Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 15 (2020), 257 – 279

ON HERMITE INTERPOLATION AND DIVIDED DIFFERENCES

François Dubeau

Abstract. This paper is a survey of topics related to Hermite interpolation. In the first part we present the standard analysis of the Hermite interpolation problem. Existence, uniqueness and error formula are included. Then some computational aspects are studied including Leibnitz' formula and devided differences for monomials. Moreover continuity and differentiation properties of divided differences are analyzed. Finally we represent Hermite polynomial with respect to different basis and give links between them.

1 Introduction

This paper is a survey of topics related to Hermite interpolation. We present an accessible treatment of the Hermite interpolation problem and some related topics. We have selected simple proofs for the results presented in the text.

In Section 2 we present the standard analysis of the Hermite interpolation problem. Existence and uniqueness of Hermite polynomial, its representation with respect to Newton basis, the definition of the divided differences, and error terms are presented. In Section 3 some computational aspects are studied. Among them are the recursive calculation of divided differences, Leibnitz' formula, and computation of divided differences for monomials. In Section 4, continuity and differentiation properties of divided differences are analyzed. In Section 5 we represent Hermite polynomial with respect to different basis and give links between them.

2020 Mathematics Subject Classification: 41A05; 65D05; 26C99

Keywords: Hermite interpolation; Hermite interpolating polynomial; divided differences; Leibnitz' formula; monomials; representation of Hermite polynomial

This work was supported by an individual discovery grant from NSERC (Natural Sciences and Engineering Research Council of Canada

2 Hermite interpolation

2.1 Hermite interpolation problem

let us consider a sequence of points x_0, x_1, …, x_n, distinct or not, such that any point z in the sequence appears $\alpha(z)+1$ times. So we have

$$\sum_{\text{distinct } z\in\{x_0,x_1,\dots,x_n\}} (\alpha(z)+1) = n+1.$$

We say that two functions $f(x)$ and $g(x)$ agree at the points x_0, x_1, …, x_n, or $g(x)$ agrees with $f(x)$, in case

$$f^{(l)}(z) = g^{(l)}(z) \qquad (l = 0, 1, \dots, \alpha(z))$$

for any point z which occurs $\alpha(z)+1$ times in the sequence x_0, x_1, …, x_n.

In fact $f(x)$ and $g(x)$ agree at the points $x_0, x_1, \dots, x_n$ if and only if the difference $f(x)-g(x)$ has the zeros x_0, x_1, …, x_n counting multiplicities. The values $f^{(l)}(z)$ could be only data not related to any function. In this case we say that the function $g(x)$ agree with the data.

Problem 1. *[4, 11] The **Hermite interpolation problem** is to find the least degree polynomial $p(x)$ which agree with $f(x)$ at the points x_0, x_1, …, x_n. If this polynomial exists it will be called the **Hermite interpolating polynomial**, or shortly **Hermite polynomial**.*

2.2 Hermite polynomial and divided differences

For the Hermite interpolation problem there are $n+1$ conditions, so it is normal to look for a polynomial $p_n(x) \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of polynomials of degree at most n. The following result about existence and uniqueness of $p_n(x)$ has some different proofs, see for example [2–4, 11].

Theorem 2. *There exists a unique polynomial $p_n(x) \in \mathcal{P}_n$ which agrees with $f(x)$ at the points x_0, x_1, …, x_n.*

Proof. Let us consider the representation of $p_n(x)$ with respect to the monomial basis 1, x, …, x^n as

$$p_n(x) = \sum_{k=0}^{n} a_k x^k.$$

The $n+1$ conditions lead to a linear system of $n+1$ equations with $n+1$ unknows, the a_k's. It is enough to show that the unique solution of the homogeneous linear system is the trivial solution $a_k = 0$ for $k = 0\dots,n$. The conditions on the homogeneous system imply that $p_n(x)$ has at least $n+1$ zeros counting the multiplicity. This is possible for $p_n(x) \in \mathcal{P}_n$ only for $p_n(x) = 0$ for all x, so $a_k = 0$ for $k = 0, \dots, n$. □

Since the coefficient a_n of x^n depends only on $f(x)$ and the sequence of points $x_0, x_1, \ldots, x_n$, we use the notation

$$a_n = f[x_0, x_1, \ldots, x_n],$$

and consider the following definition.

Definition 3. *[6, 8] The **n-th order divided difference**, $f[x_0, \ldots, x_n]$, is the coefficient of x^n of the Hermite polynomial $p_n(x)$ which agree with $f(x)$ at the points $x_0, x_1, \ldots, x_n$.*

If $\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ is a permutation, from the uniqueness of $p_n(x)$, and since the conditions on $x_0, x_1, \ldots, x_n$ and on $x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n)}$, are the same, we have

$$f[x_0, x_1, \ldots, x_n] = f\left[x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right],$$

which means that the coefficient of x^n does not depends on the order of the x_i's.

Because a_n comes from the solution of a linear system, we have directly the following linearity property.

Theorem 4. *Linearity: let $f(x)$ and $g(x)$ be two functions and λ a constant, then*

$$(f + \lambda g)[x_0, x_1, \ldots, x_n] = f[x_0, x_1, \ldots, x_n] + \lambda g[x_0, x_1, \ldots, x_n].$$

2.3 Newton form of the Hermite polynomial

There are several possible representations of $p_n(x)$, each representation depends on the choice of the basis for $\mathcal{P}_n$. One basis if well suited for a recursive computation of $p_n(x)$, it is the Newton basis.

The Newton representation of $p_n(x)$ is based on the set of polynomials $\{\pi_k(x)\}_{k=0}^n$, given by

$$\pi_0(x) = 1,$$

and

$$\pi_k(x) = \prod_{j=0}^{k-1}(x - x_j) \qquad \text{for} \quad k = 1, \ldots, n,$$

such that the degree of $\pi_k(x)$ is k. These $n + 1$ polynomials of increasing degree from 0 up to n form a basis of $\mathcal{P}_n$. So we can write

$$p_n(x) = \sum_{k=0}^{n} \gamma_k \pi_k(x).$$

Expanding $\pi_n(x)$, we get

$$p_n(x) = \gamma_n x^n + u_{n-1}(x)$$

where $u_{n-1}(x)$ is a polynomial of degree at most $n-1$. It follows that

$$\gamma_n = a_n.$$

The Hermite polynomial $p_n(x)$ can be determined recursively. Indeed if $p_{n-1}(x)$ is already determined with the conditions on x_0, x_1, $\dots$, x_{n-1}, then we set

$$p_n(x) = p_{n-1}(x) + f\,[x_0, x_1, \dots, x_n]\,\pi_n(x).$$

This $p_n(x)$ satisfies all the conditions on x_0, x_1, $\dots$, x_{n-1} because the last term will be zero for these conditions. The new condition at x_n is used to find $f\,[x_0, x_1, \dots, x_n]$. If $z = x_n$ occurs $\alpha+1$ times in the sequence x_0, x_1, $\dots$, x_n, we have to consider the supplementary condition

$$p_n^{(\alpha)}(z) = f^{(\alpha)}(z)$$

for $z = x_n$. Since

$$p_n^{(\alpha)}(x_n) = p_{n-1}^{(\alpha)}(x_n) + f\,[x_0, x_1, \dots, x_n]\,\pi_n^{(\alpha)}(x_n)$$

with $\pi_n^{(\alpha)}(x_n) \neq 0$, then

$$f\,[x_0, x_1, \dots, x_n] = \frac{f^{(\alpha)}(x_n) - p_{n-1}^{(\alpha)}(x_n)}{\pi_n^{(\alpha)}(x_n)}.$$

So, using a recursive argument, we can write

$$p_n(x) = \sum_{k=0}^{n} f\,[x_0, x_1, \dots, x_k]\,\pi_k(x).$$

Let us remark that for the Taylor expansion, which is the case $x_0 = x_1 = \dots = x_n = \xi$ with $\alpha(\xi) = n$, we have

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\xi)}{k!}(x-\xi)^k,$$

so

$$f[\underbrace{\xi, \dots, \xi}_{(k+1)-times}] = \frac{f^{(k)}(\xi)}{k!}$$

for $k = 0, \dots, n$.

2.4 The error of the Hermite polynomial

In this section we recall the standard result about the error of interpolation which appears for example in [2–4].

Theorem 5. *Let us suppose $f(x) \in \mathcal{C}^{n+1}(\mathbb{R})$ and x_0, x_1, $\ldots$, x_n be given, distinct or not. For any $x \in \mathbb{R}$ there exists*

$$\xi_x \in [\min\{x, x_0, x_1, \ldots, x_n\}, \max\{x, x_0, x_1, \ldots, x_n\}]$$

such that

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \pi_{n+1}(x).$$

Proof. The result clearly holds for $x = x_i$. Let us fix $x \neq x_i$ for $i = 0, \ldots, n$. For the Hermite polynomial $p_{n+1}(t)$, which agrees with $f(x)$ on x, x_0, $\ldots$, x_n, we have

$$p_{n+1}(t) = p_n(t) + \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t).$$

Hence $F(t) = f(t) - p_{n+1}(t)$ has $n+2$ zeros at x, x_0, $\ldots$, x_n counting the multiplicity. It follows that there exists ξ_x such that $F^{(n+1)}(\xi_x) = 0$. But

$$\begin{aligned} F^{(n+1)}(t) &= f^{(n+1)}(t) - p_{n+1}^{(n+1)}(t) \\ &= f^{(n+1)}(t) - (n+1)! \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}. \end{aligned}$$

The result follows when we set $t = \xi_x$ in this last expression and $t = x$ in $p_{n+1}(t)$. □

Let x be different from any x_i's. The Hermite polynomial of $f(x)$ on x, x_0, $\ldots$, x_n is

$$p_{n+1}(t) = p_n(t) + f[x, x_0, \ldots, x_n] \pi_{n+1}(t).$$

Hence for $t = x$ we get

$$f(x) = p_n(x) + f[x, x_0, \ldots, x_n] \pi_{n+1}(x).$$

From the preceding result we also have

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \pi_{n+1}(x)$$

for $\xi_x \in [\min\{x, x_0, \ldots, x_n\}, \max\{x, x_0, \ldots, x_n\}]$. So we have

$$f[x, x_0, \ldots, x_n] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

As a consequence, if the x_i's are not all equal we get

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for $\xi \in [\min\{x_0, \ldots, x_n\}, \max\{x_0, \ldots, x_n\}]$. So we have proved the following result.

Theorem 6. *Assume that $f(x) \in \mathcal{C}^n(\mathbb{R})$, and let x_0, ..., x_n be points distinct or not. Then there exists*

$$\xi \in [\min \{x_0, \dots, x_n\}, \max \{x_0, \dots, x_n\}]$$

such that

$$f[x_0, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

3 On the computation of Divided differences

3.1 Recursive computation of divided differences

A recursive way to compute the divided differences is a simple consequence of the following result.

Theorem 7. *[1, 8] We always have*

$$((\bullet - x_{n+1})f)[x_0, x_1, \dots, x_{n+1}] = f[x_0, x_1, \dots, x_n],$$

or equivalently

$$((\bullet)f)[x_0, x_1, \dots, x_{n+1}] = f[x_0, x_1, \dots, x_n] + x_{n+1} f[x_0, x_1, \dots, x_{n+1}].$$

In this theorem and below, an expression like $(\bullet - x_k)f)$ applied to x means $(x - x_k)f(x)$.

Proof. Suppose that $f(x)$ and $g(x)$ agree on x_0, x_1, ..., x_n, then

$$f[x_0, x_1, \dots, x_n] = g[x_0, x_1, \dots, x_n].$$

Let x_{n+1} be arbitrary. Then $(x - x_{n+1})f(x)$ and $(x - x_{n+1})g(x)$ agree also on x_0, x_1, ..., x_n, x_{n+1} because, using Leibnitz' rule to compute the l-th derivative, we get

$$\begin{aligned} \frac{d^l}{dz^l}(z - x_{n+1})f(z) &= (z - x_{n+1})f^{(l)}(z) + l f^{(l-1)}(z) \\ &= (z - x_{n+1})g^{(l)}(z) + l g^{(l-1)}(z) \\ &= \frac{d^l}{dz^l}(z - x_{n+1})g(z) \end{aligned}$$

for any z in x_0, x_1, ..., x_n, x_{n+1} and $l = 0, \dots, \alpha(z)$. It follows that

$$((\bullet - x_{n+1})f)[x_0, x_1, \dots, x_{n+1}] = ((\bullet - x_{n+1})g)[x_0, x_1, \dots, x_{n+1}].$$

Now, replace $g(x)$ by the Hermite polynomial $p_n(x)$ of $f(x)$ on $x_0, x_1, \ldots, x_n$. Then $(x-x_{n+1})f(x)$ and $(x-x_{n+1})p_n(x)$ agree on $x_0, x_1, \ldots, x_n, x_{n+1}$, so $(x-x_{n+1})p_n(x)$ is the Hermite polynomial of $(x-x_{n+1})f(x)$ on $x_0, x_1, \ldots, x_n, x_{n+1}$. Then, we have

$$((\bullet - x_{n+1})p_n)\,[x_0, x_1, \ldots, x_{n+1}] = f\,[x_0, x_1, \ldots, x_n]\,,$$

because the coefficient of x^{n+1} of $(x-x_{n+1})p_n(x)$ is the coefficient of x^n of $p_n(x)$. The result follows. □

Now considering the following two relations

$$\begin{aligned} ((\bullet - x_n)f)\,[x_0, x_1, \ldots, x_n] &= f\,[x_0, x_1, \ldots, x_{n-1}]\,, \\ ((\bullet - x_0)f)\,[x_0, x_1, \ldots, x_n] &= f\,[x_1, \ldots, x_n]\,, \end{aligned}$$

by substraction and linearity we get

$$(x_n - x_0)f\,[x_0, x_1, \ldots, x_n] = f\,[x_1, \ldots, x_n] - f\,[x_0, x_1, \ldots, x_{n-1}]\,.$$

In summary, we have

$$f\,[x_0, \ldots, x_n] = \begin{cases} \frac{f^{(n)}(x)}{n!} & \text{if} \quad x_0 = x_1 = \cdots = x_n = x, \\ \frac{f[x_1,\ldots,x_n] - f[x_0,x_1,\ldots,x_{n-1}]}{x_n - x_0} & \text{if} \quad x_0 \neq x_n, \end{cases}$$

which is a way to recursively generates $f\,[x_0, \ldots, x_n]$ for $n = 0, 1, 2, \ldots$

3.2 An identity

An interesting identity, related to multivariate B-spline, was obtained in [9] as a consequence of Theorem 7. A simple proof, which we present here, was given in [8].

Theorem 8. *let* $\sum_{k=0}^{n} \lambda_k = 1$, *and* $\sum_{k=0}^{n} \lambda_k x_k = x$, *then*

$$\sum_{k=0}^{n} \lambda_k f\,[x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n] = f\,[x_0, \ldots, x_n]\,.$$

Proof. We have

$$\begin{aligned} f\,[x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n] &= f\,[x, x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n] \\ &= ((\bullet - x_k)f)\,[x, x_0, \ldots, x_n]\,. \end{aligned}$$

So, with the assumption on the λ_k's we obtain

$$\begin{aligned} \sum_{k=0}^{n} \lambda_k f\,[x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n] &= \left(\sum_{k=0}^{n} \lambda_k (\bullet - x_k) f \right) [x, x_0, \ldots, x_n] \\ &= ((\bullet - x)f)\,[x, x_0, \ldots, x_n] \\ &= f\,[x_0, \ldots, x_n] \end{aligned}$$

□

3.3 Leibnitz' formula

Leibnitz' formula for divided differences was obtained in [5]. A simpler way to obtain this rule, suggested in [8], is presented below.

Theorem 9. *Leibnitz' formula. For any two functions $f(x)$ and $g(x)$, and a sequence of points x_0, x_1, $\dots$, x_n distinct or not, we have*

$$(fg)\left[x_0, x_1, \dots, x_n\right] = \sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] g\left[x_k, \dots, x_n\right].$$

Proof. Using $\pi_k(x) = \prod_{j=0}^{k-1}(x - x_j)$, and applying Theorem 7, we get

$$(\pi_k f)\left[x_0, x_1, \dots, x_n\right] = \left(\prod_{j=0}^{k-1}(\bullet - x_j) f\right)\left[x_0, x_1, \dots, x_n\right] = f\left[x_k, \dots, x_n\right].$$

Now, let us consider the Hermite polynomial $p_n(x) = \sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] \pi_k(x)$, which agrees with $f(x)$ on x_0, x_1, $\dots$, x_n. For any function $g(x)$, $f(x)g(x)$ and $p_n(x)g(x)$ also agree on x_0, x_1, $\dots$, x_n. We have

$$\begin{aligned}
(fg)\left[x_0, x_1, \dots, x_n\right] &= (p_n g)\left[x_0, x_1, \dots, x_n\right] \\
&= \left(\sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] \pi_k g\right)\left[x_0, x_1, \dots, x_n\right] \\
&= \sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] (\pi_k g)\left[x_0, x_1, \dots, x_n\right] \\
&= \sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] g\left[x_k, \dots, x_n\right],
\end{aligned}$$

hence we have the result. □

Not only we get the Leibnitz' formula but we also have the Hermite polynomial of $(fg)(x)$ on x_0, x_1, $\dots$, x_n.

Theorem 10. *For any two functions $f(x)$ and $g(x)$ and a sequence of points x_0, x_1, $\dots$, x_n distinct or not, let $q_{n-k}(x)$ be the Hermite polynomial of $g(x)$ on x_k, $\dots$, x_n. Then*

$$P_n(x) = \sum_{k=0}^{n} f\left[x_0, \dots, x_k\right] \pi_k(x) q_{n-k}(x)$$

is the Hermite polynomial of $(fg)(x)$ on x_0, $\dots$, x_n.

Proof. Indeed, suppose z appears $\alpha(z)+1$ times in $x_0, \ldots, x_n$. Take any l such that $0 \le l \le \alpha(z)$. Applying the Leibnitz' rule for the derivatives, we get

$$\begin{aligned} P_n^{(l)}(z) &= \sum_{k=0}^{n} f\left[x_0, \ldots, x_k\right] \left(\pi_k q_{n-k}\right)^{(l)}(z) \\ &= \sum_{k=0}^{n} f\left[x_0, \ldots, x_k\right] \sum_{j=0}^{l} \binom{l}{j} \pi_k^{(j)}(z) q_{n-k}^{(l-j)}(z) \\ &= \sum_{j=0}^{l} \binom{l}{j} \sum_{k=0}^{n} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) q_{n-k}^{(l-j)}(z). \end{aligned}$$

For each j, let x_i be the $(j+1)$'th occurence of z in $x_0, \ldots, x_n$. Obviously $i \ge j$. For $k \ge i+1$

$$\pi_k(x) = (x-z)^{j+1} u_{k-(j+1)}(x)$$

where $u_{k-(l+1)}(x)$ is a polynomial of degree $k-(j+1) \ge 0$, hence $\pi_k^{(j)}(z) = 0$. Moreover, for each $k \le i$, z occurs at least $\alpha(z)-j+1$ on $x_k, \ldots, x_n$, and since $l-j \le \alpha(z)-j$, we have $q_{n-k}^{(l-j)}(z) = g^{(l-j)}(z)$. So we have successively

$$\begin{aligned} \sum_{k=0}^{n} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) q_{n-k}^{(l-j)}(z) &= \sum_{k=0}^{i} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) q_{n-k}^{(l-j)}(z) \\ &= \sum_{k=0}^{i} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) g^{(l-j)}(z) \\ &= g^{(l-j)}(z) \sum_{k=0}^{i} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) \\ &= g^{(l-j)}(z) \sum_{k=0}^{n} f\left[x_0, \ldots, x_k\right] \pi_k^{(j)}(z) \\ &= g^{(l-j)}(z) p_n^{(j)}(z) \\ &= g^{(l-j)}(z) f^{(j)}(z). \end{aligned}$$

It follows that

$$\begin{aligned} P_n^{(l)}(z) &= \sum_{j=0}^{l} \binom{l}{j} f^{(j)}(z) g^{(l-j)}(z) \\ &= (fg)^{(l)}(z) \end{aligned}$$

for $l = 0, \ldots, \alpha(z)$. □

3.4 Divided differences for monomials

In this section we present expressions for the divided differences of x^l not only for $l \geq 0$ as it is done in [10] for distinct points, but also for $l < 0$. These expressions are also based on Theorem 7.

Theorem 11. *For $n \geq 0$ and $l \geq 0$ we have*

$$\left((\bullet)^{n+l+1} f\right)[x_0, \dots, x_n] = \sum_{i=0}^{n} x_i \left((\bullet)^{i+l} f\right)[x_0, \dots, x_i].$$

Proof. From the following expression

$$x^{n+l+1} f(x) = (x - x_n) x^{n+l} f(x) + x_n x^{n+l} f(x),$$

and using Theorem 7, we obtain the following formula

$$\begin{aligned} \left((\bullet)^{n+l+1} f\right)[x_0, \dots, x_n] &= \left((\bullet - x_n)(\bullet)^{n+l} f\right)[x_0, \dots, x_n] + x_n \left((\bullet)^{n+l} f\right)[x_0, \dots, x_n] \\ &= \left((\bullet)^{n+l} f\right)[x_0, \dots, x_{n-1}] + x_n \left((\bullet)^{n+l} f\right)[x_0, \dots, x_n] \\ &\vdots \\ &= \sum_{i=0}^{n} x_i \left((\bullet)^{i+l} f\right)[x_0, \dots, x_i]. \end{aligned}$$

□

As an application of both the definition of divided differences and the preceding result, we get the following expressions for the powers x^l for $l \geq 0$. Since we have $f[x_0] = f(x_0)$, we consider $n > 0$.

Theorem 12. *For $n \geq 1$ and $l \geq 0$ we have*

(i) $(\bullet)^l [x_0, \dots, x_n] = 0$ *for* $l = 0, \dots, n-1$;

(ii) $(\bullet)^n [x_0, \dots, x_n] = 1$;

(iii) $(\bullet)^{n+l+1} [x_0, \dots, x_n] = \sum_{\substack{(l_0 \geq 0, \dots, l_n \geq 0) \\ l_0 + \dots + l_n = l+1}} \prod_{i=0}^{n} x_i^{l_i}$ *for* $l = 0, 1, \dots$.

Proof. (i) and (ii) follow directly from the definition of the divided differences, while (iii) is a direct consequence of the preceding result with $f(x) = 1$. □

For the negative power of x, the x_i's must not vanish, so we assume $x_i \neq 0$ for $i = 0, \ldots, n$.

Theorem 13. *For $f(x) = 1/x$ we have*

$$\left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n] = \frac{(-1)^n}{\prod_{i=0}^n x_i}.$$

Proof. Using Leibnitz' formula and the preceding result, we have

$$\begin{aligned}
1\,[x_0, \ldots, x_n] &= \left(\left(\frac{1}{\bullet}\right)\bullet\right)[x_0, \ldots, x_n] \\
&= \sum_{i=0}^n \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_i]\,(\bullet)\,[x_i, \ldots, x_n] \\
&= \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_{n-1}]\,(\bullet)\,[x_{n-1}, x_n] + \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]\,(\bullet)\,[x_n] \\
&= \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_{n-1}] + x_n \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]
\end{aligned}$$

so for $n \geq 1$ we have

$$\begin{aligned}
\left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n] &= -\frac{1}{x_n}\left(\frac{1}{\bullet}\right)[x_0, \ldots, x_{n-1}] \\
&\ \ \vdots \\
&= \frac{(-1)^n}{\prod_{i=0}^n x_i}.
\end{aligned}$$

This formula also holds for $n = 0$. □

Let us observe that from Theorem 11 we have

$$\left((\bullet)^{n+l+1} f\right)\left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right] = \sum_{i=0}^n \frac{1}{x_i}\left((\bullet)^{i+l} f\right)\left[\frac{1}{x_0}, \ldots, \frac{1}{x_i}\right].$$

Divided differences for the negative powers of x are consequences of next two results.

Theorem 14. *For $l \geq 0$ and any $n \geq 0$ we have*

$$\left(\frac{1}{\bullet^{l+1}}\right)[x_0, \ldots, x_n] = \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]\,(\bullet)^{n+l}\left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right].$$

Proof. The proof is by induction on l.

Step 1. For $l = 0$ and any $n \geq 0$ we have

$$\left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n] = \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]\,(\bullet)^n \left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right],$$

because

$$(\bullet)^n \left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right] = 1.$$

Step 2. Assume we have the result for an $l - 1 \geq 0$ and any $n \geq 0$, so

$$\left(\frac{1}{\bullet^l}\right)[x_0, \ldots, x_n] = \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]\,(\bullet)^{n+l-1} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right].$$

For l, using Leibnitz' formula and the induction assumption, we have

$$\begin{aligned}
\left(\frac{1}{\bullet^{l+1}}\right)[x_0, \ldots, x_n] &= \sum_{i=0}^{n} \left(\frac{1}{\bullet^l}\right)[x_0, \ldots, x_i] \left(\frac{1}{\bullet}\right)[x_i, \ldots, x_n] \\
&= \sum_{i=0}^{n} \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_i]\,(\bullet)^{i+l-1} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_i}\right] \left(\frac{1}{\bullet}\right)[x_i, \ldots, x_n] \\
&= \sum_{i=0}^{n} \frac{(-1)^i}{\prod_{j=0}^{i} x_j} \frac{(-1)^{n-i}}{\prod_{j=i}^{n} x_j} (\bullet)^{i+l-1} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_i}\right] \\
&= \frac{(-1)^n}{\prod_{j=0}^{n} x_j} \sum_{i=0}^{n} \frac{1}{x_i} (\bullet)^{i+l-1} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_i}\right] \\
&= \frac{(-1)^n}{\prod_{j=0}^{n} x_j} (\bullet)^{n+l} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right] \\
&= \left(\frac{1}{\bullet}\right)[x_0, \ldots, x_n]\,(\bullet)^{n+l} \left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right].
\end{aligned}$$

□

Finally, from Theorems 12, 13, and 14, we obtain the following expression for the divided differences of $f(x) = 1/x^{l+1}$.

Theorem 15. *For $l \geq 0$ and any $n \geq 0$ we have*

$$\left(\frac{1}{\bullet^{l+1}}\right)[x_0, \ldots, x_n] = \frac{(-1)^n}{\prod_{i=0}^{n} x_i} \sum_{\substack{(l_0 \geq 0, \ldots, l_n \geq 0) \\ l_0 + \cdots + l_n = l}} \frac{1}{\prod_{i=0}^{n} x_i^{l_i}}.$$

□

4 On the regularity of divided differences

4.1 Continuity

The next result concerns the continuity of the divided differences with respect to the points $x_0, \ldots, x_n$. The proof we present is an adaptation of the proof given in [2].

Theorem 16. *Assume that $f(x) \in \mathcal{C}^n(\mathbb{R})$ and let $x_0, \ldots, x_n$ be points, distinct or not. If for each k, $x_{0k}, \ldots, x_{nk}$ are $n+1$ points and $\lim_{k\to\infty} x_{ik} = x_i$, $i = 0, \ldots, n$, then*

$$\lim_{k\to\infty} f[x_{0k}, \ldots, x_{nk}] = f[x_0, \ldots, x_n].$$

Proof. We proceed by induction on n.

Step 1. For $n = 0$ the result holds because $f[x] = f(x)$.

Step 2. Suppose the result holds for $n-1$. Now for n we proceed as follows.

Step 2a. We first show the result for x_i's not all equal. We suppose $x_0 \neq x_n$ and we assume that $x_{0k} \neq x_{nk}$ for large k. Then

$$\begin{aligned}
\lim_{k\to\infty} f[x_{0k}, \ldots, x_{nk}] &= \lim_{k\to\infty} \frac{f[x_{1k}, \ldots, x_{nk}] - f\left[x_{0k}, \ldots, x_{(n-1)k}\right]}{x_{nk} - x_{0k}} \\
&= \frac{\lim_{k\to\infty} f[x_{1k}, \ldots, x_{nk}] - \lim_{k\to\infty} f\left[x_{0k}, \ldots, x_{(n-1)k}\right]}{\lim_{k\to\infty}(x_{nk} - x_{0k})} \\
&= \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \\
&= f[x_0, \ldots, x_n]
\end{aligned}$$

where the limits exist by induction hypothesis.

Step 2b. We show now the result for $x = x_0 = \ldots = x_n$. We have

$$\begin{aligned}
\lim_{k\to\infty} f[x_{0k}, \ldots, x_{nk}] &= \lim_{k\to\infty} \frac{f^{(n)}(\xi_k)}{n!} \\
&= \frac{f^{(n)}(x)}{n!} \\
&= f[x_0, \ldots, x_n]
\end{aligned}$$

because $\xi_k \in [\min\{x_{0k}, \ldots, x_{nk}\}, \max\{x_{0k}, \ldots, x_{nk}\}]$, and

$$\lim_{k\to\infty} [\min\{x_{0k}, \ldots, x_{nk}\}, \max\{x_{0k}, \ldots, x_{nk}\}] = \{x\}.$$

So the result holds for n. $\square$

4.2 Differentiation

Divided differences are also differentiable with repect to the points x_i's.

Theorem 17. *Let* $g_n(x) = f[x_0, \dots, x_n, x]$ *continuous for* $f(x) \in \mathcal{C}^{n+1}(\mathbb{R})$. *For* $l \geq 0$ *and* $f(x) \in \mathcal{C}^{n+l+1}(\mathbb{R})$, *we have*

$$f[x_0, \dots, x_n, \underbrace{x, \dots, x}_{(l+1)-times}] = \frac{g_n^{(l)}(x)}{l!}.$$

Proof. Since

$$\begin{aligned} \frac{g_n(x+h) - g_n(x)}{h} &= \frac{f[x_0, \dots, x_n, x+h] - f[x_0, \dots, x_n, x]}{h} \\ &= f[x_0, \dots, x_n, x, x+h], \end{aligned}$$

we have

$$\begin{aligned} g_n'(x) &= \lim_{h \to 0} \frac{g_n(x+h) - g_n(x)}{h} \\ &= \lim_{h \to 0} f[x_0, \dots, x_n, x, x+h] \\ &= f[x_0, \dots, x_n, x, x]. \end{aligned}$$

Now let us suppose the result holds for $l-1$,

$$g_n^{(l-1)}(x) = (l-1)! f[x_0, \dots, x_n, \underbrace{x, \dots, x}_{l-times}],$$

then

$$g_n^{(l-1)}(x+h) - g_n^{(l-1)}(x) = (l-1)! \left[f[x_0, ., x_n, \underbrace{x+h, ., x+h}_{l-times}] - f[x_0, ., x_n, \underbrace{x, ., x}_{l-times}] \right]$$

$$= \quad (l-1)! \times$$

$$\sum_{i=0}^{l-1} \left[f[x_0, ., x_n, \underbrace{x, ., x}_{i-times}, \underbrace{x+h, ., x+h}_{(l-i)-times}] - f[x_0, ., x_n, \underbrace{x, ., x}_{(i+1)-times}, \underbrace{x+h, ., x+h}_{(l-(i+1))-times}] \right]$$

$$= \quad (l-1)! h \sum_{i=0}^{l-1} f[x_0, ., x_n, \underbrace{x, ., x}_{(i+1)-times}, \underbrace{x+h, ., x+h}_{(l-i)-times}].$$

Hence

$$\begin{aligned}
g_n^{(l)}(x) &= \lim_{h\to 0}\frac{g_n^{(l-1)}(x+h)-g_n^{(l-1)}(x)}{h}\\
&= (l-1)!\sum_{i=0}^{l-1}\lim_{h\to 0} f[x_0,\dots,x_n,\ \underbrace{x,\dots,x}_{(i+1)-times},\underbrace{x+h,\dots,x+h}_{(l-i)-times}]\\
&= (l-1)!\sum_{i=0}^{l-1} f[x_0,\dots,x_n,\ \underbrace{x,\dots,x}_{(l+1)-times}\ ]\\
&= l!f[x_0,\dots,x_n,\ \underbrace{x,\dots,x}_{(l+1)-times}\ ].
\end{aligned}$$

□

Now a direct consequence of this last result is the next proposition.

Theorem 18. *Let us suppose that there are $r+1$ different values x_{i_j} for $j=0,\dots,r$ which each appears $\alpha(x_{i_j})+1$ times in the sequence $x_0,\ \dots,\ x_n$, in such a way that*

$$\sum_{j=0}^{r}\big(\alpha(x_{i_j})+1\big)=\sum_{j=0}^{r}\alpha(x_{i_j})+(r+1)=n+1.$$

For $f(x)\in\mathcal{C}^n(\mathbb{R})$, we have

$$\begin{aligned}
\frac{\partial^{\alpha(x_{i_0})}}{\partial x_{i_0}^{\alpha(x_{i_0})}}\cdots\frac{\partial^{\alpha(x_{i_r})}}{\partial x_{i_r}^{\alpha(x_{i_r})}} f[x_{i_0},\dots,x_{i_r}] &= f[\ \underbrace{x_{i_0},\dots,x_{i_0}}_{(\alpha(x_{i_0})+1)-times}\ ,\dots,\ \underbrace{x_{i_r},\dots,x_{i_r}}_{(\alpha(x_{i_r})+1)-times}\ ]\\
&= f[x_0,\dots,x_n].
\end{aligned}$$

5 Bases and representations of the Hermite polynomial

5.1 Reformulation of the problem

The Hermite interpolation problem we have solved can be restated as presented in the last section. We consider $r+1$ distinct points $x_0,\ x_1,\ \dots,\ x_r$, and associated $r+1$ integers $\alpha_0,\ \alpha_1,\ \dots,\ \alpha_r$. We look for a polynomial $p_n(x)\in\mathcal{P}_n$ such that

$$p_n^{(l)}(x_i)=f^{(l)}(x_i)\quad\text{for}\quad l=0,\dots,\alpha_i;i=0,\dots,r.$$

We have $\sum_{i=0}^{r}(\alpha_i+1)=\sum_{i=0}^{r}\alpha_i+r+1=n+1$. We are going to represent $p_n(x)$ with respect to different bases as was done in [7] for Lagrange interpolation ($\alpha_i=0$ for all i, so $r=n$, all distinct x_i).

5.2 Monomial basis

Using the monomial basis $\{1, x, x^2, \dots, x^n\}$, we can write

$$p_n(x) = \sum_{j=0}^{n} a_j x^j.$$

The coefficients a_j's are obtained by considering the linear system obtained by the $\alpha_i + 1$ conditions at each x_i. So we have

$$p_n^{(l)}(x_i) = \sum_{j=l}^{n} a_j \frac{j!}{(j-l)!} x^{j-l} = f^{(l)}(x_i) \quad (l = 0, \dots, \alpha_i).$$

Under matrix form, for each index i we have a $(\alpha_i + 1, n+1)$-matrix V_i and a $(\alpha_i + 1, 1)$-matrix F_i

$$V_i = \begin{bmatrix} 1 & \frac{1!}{1!}x_i & \frac{2!}{2!}x_i^2 & \cdots & \frac{\alpha_i!}{\alpha_i!}x_i^{\alpha_i} & \cdots & \frac{n!}{n!}x_i^n \\ & 1 & \frac{2!}{1!}x_i & \cdots & \frac{\alpha_i!}{(\alpha_i-1)!}x_i^{\alpha_i-1} & \cdots & \frac{n!}{(n-1)!}x_i^{n-1} \\ & & 2 & \cdots & \frac{\alpha_i!}{(\alpha_i-2)!}x_i^{\alpha_i-2} & \cdots & \frac{n!}{(n-2)!}x_i^{n-2} \\ & & & \ddots & & & \\ & & & & \alpha_i! & \cdots & \frac{n!}{(n-\alpha_i)!}x_i^{n-\alpha_i} \end{bmatrix}, F_i = \begin{bmatrix} f(x_i) \\ f'(x_i) \\ f''(x_i) \\ \vdots \\ f^{(\alpha_i)}(x_i) \end{bmatrix}.$$

We form the global $(n+1, n+1)$-matrix V and the global $(n+1, 1)$-matrix F

$$V = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_r \end{bmatrix}, F = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_r \end{bmatrix}.$$

The V matrix is the confluent Vandermonde matrix. The a_i's are the solution of the system

$$Va = F$$

with $a = [a_0, a_1, \dots, a_n]^\intercal$. We know that this matrix is invertible because the homogeneous system $Va = 0$ has only the zero solution $a = 0$. So

$$a = V^{-1}F.$$

If we set $M(x) = [1, x, \cdots, x^n]^\intercal$ we have

$$p_n(x) = M^\intercal(x)a = M^\intercal(x)V^{-1}F.$$

5.3 Generalized Lagrange basis

To form this basis, we look for polynomials $\ell_{il}(x) \in \mathcal{P}_n$ $(l = 0, \ldots, \alpha_i;\ i = 0, \ldots, r)$, such that

$$\ell_{il}^{(k)}(x_j) = \delta_{ij}\delta_{lk},$$

where

$$\delta_{rs} = \begin{cases} 0 & \text{if} \quad r \neq s, \\ 1 & \text{if} \quad r = s. \end{cases}$$

For each i, $l_{i\alpha_i}(x)$ must have a zero of multiplicity $\alpha_j + 1$ at x_j for $j \neq i$ and a zero of multiplicity α_i at x_i. So we write

$$\ell_{i\alpha_i}(x) = \frac{(x - x_i)^{\alpha_i}}{\alpha_i!} v_i(x),$$

where

$$v_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{r} \left(\frac{x - x_j}{x_i - x_j} \right)^{\alpha_j + 1}.$$

Hence $\ell_{i\alpha_i}(x)$ verifies all the desired conditions. We now construct recursively, in a decreasing order for the index l, the polynomials $\ell_{il}(x)$ for $l = \alpha_i - 1, \alpha_i - 2, \ldots, 1, 0$. Suppose $\ell_{ik}(x)$ is already defined for $k = \alpha_i, \alpha_i - 1, \ldots, l + 1$, for an $l \in \{\alpha_i - 1, \alpha_i - 2, \ldots, 1, 0\}$. To determine $\ell_{il}(x)$, we start with

$$\tilde{\ell}_{il}(x) = \frac{(x - x_i)^l}{l!} v_i(x),$$

which verifies the conditions for each x_j $(j \neq i)$. For x_i we have

$$\tilde{\ell}_{il}^{(k)}(x_i) = 0 \quad \text{if} \quad k < l,$$

and

$$\tilde{\ell}_{il}^{(l)}(x_i) = 1.$$

Moreover, for $k > l$, we have

$$\tilde{\ell}_{il}^{(k)}(x)|_{x=x_i} = \frac{d^k}{dx^k} \frac{(x - x_i)^l}{l!} v_i(x)|_{x=x_i} = \sum_{\sigma=0}^{k} \left(\begin{array}{c} k \\ \sigma \end{array} \right) v_i^{(k-\sigma)}(x) \frac{d^\sigma}{dx^\sigma} \frac{(x - x_i)^l}{l!}|_{x=x_i}.$$

But

$$\frac{d^\sigma}{dx^\sigma} \frac{(x - x_i)^l}{l!}|_{x=x_i} = \delta_{\sigma l},$$

then

$$\tilde{\ell}_{il}^{(k)}(x_i) = \left(\begin{array}{c} k \\ l \end{array} \right) v_i^{(k-l)}(x_i).$$

So we set

$$\ell_{il}(x) = \tilde{\ell}_{il}(x) - \sum_{j=l+1}^{\alpha_i} \begin{pmatrix} j \\ l \end{pmatrix} v_i^{(j-l)}(x_i)\ell_{ij}(x).$$

A similar construction of the $\ell_{il}(x)$ for $i = 0, \dots, r$ and each i for $l = 0, \dots, \alpha_i$ has been presented in [11]. Finally, we have

$$p_n(x) = \sum_{i=0}^{r} \sum_{l=0}^{\alpha_i} f^{(l)}(x_i)\ell_{il}(x).$$

If we set

$$L_i(x) = \begin{bmatrix} \ell_{i0}(x) \\ \ell_{i1}(x) \\ \ell_{i2}(x) \\ \vdots \\ \ell_{i\alpha_i}(x) \end{bmatrix}, \quad \text{for} \quad i = 0, \dots, r, \quad \text{and} \quad L(x) = \begin{bmatrix} L_0(x) \\ L_1(x) \\ L_2(x) \\ \vdots \\ L_r(x) \end{bmatrix},$$

we have

$$p_n(x) = L^{\intercal}(x)F.$$

We also obtain

$$V^{\intercal}L(x) = M(x).$$

5.4 Newton basis

We construct a family of $n+1$ polynomials of increasing degree $\pi_{il}(x) \in \mathcal{P}_n$ ($l = 0, \dots, \alpha_i$; $i = 0, \dots, r$) by considering successively the x_i. They are linearly independant and form a basis of $\mathcal{P}_n$. We set

$$\pi_{il}(x) = (x - x_i)^l \prod_{j=0}^{i-1} (x - x_j)^{\alpha_j + 1},$$

such that $\pi_{00}(x) = 1$. We can write

$$p_n(x) = \sum_{i=0}^{r} \sum_{l=0}^{\alpha_i} \lambda_{il}\pi_{il}(x).$$

We observe that

$$p_n^{(l)}(x_i) = \sum_{j=0}^{i-1} \sum_{k=0}^{\alpha_j} \lambda_{jk}\pi_{jk}^{(l)}(x_i) + \sum_{k=0}^{l} \lambda_{ik}\pi_{ik}^{(l)}(x_i),$$

because for any fixed $l = 0, \dots, \alpha_i$

$$\pi_{jk}^{(l)}(x_i) = 0 \quad \text{for} \quad \begin{cases} j = i \quad \text{and} \quad k = l+1, \dots, \alpha_i, \\ \\ j > i \quad \text{and} \quad k = 0, \dots, \alpha_i. \end{cases}$$

Since

$$\pi_{il}^{(l)}(x_i) = l! \prod_{j=0}^{i-1} (x_i - x_j)^{\alpha_j + 1} \neq 0,$$

we can solve for λ_{il} to get

$$\lambda_{il} = \frac{f^{(l)}(x_i) - \sum_{j=0}^{i-1} \sum_{k=0}^{\alpha_j} \lambda_{jk} \pi_{jk}^{(l)}(x_i) - \sum_{k=0}^{l-1} \lambda_{ik} \pi_{ik}^{(l)}(x_i)}{\pi_{il}^{(l)}(x_i)}.$$

So we can find $p_n(x)$ by solving recursively one equation. Since $\pi_{r\alpha_r}(x)$ is the unique polynomial of degree n we get $a_n = \lambda_{r\alpha_r}$.

Let us observe that with the Newton basis and an ordering of the points (with their multiplicities), we can compute the divided differences with an appropriate table (as the usual table for the Lagrange case).

From a matrix point of view, let us set the $(\alpha_i + 1, \alpha_j + 1)$-matrices

$$\Pi_{ij} = \begin{bmatrix} \\ \quad \pi_{jk}^{(l)}(x_i) \quad \\ \\ \end{bmatrix}_{\substack{l = 0, \dots, \alpha_i \\ k = 0, \dots, \alpha_j}}$$

for $i = 0, \dots, r$ and $j = 0, \dots, r$. Let us observe that

(i) if $j > i$ then $\Pi_{ij} = 0$,
(ii) if $j = i$ then Π_{ij} is a lower triangular matrix because $\pi_{ik}^{(l)}(x_i) = 0$ for $k > l$,
(iii) if $j < i$ then Π_{ij} is a full matrix.

Let

$$\Lambda_i = \begin{bmatrix} \lambda_{i0} \\ \lambda_{i1} \\ \lambda_{i2} \\ \vdots \\ \lambda_{i\alpha_i} \end{bmatrix} \quad \text{for} \quad i = 0, \dots, r, \quad \text{and} \quad \Lambda = \begin{bmatrix} \Lambda_0 \\ \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_r \end{bmatrix},$$

and

$$\Pi = \begin{bmatrix} \Pi_{00} & \Pi_{01} & \dots & \Pi_{0r} \\ \Pi_{10} & \Pi_{11} & \dots & \Pi_{1r} \\ \vdots & \vdots & & \vdots \\ \Pi_{r0} & \Pi_{r1} & \dots & \Pi_{rr} \end{bmatrix}.$$

So we have $\Pi\Lambda = F$ and $\Lambda = \Pi^{-1}F$. If

$$\Pi_i(x) = \begin{bmatrix} \pi_{i0}(x) \\ \pi_{i1}(x) \\ \pi_{i2}(x) \\ \vdots \\ \pi_{i\alpha_i}(x) \end{bmatrix} \quad \text{for} \quad i = 0, \ldots, r, \quad \text{and} \quad \Pi(x) = \begin{bmatrix} \Pi_0(x) \\ \Pi_1(x) \\ \Pi_2(x) \\ \vdots \\ \Pi_r(x) \end{bmatrix},$$

we have

$$p_n(x) = \Pi^\intercal(x)\Lambda = \Pi^\intercal(x)\Pi^{-1}F.$$

5.5 An orthogonal basis

Let us consider the inner product defined by

$$\langle p(\bullet), q(\bullet)\rangle = \sum_{i=0}^{r}\sum_{l=0}^{\alpha_i} p^{(l)}(x_i)q^{(l)}(x_i)$$

for two polynomials $p(x)$ and $q(x)$ in $\mathcal{P}_n$. We can construct recursively an orthogonal basis with this inner product. Let us start with $q_0(x) = 1$, and assume by induction that the set $\{q_j(x)\}_{j=0}^{k}$ form a sequence of orthogonal polynomials such that the degree of $q_j(x)$ is j. We get $q_{k+1}(x)$ by

$$q_{k+1}(x) = xq_k(x) - \sum_{j=0}^{k}\gamma_{k+1,j}q_j(x)$$

where

$$\gamma_{k+1,j} = \frac{\langle(\bullet)q_k(\bullet), q_j(\bullet)\rangle}{\langle q_j(\bullet), q_j(\bullet)\rangle}$$

for $j = 0, \ldots, k$. Let us observe that we have

$$\langle(\bullet)p(\bullet), q(\bullet)\rangle = \langle p(\bullet), (\bullet)q(\bullet)\rangle$$

only in the case $\alpha_i = 0$ for $i = 0, \ldots, r$, for which we have the usual three term relation for the orthogonal $q_k(x)$.

The norm associated to the inner product is

$$\|p\| = \sqrt{\langle p(\bullet), p(\bullet)\rangle}.$$

Now we look for a polynomial

$$p_n(x) = \sum_{k=0}^{n} w_k q_k(x)$$

which minimize $\|f - p_n\|$. Using the normal equations

$$\langle f(\bullet) - p_n(\bullet), q_k(\bullet)\rangle = 0$$

for $k = 0, \ldots, n$. The solution is given by

$$p_n(x) = \sum_{k=0}^{n} \langle f(\bullet), q_k(\bullet)\rangle \, q_k(x).$$

We have for $j = 0, \ldots, n$

$$Q_{ij} = \begin{bmatrix} q_j(x_i) \\ q_j'(x_i) \\ q_j''(x_i) \\ \vdots \\ q_j^{(\alpha_i)}(x_i) \end{bmatrix} \quad \text{for} \quad i = 0, \ldots, r, \quad \text{and} \quad Q_j = \begin{bmatrix} Q_{0j} \\ Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{rj} \end{bmatrix}.$$

Then, let us set

$$Q = \begin{bmatrix} Q_0 & \ldots & Q_n \end{bmatrix} = \begin{bmatrix} Q_{00} & \ldots & Q_{0n} \\ Q_{10} & \ldots & Q_{1n} \\ \vdots & & \vdots \\ Q_{r0} & \ldots & Q_{rn} \end{bmatrix}.$$

With $w = \begin{bmatrix} w_0 & \ldots & w_n \end{bmatrix}^\intercal$, we have to solve the linear system

$$Qw = F.$$

But $Q^\intercal Q = \mathrm{Diag}\left(\|q_j\|^2\right)$, so from $Q^\intercal Q w = Q^\intercal F$ we get

$$w = \mathrm{Diag}\left(\frac{1}{\|q_j\|^2}\right) Q^\intercal F.$$

Then, for $Q(x) = \begin{bmatrix} q_0(x) \ldots q_n(x) \end{bmatrix}^\intercal$, we have

$$p_n(x) = Q^\intercal(x) w = Q^\intercal(x) \mathrm{Diag}\left(\frac{1}{\|q_j\|^2}\right) Q^\intercal F.$$

5.6 Links between bases

We have

$$p_n(x) = M^\intercal(x) V^{-1} F = L^\intercal(x) F = \Pi^\intercal(x) \Pi^{-1} F = Q^\intercal(x) \mathrm{Diag}\left(\frac{1}{\|q_j\|^2}\right) Q^\intercal F,$$

so obtain

$$M^{\mathsf{T}}(x)V^{-1} = L^{\mathsf{T}}(x) = \Pi^{\mathsf{T}}(x)\Pi^{-1} = Q^{\mathsf{T}}(x)\mathrm{Diag}\left(\frac{1}{\|q_j\|^2}\right)Q^{\mathsf{T}}.$$

Acknowledgement. This work has been financially supported by an individual discovery grant from NSERC (Natural Sciences and Engineering Research Council of Canada).

Conflict of interest disclosure The author declares that there is no conflict of interest regarding the publication of this paper.

References

[1] D.F. Bailey, G.J. Fix, *A generalization of the mean value theorem*, Appl. Math. Lett., **1(4)**(1988), 327-330. MR0975800(90c:26051). Zbl 0728.39004.

[2] S.D. Conte, C. de Boor, *Elementary Numerical Analysis, an Algorithmic Approach*, McGraw-Hill, New York, 1980. MR3829062. Zbl 0496.65001.

[3] M. Crouzeix, A.L. Mignot, *Analyse numérique des équations différentielles*, Masson, Paris, 1984. MR0762089(86m:65002). Zbl 0635.65079.

[4] P.J. Davis, *Interpolation & Approximation*, Dover, New York, 1963. MR0157156(28 # 393). Zbl 0111.06003.

[5] C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York, 1978. MR0507062(80a:65027). Zbl 0406.41003.

[6] C. de Boor, *Divided Differences*, Surveys in Approximation Theory, **1** (2005), 46-69. MR2221566(2006k:41001). Zbl 1071.65027.

[7] W. Gander, *Change of basis in polynomial interpolation*, Numerical Linear Algebra with Applications, **12** (2005), 769-778. MR2172676(2006g:41005). Zbl 1164.41302.

[8] E.T.Y. Lee, *A remark on divided differences*, The American Mathematical Monthly, **96(7)** (1989), 618-622. MR1008794(90i:39008). Zbl 0703.41001.

[9] C. Micchelli, *A constructive approach to Kergin interpolation*, Rocky Mountain J. Math., **10** (1980), 485-497. MR0590212(84i:41002). Zbl 0456.41003.

[10] J. Schwaiger, *On a characterization of polynomials by divided differences*, Aequationes Mathematicae, **48(2-3)** (1994), 317-323. MR1295099(95i:39019). Zbl 0810.39007.

[11] A. Spitzbart, *A generalization of Hermite's interpolation formul*a, The American Mathematical Monthly, **67(1)** (1960), 42-46. MR0137945(25 # 1393). Zbl 0097.04702

François Dubeau
Département de mathématiques,
Faculté des sciences, Université de Sherbrooke,
2500, boul. de l'Université,
Sherbrooke (Qc), Canada, J1K 2R1.
e-mail: francois.dubeau@usherbrooke.ca
https://www.usherbrooke.ca/mathematiques/personnel/corps-professoral/professeurs/francois-dubeau/

License