Ω, then*

$$\begin{aligned}
&\Delta(z_k^{(1)})\\
= &\int_0^1 \Delta''(u_k + r(z_k^{(1)} - u_k))(1-r)dr(z_k^{(1)} - u_k)^2\\
&- \int_0^1 \Delta''(y_k + r(u_k - y_k))(u_k - y_k)dr[\Delta'(y_k)]^{-1}[\Delta'(y_k) - \Delta'(x_k)]\\
&\times(u_k - x_k)\\
&+ \int_0^1 \left[\Delta''(x_k + r(u_k - x_k)) - \Delta''(x_k)\right](1-r)dr(u_k - x_k)^2\\
&+\frac{1}{2}\int_0^1 \left[\Delta''(x_k) - \Delta''(x_k + \frac{1}{2}r(u_k - x_k))\right] dr(u_k - x_k)^2. \qquad (2.1)
\end{aligned}$$

Lemma 2. *[6] Suppose the hypotheses of Lemma 1 hold, then*

$$\begin{aligned}
&\Delta(z_k^{(2)})\\
= &\frac{1}{2}\Delta''(x_k)(u_k - x_k)[\Delta'(y_k)]^{-1}\\
&\times\Big\{\Delta''(x_k)(u_k - x_k) + \int_0^1 \left[\Delta''\left(x_k + \frac{1}{2}r(u_k - x_k)\right) - \Delta''(x_k)\right] dr\\
&\times(u_k - x_k)\Big\}\Gamma_k\Delta(z_k^{(1)})\\
&+\frac{1}{2}\int_0^1 \left[\Delta''\left(x_k + \frac{1}{2}r(u_k - x_k)\right) - \Delta''(x_k)\right] dr(u_k - x_k)\Gamma_k\Delta(z_k^{(1)})\\
&+ \int_0^1 \left[\Delta''\left(x_k + r(u_k - x_k)\right) - \Delta''(x_k)\right] dr(u_k - x_k)(z_k^{(2)} - z_k^{(1)})\\
&-\frac{1}{2}\int_0^1 \left[\Delta''\left(x_k + \frac{1}{2}r(u_k - x_k)\right) - \Delta''(x_k)\right] dr(u_k - x_k)(z_k^{(2)} - z_k^{(1)})\\
&+ \int_0^1 \left[\Delta'\left(z_k^{(1)} + r(z_k^{(2)} - z_k^{(1)})\right) - \Delta'(u_k)\right] dr(z_k^{(2)} - z_k^{(1)}). \qquad (2.2)
\end{aligned}$$

Lemma 3. *[6] Let the hypotheses of Lemma 1 are true, then*

$$
\begin{aligned}
&\Delta(x_{k+1})\\
=\;& \frac{1}{2}\Delta''(x_k)(u_k-x_k)[\Delta'(y_k)]^{-1}\\
&\times\left\{\Delta''(x_k)(u_k-x_k)+\int_0^1\left[\Delta''\left(x_k+\frac{1}{2}r(u_k-x_k)\right)-\Delta''(x_k)\right]dr\right.\\
&\left.\times(u_k-x_k)\right\}\Gamma_k\Delta(z_k^{(2)})\\
&+\frac{1}{2}\int_0^1\left[\Delta''\left(x_k+\frac{1}{2}r(u_k-x_k)\right)-\Delta''(x_k)\right]dr(u_k-x_k)\Gamma_k\Delta(z_k^{(2)})\\
&+\int_0^1\left[\Delta''\left(x_k+r(u_k-x_k)\right)-\Delta''(x_k)\right]dr(u_k-x_k)(x_{k+1}-z_k^{(2)})\\
&-\frac{1}{2}\int_0^1\left[\Delta''\left(x_k+\frac{1}{2}r(u_k-x_k)\right)-\Delta''(x_k)\right]dr(u_k-x_k)(x_{k+1}-z_k^{(2)})\\
&+\int_0^1\left[\Delta'\left(z_k^{(2)}+r(x_{k+1}-z_k^{(2)})\right)-\Delta'(u_k)\right]dr(x_{k+1}-z_k^{(2)}).
\end{aligned}
\tag{2.3}
$$

Now we designate the following expressions, which will be use later in the proofs of the results. So, let

$$
\begin{aligned}
&\phi(r)\\
=\;&\frac{r^8-4r^7-12r^6-8r^5+336r^4-672r^3+1024r^2-512r+256}{2(2-r)^7},
\end{aligned}
\tag{2.4}
$$

$$\psi(r)=\frac{1}{1-r\phi(r)}, \tag{2.5}$$

$$\varphi_1(r,s)=\frac{r^3}{2(2-r)^2}+\frac{r^2}{2(2-r)}+(H_1+H_2)s, \tag{2.6}$$

$$
\begin{aligned}
\varphi_2(r,s)\;=\;&\varphi_1(r,s)\left[\frac{r}{2-r}(r+2H_2s)+H_2s+(H_3+H_2)\left(\frac{2+r}{2-r}\right)s\right.\\
&\left.+\frac{r^2(2+r)}{(2-r)^2}+\frac{r}{2}\left(\frac{2+r}{2-r}\right)^2\varphi_1(r,s)\right],
\end{aligned}
\tag{2.7}
$$

$$\begin{aligned}\varphi_3(r,s) &= \varphi_2(r,s)\Bigg[\frac{r}{2-r}(r+2H_2s)+H_2s+(H_2+H_3)\left(\frac{2+r}{2-r}\right)s \\ &\quad +\frac{r^2(2+r)}{(2-r)^2}+\frac{r^2(4-r)(2+r)^2}{(2-r)^4}+\frac{r}{2}\left(\frac{2+r}{2-r}\right)^2\varphi_2(r,s)\Bigg],\end{aligned} \tag{2.8}$$

where $H_1=\int_0^1\varphi(r)(1-r)dr$, $H_2=\frac{1}{2}\int_0^1\varphi\left(\frac{r}{2}\right)dr$ and $H_3=\int_0^1\varphi(r)dr$. Let us assume $z(r)=\phi(r)r-1$, then clearly $z(0)=-1$ and $z(2)=+\infty$ and hence $z(r)$ has at least one real root in $(0,2)$ say, ρ be the least positive root. Now, we define some of the properties of the equations ϕ,ψ and φ_3 which are given in the equations (2.4), (2.5) and (2.8), respectively.

Lemma 4. *Consider the functions ϕ,ψ and φ_3 are mentioned by the expression (2.4), (2.5) and (2.8) respectively then,*
(i) $\phi(r)>1$ and $\psi(r)>1$ are increasing, for $r\in(0,\rho)$ and,
(ii) $\varphi_3(r,s)$ is increasing for $s>0$ and $r\in(0,\rho)$.

Proof. The proof can be done with the help of the expressions of the functions ϕ,ψ and φ_3. □

Let

$$\begin{aligned}\sigma_0 &= \sigma,\\ \tau_0 &= \tau,\\ p_0 &= \Theta\tau\sigma,\\ q_0 &= \sigma\tau\omega(\sigma),\\ \zeta &= \frac{1}{\psi(p_0)},\\ \vartheta &= \psi^2(p_0)\varphi_3(p_0,q_0),\\ \kappa &= \frac{\phi(p_0)}{1-\psi(p_0)\varphi_3(p_0,q_0)}.\end{aligned} \tag{2.9}$$

Clearly $\delta,\vartheta<1$. Moreover, we assume the following sequences holds for $k\geq 0$,

$$\tau_{k+1} = \tau_k\psi(p_k), \tag{2.10}$$

$$\sigma_{k+1} = \sigma_k\psi(p_k)\varphi_3(p_k,q_k), \tag{2.11}$$

$$p_{k+1} = p_k\psi(p_k)^2\varphi_3(p_k,q_k), \tag{2.12}$$

$$q_{k+1} = \sigma_{k+1}\tau_{k+1}\omega(\sigma_{k+1})\leq q_k\psi(p_k)^2\varphi_3(p_k,q_k)\varphi(\delta_k), \tag{2.13}$$

where $\delta_k=\psi(p_k)\varphi_3(p_k,q_k)$.

Lemma 5. *Let the functions ϕ, ψ and φ_3 are given by the equations (2.4), (2.5) and (2.8) respectively. If*

$$\psi(p_0)^2\varphi_3(p_0, q_0) < 1,\ \ 0 < p_0 < \rho, \tag{2.14}$$

then, the following results hold
(i) $\psi(p_k) > 1$ and $\psi(p_k)\varphi_3(p_k, q_k) < 1$, for all $k \geq 0$,
(ii) the sequences $\{\sigma_k\}$, $\{p_k\}$, $\{q_k\}$ and $\{\psi(p_k)\varphi_3(p_k, q_k)\}$ are decreasing and,
(iii) $\phi(p_k)a_k < 1$ and $\psi^2(p_k)\varphi_3(p_k, q_k) < 1$, $\forall\ k \geq 0$.

Proof. From the Lemma 4 and the equations (2.6) to (2.14), it can be easily prove that the results hold for $k = 0$ and then, on applying induction, it can be showed that it is also true for all $k \geq 0$. □

Lemma 6. *Consider the expressions ϕ, ψ and φ_3 given in the equations (2.4), (2.5) and (2.8), respectively. Also suppose that $\theta \in (0, 1)$, then $\phi(\theta r) < \phi(r)$, $\psi(\theta r) < \psi(r)$ and $\varphi_3(\theta r, \theta s) < \theta^3\varphi_3(r, s)$ for $r \in (0, \rho)$ and $\varphi_3(\theta r, \theta^{1+q}s) \leq \theta^{3+3q}\varphi_3(r, s)$.*

Proof. The proof quickly follows from the equations (2.4) to (2.8) and the fact that $r \in (0, \rho)$ and $\theta \in (0, 1)$. □

3 Recurrence relations

We shall define the following recurrence relations by assuming that the hypotheses presumed in Section 2 holds. For $k = 0$, the existence of Γ_0 signifies the existence of y_0 and u_0. So, we have

$$\|y_0 - x_0\| \leq \frac{1}{2}\sigma_0, \tag{3.1}$$

Also

$$\|u_0 - x_0\| \leq \sigma_0. \tag{3.2}$$

Hence $y_0,\ u_0 \in D(x_0, \kappa\sigma)$. And hence

$$\|H(x_0)\| \leq \frac{1}{2}p_0. \tag{3.3}$$

Using Banach lemma and the fact that $p_0 < 2$ we can conclude that $[\Delta'(y_0)]^{-1}$ exists and

$$\|[\Delta'(y_0)]^{-1}\| \leq \frac{2\tau_0}{2 - p_0}. \tag{3.4}$$

Also

$$\|J_\Delta(x_0)\| \leq \frac{2 + p_0}{2 - p_0}\tau_0. \tag{3.5}$$

Therefore one can attain

$$\|z_0^{(1)} - x_0\| \leq \frac{2}{2-p_0}\sigma_0, \tag{3.6}$$

$$\|z_0^{(1)} - u_0\| \leq \frac{p_0}{2-p_0}\sigma_0, \tag{3.7}$$

and

$$\|z_0^{(2)} - x_0\| \leq \frac{-p_0^3 + 4p_0^2 + 8}{(2-p_0)^3}\sigma_0. \tag{3.8}$$

Also, the following can be quickly verified

$$\|\Delta(z_0^{(2)})\| \leq \frac{4-p_0}{(2-p_0)^2}\left\{\frac{2(4p_0^2 - 8p_0 + 16)}{(2-p_0)^2} + \frac{p_0^2(2+p_0)^2(4-p_0)}{(2-p_0)^4}\right\}\frac{\Theta\sigma_0^2}{2}. \tag{3.9}$$

The last step of the scheme (1.2) implies

$$\|x_1 - z_0^{(2)}\| \leq \frac{2+p_0}{2-p_0}\tau_0\|\Delta(z_0^{(2)})\|. \tag{3.10}$$

On using triangle inequality and the equations (3.8)-(3.10), we have that

$$\|x_1 - x_0\| \leq \phi(p_0)\sigma_0. \tag{3.11}$$

Thus $x_1 \in D(x_0, \kappa\sigma)$. Now since $\phi(p_0) < \phi(\rho)$, we can obtain

$$\|I - \Gamma_0\Delta'(x_1)\| \quad \leq \quad p_0\phi(p_0) < 1.$$

Therefore $\Gamma_1 = [\Delta'(x_1)]^{-1}$ exists by Banach lemma and hence it follows that

$$\|\Gamma_1\| \quad \leq \quad \frac{\tau_0}{1 - p_0\phi(p_0)} = \tau_1. \tag{3.12}$$

Using the involved relations in Lemma 1 - Lemma 3, we can calculate

$$\|\Delta(z_0^{(1)})\| \leq \varphi_1(p_0, q_0)\frac{\sigma_0}{\tau_0}, \tag{3.13}$$

$$\|\Delta(z_0^{(2)})\| \leq \varphi_2(p_0, q_0)\frac{\sigma_0}{\tau_0}, \tag{3.14}$$

and

$$\|\Delta(x_1)\| \leq \varphi_3(p_0, q_0)\frac{\sigma_0}{\tau_0}. \tag{3.15}$$

And hence,

$$\|u_1 - x_1\| \leq \psi(p_0)\varphi_3(p_0, q_0)\sigma_0 = \sigma_1. \tag{3.16}$$

Also, because $\phi(p_0) > 1$ and by the triangle inequality, we find

$$\|u_1 - x_0\| \leq \kappa\sigma. \tag{3.17}$$

which implies that $u_1, y_1 \in D(x_0, \kappa\sigma)$. Furthermore, we have

$$\begin{aligned} \Theta\|\Gamma_1\|\|\Gamma_1\Delta(x_1)\| &\leq \psi^2(p_0)\varphi_3(p_0, q_0)p_0 = p_1, \\ \|\Gamma_1\|\|\Gamma_1\Delta(x_1)\|\omega(\|\Gamma_1\Delta(x_1)\|) &\leq \psi^2(p_0)\varphi_3(p_0, q_0)\varphi(\delta_0)q_0 = q_1. \end{aligned} \tag{3.18}$$

On using mathematical induction, we can define the following system of recurrence relations as a lemma:

Lemma 7. *Assume the hypotheses of Lemma 5 hold and the assumptions $(A1)-(A4)$ are satisfying then the subsequent inequalities also hold $\forall\ k \geq 0$,*

$$\begin{aligned} &(i)\Gamma_k = [\Delta'(x_k)]^{-1} \textit{exists and } \|\Gamma_k\| \leq \tau_k, \\ &(ii)\|\Gamma_k\Delta(x_k)\| \leq \sigma_k, \\ &(iii)\Theta\|\Gamma_k\|\|\Gamma_k\Delta(x_k)\| \leq p_k, \\ &(iv)\|\Gamma_k\|\|\Gamma_k\Delta(x_k)\|\omega(\|\Gamma_k\Delta(x_k)\|) \leq q_k, \\ &(v)\|x_{k+1} - x_k\| \leq \phi(p_k)\sigma_k, \\ &(vi)\|x_{k+1} - x_0\| \leq \kappa\sigma. \end{aligned} \tag{3.19}$$

Proof. The proof of $(i)-(v)$ is simple. So, we shall show part (vi). From the relation (v), Lemma 6 and Lemma 9, we obtain the following relations:

$$\delta_k \leq \zeta\vartheta^{4^k}, \tag{3.20}$$

$$\prod_{i=0}^{k} \delta_i \leq \zeta^{k+1}\vartheta^{\frac{4^{k+1}-1}{3}}, \tag{3.21}$$

$$\sigma_i \leq \sigma\zeta^k\vartheta^{\frac{4^k-1}{3}}, \tag{3.22}$$

$$\sum_{i=k}^{k+m} \sigma_i \leq \sigma\zeta^k\vartheta^{\frac{4^k-1}{3}} \left(\frac{1 - \zeta^{m+1}\vartheta^{4^k(\frac{4^m+2}{3})}}{1 - \zeta\vartheta^{4^k}} \right), \tag{3.23}$$

for $k \geq 0, m \geq 0$. Now, by using (v) and the above relation, we have

$$\begin{aligned}
\|x_{k+1} - x_0\| &\leq \sum_{i=0}^{k} \phi(p_i)\sigma_i \\
&\leq \phi(p_0)\sigma \left(\frac{1 - \zeta^{k+1}\vartheta^{\frac{2+4^k}{3}}}{1 - \psi(p_0)\varphi_3(p_0, q_0)} \right) \leq \kappa\sigma.
\end{aligned}$$

□

4 Semi-local convergence

In this portion, we state and prove the convergence theorem along with the error bound expression for the method (1.2).

Theorem 8. *Let $\Delta : \Omega \subseteq X_1 \to X_2$ be a continuously twice Fréchet differentiable on Ω. Let the assumptions $(A1)-(A4)$ holds and $x_0 \in \Omega$. Assume that the functions ϕ, ψ and φ_3 and the sequences p_0, q_0 are given by the equations (2.4) − (2.9), respectively and the inequality (2.14) also holds. Likewise, let $\overline{D(x_0, \kappa\sigma)} \subseteq \Omega$. Then initiating with x_0, the iterative sequence $\{x_k\}$ generating from the scheme (1.2) converges to a zero x^* of $\Delta(x) = 0$ with $x_k, x^* \in \overline{D(x_0, \kappa\sigma)}$ and x^* is an exclusive zero of $\Delta(x) = 0$ in $D(x_0, \frac{2}{\Theta\tau} - \kappa\sigma) \cap \Omega$. Furthermore, its error bound is defined as*

$$\|x_k - x^*\| \leq \phi(p_0)\sigma\zeta^k\vartheta^{\frac{4^k-1}{3}} \left(\frac{1}{1 - \zeta\vartheta^{4^k}} \right). \tag{4.1}$$

Proof. The sequence $\{x_k\}$ is well established in $\overline{D(x_0, \kappa\sigma)}$. Now

$$\begin{aligned}
\|x_{k+m} - x_k\| &\leq \sum_{i=k}^{k+m-1} \|x_{i+1} - x_i\| \\
&\leq \phi(p_0)\sigma\zeta^k\vartheta^{\frac{4^k-1}{3}} \left(\frac{1 - \zeta^m\vartheta^{\frac{4^k(4^{m-1}+2)}{6}}}{1 - \zeta\vartheta^{4^k}} \right),
\end{aligned} \tag{4.2}$$

which shows that $\{x_k\}$ is a cauchy sequence. Hence, there exists an x^* satisfying $\lim_{k\to\infty} x_k = x^*$. Let $k = 0$, $m \to \infty$ in the equation (4.2), and thus we obtain

$$\|x^* - x_0\| \leq \kappa\sigma, \tag{4.3}$$

which implies that, $x^* \in \overline{D(x_0, \kappa\sigma)}$. Next, we show that x^* is a zero of $\Delta(x) = 0$. Because

$$\|\Gamma_0\|\|\Delta(x_k)\| \leq \|\Gamma_k\|\|\Delta(x_k)\|. \tag{4.4}$$

By tending $k \to \infty$ in the equation (4.4) and assuming the continuity of Δ in Ω, we find that $\Delta(x^*) = 0$. At last, we verify the uniqueness of x^* in $D(x_0, \frac{2}{\Theta\tau} - \kappa\sigma) \cap \Omega$. It is obvious that

$$\frac{2}{\Theta\tau} - \kappa\sigma > \kappa\sigma,$$

since $\kappa < 1/p_0$. Let x^{**} be the other solution of $\Delta(x)$ in $D(x_0, \frac{2}{\Theta\tau} - \kappa\sigma) \cap \Omega$. On applying Taylor's theorem, we attain

$$0 = \Delta(x^{**}) - \Delta(x^*) = \int_0^1 \Delta'((1-r)x^* + rx^{**})dr(x^{**} - x^*). \tag{4.5}$$

Also, since

$$\begin{aligned} & \|\Gamma_0\| \left\| \int_0^1 [\Delta'((1-r)x^* + rx^{**}) - \Delta'(x_0)]dr \right\| \\ \leq \quad & \Theta\tau \int_0^1 [(1-r)\|x^* - x_0\| + r\|x^{**} - x_0\|]dr \leq \frac{\Theta\tau}{2}\left[\kappa\sigma + \frac{2}{\Theta\tau} - \kappa\sigma\right] < 1, \end{aligned}$$

which implies $\int_0^1 \Delta'((1-r)x^* + rx^{**})dr$ is invertible and thus $x^{**} = x^*$. □

5 R-order of convergence

This segment includes analysis of the R-order of convergence of the method (1.2), under the following condition which is defined as

$$\|\Delta''(x) - \Delta''(y)\| \leq \sum_{i=1}^{m} K_i\|x - y\|^{q_i}, K_i \geq 0, q_i \in [0, 1], x, y \in \Omega, \tag{5.1}$$

choosing $\omega(s) = \sum_{i=1}^m K_i s^{q_i}$, then $\omega(ts) = \sum_{i=1}^m K_i t^{q_i} s^{q_i}$. Since, $t \in [0, 1]$ and $q_i \in [0, 1]$, we get $\varphi(t) = t^q$, where $q = min(q_1, q_2, \cdots, q_m)$.

Lemma 9. *Suppose the hypotheses of Lemma 5 are true and let* $\delta_k = \psi(p_k)\varphi_3(p_k, q_k), \vartheta = \psi(p_0)\delta_0$ *and* $\zeta = \frac{1}{\psi(p_0)}$, *then*

$$\delta_k \leq \zeta\vartheta^{(4+3q)^k}, \tag{5.2}$$

$$\prod_{i=0}^{k} \delta_i \leq \zeta^{k+1}\vartheta^{\frac{(4+3q)^{k+1}-1}{3+3q}}, \tag{5.3}$$

$$\sigma_i \leq \sigma\zeta^k\vartheta^{\frac{(4+3q)^k-1}{3+3q}}, \tag{5.4}$$

$$\sum_{i=k}^{k+m} \sigma_i \leq \sigma \zeta^k \vartheta^{\frac{(4+3q)^k-1}{3+3q}} \left(\frac{1-\zeta^{m+1}\vartheta^{(4+3q)^k(\frac{(4+3q)^m+2+3q}{3+3q})}}{1-\zeta\vartheta^{(4+3q)^k}} \right), \tag{5.5}$$

for $k \geq 0, m \geq 0$.

Proof. Because $p_1 = p_0\psi(p_0)^2\varphi_3(p_0,q_0) = \vartheta p_0$ and $q_1 = q_0\psi(p_0)^3\varphi_3(p_0,q_0)\varphi(\delta_0)$ $\leq\ q_0\vartheta^{1+q}$, so that we can write

$$\psi(p_1)\varphi_3(p_1,q_1) < \psi(\vartheta p_0)\varphi_3(\vartheta p_0, \vartheta^{1+q}q_0) < \vartheta^{3+3q}\psi(p_0)\varphi_3(p_0,q_0) < \zeta\vartheta^{(4+3q)^1}.$$

Now, assume $\psi(p_i)\varphi_3(p_i,q_i) \leq \zeta\vartheta^{(4+3q)^i}, i \geq 1$, hence by using the Lemma 6, we get

$$\begin{aligned} \psi(p_{i+1})\varphi_3(p_{i+1},q_{i+1}) &< \psi(p_i)\varphi_3(p_i\psi(p_i)\delta_i, q_i\psi(p_i)\delta_i^{1+q}) \\ &< \psi(p_i)^{3+3q}\delta_i^{4+3q} \\ &< \zeta\vartheta^{(4+3q)^{i+1}}. \end{aligned}$$

Hence $\psi(p_k)\varphi_3(p_k,q_k) \leq \zeta\vartheta^{(4+3q)^k}$ holds for all $k \geq 0$. By using this inequality, we derive

$$\prod_{i=0}^{k}\psi(p_i)\varphi_3(p_i,q_i) \leq \prod_{i=0}^{k}\zeta\vartheta^{(4+3q)^k} \leq \zeta^{k+1}\vartheta^{\left(\frac{(4+3q)^{k+1}-1}{3+3q}\right)},$$

which is same as the equation (3.21). Now, from the relations (2.6) and (3.21), we have

$$\sigma_k = \sigma_{k-1}\psi(p_{k-1})\varphi_3(p_{k-1},q_{k-1}) = \sigma\prod_{i=0}^{k-1}\psi(p_i)\varphi_3(p_i,q_i) \leq \zeta^{k+1}\vartheta^{\left(\frac{(4+3q)^k-1}{3+3q}\right)}.$$

Since $\zeta, \vartheta < 1$, then $\sigma_k \to 0$ as $k \to \infty$. Suppose, we presume

$$\varpi = \sum_{i=k}^{k+m}\zeta^i\vartheta^{\frac{(4+3q)^i}{3+3q}},\ \ k \geq 0, m \geq 1.$$

On rewriting the above equation, we have

$$\begin{aligned} \varpi &\leq \zeta^k\vartheta^{\frac{(4+3q)^k}{3+3q}} + \vartheta^{(4+3q)^k}\left(\sum_{i=k+1}^{k+m}\zeta^k\vartheta^{\frac{(4+3q)^{i-1}}{3+3q}}\right) \\ &\leq \zeta^k\vartheta^{\frac{(4+3q)^k}{3+3q}} + \zeta\vartheta^{(4+3q)^k}\left(\varpi - \zeta^{k+m}\vartheta^{\frac{(4+3q)^{k+m}}{3+3q}}\right) \end{aligned}$$

Also, we have

$$\varpi \leq \zeta^k\vartheta^{\frac{(4+3q)^k}{3+3q}}\left(\frac{1-\zeta^{m+1}\vartheta^{\frac{(4+3q)^k((4+3q)^m+2+3q)}{3+3q}}}{1-\zeta\vartheta^{(4+3q)^k}}\right).$$

Furthermore, we can derive apriori error estimate

$$\|x_k - x^*\| \leq \frac{\phi(p_0)\sigma}{\vartheta^{\frac{1}{3+3q}}\left(1-\psi(p_0)\varphi_3(p_0,q_0)\right)}\vartheta^{\frac{(4+3q)^k}{3+3q}},$$

which states that the R-order of convergence of the scheme (1.2) is at least $4+3q, q \in [0,1]$ and particularly, when Δ'' satisfies Lipschitz condition then the R-order turns out to be seven.

□

Example 10. [15] Consider a mixed Hammerstein type nonlinear integral equation which is given as

$$x(s) = 1 + \int_0^1 M(s,t)\left(\frac{1}{3}x(t)^{\frac{7}{3}} + \frac{5}{8}x(t)^3\right)dt, s \in [0,1], \tag{5.6}$$

where $x \in [0,1], t \in [0,1]$ and M is the Green function defined as

$$M(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t) & s \leq t. \end{cases} \tag{5.7}$$

Solving the equation (5.6) is same as to get the solution for $\Delta(x) = 0$ where, $\Delta : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[\Delta(x)](s) = x(s) - 1 - \int_0^1 M(s,t)\left(\frac{1}{3}x(t)^{\frac{7}{3}} + \frac{5}{8}x(t)^3\right)dt, s \in [0,1], \tag{5.8}$$

On choosing $\Omega_0 = D(0,2)$. The Fréchet derivatives of Δ are defined as,

$$\Delta'(x)y(s) = y(s) - 1 - \int_0^1 M(s,t)\left(\frac{7}{9}x(t)^{\frac{4}{3}} + \frac{15}{8}x(t)^2\right)y(t)dt, y \in \Omega_0, \tag{5.9}$$

$$\Delta''(x)yz(s) = -\int_0^1 M(s,t)\left(\frac{28}{27}x(t)^{\frac{1}{3}} + \frac{15}{4}x(t)\right)y(t)z(t)dt, y,z \in \Omega_0. \tag{5.10}$$

Thus, we have

$$\|\Delta''(x)\| \leq \frac{7 \times 2^{\frac{1}{3}}}{54} + \frac{15}{16},$$

and

$$\|\Delta''(x) - \Delta''(u)\| \leq \frac{7}{54}\|x-u\|^{\frac{1}{3}} + \frac{15}{32}\|x-u\|,$$

Here, we are assuming the max norm. We can let $\omega(z) = \frac{7}{54}z^{\frac{1}{3}} + \frac{15}{32}z, \varphi(z) = z^{\frac{1}{3}}$ and $q = \frac{1}{3}$. Choosing $x_0(t) = 1$ to be an initial approximation, we can find

$$\|\Delta(x_0)\| \leq \frac{23}{192}, \|\Gamma_0\| \leq \frac{576}{385} \equiv \tau, \|\Gamma_0\Delta(x_0)\| \leq \frac{69}{385} \equiv \sigma.$$

Thus, $p_0\phi(p_0) \leq 0.632298 < 1$ and $\psi(p_0)^2\varphi_3(p_0, q_0) \leq 0.035077 < 1$. This shows that, the assumptions of Theorem 8 are true. Hence, the solution $x^* \in \overline{D(x_0, \kappa\sigma)} = \overline{D(1, 0.388938)} \subset \Omega$ and it is unique in $D(1, 0.825431) \cap \Omega$.

Example 11. [15] Consider the problem that find the minimizer of the chained Rosenbrock function,

$$R(X) = \sum_{i=1}^{m}\left[4\left(x_i - x_{i+1}^2\right)^2 + \left(1 - x_{i+1}^2\right)\right], X \in R^m. \tag{5.11}$$

To find the minimum of R, we need to calculate the nonlinear system $\Delta(x) = 0$, where $\Delta(x) = \nabla R(X)$. Here, we will use the algorithm (1.2) and compare it with the method given by Wang and Kou presented in [15] which is denoted by WKM. Let the initial approximation $x_i = 1.3, i = 1, 2, ..., m$. Also, we take $m = 100$. Displayed in Table 1 are the norms $\|x_k - x^*\|_2$ at each iterative step, where $x^* = (1, 1, \cdots 1)$.

Table 1: Comparison of the norm for vector function

k	Method (1.2)	WKM
0	3.0000e+00	3.0000e+00
1	0.59254e-01	1.4293e-01
2	2.7482e-10	1.9326e-05

From the above table, we can see that the norm for vector function for the method (1.2) seems to be more finer than the method WKM, which is also of seventh order. Hence, the considered method is more efficient.

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