Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 15 (2020), 233 – 255

REFINEMENTS OF SOME RETARDED INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN-BIHARI TYPE AND THEIR APPLICATIONS

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Abstract. In this study, some generalizations and refinements of some retarded integral inequalities of Gronwall-Bellman-Bihari type are established. To show the feasibility of the obtained inequalities, some illustrative examples are also introduced.

1 Introduction

It is well known that integral inequalities which were introduced by Gronwall–Bellman [8], [12], and their various generalizations [4–9] play a very important role in the study of qualitative properties of solutions of differential equations, integral equations and integral differential equations. Recently, many versions of Gronwall–Bellman type nonlinear inequalities can be found in [1], [10], [11], [20].

In 2016, Abdeldaim [3], discussed the following nonlinear integral inequalities

$$u^p(t) \le n(t) + \int_0^t g(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)u^q(s)ds, \tag{1.1}$$

$$u(t) \le u_0 + \int_0^{\alpha(t)} f(s)\left[u^{(2-p)}(s) + \int_0^s g(\lambda)u^q(\lambda)\, d\lambda\right]^p ds,\ \ q, u_0 > 0, 0 < p \le 1. \tag{1.2}$$

In this paper, motivated by the work of Abdeldaim [3] and the papers [11–13, 17–19] we establish a new nonlinear retarded integral inequalities which can be used as handy tools to study the qualitative behavior of certain retarded differential and integral equations.

2020 Mathematics Subject Classification: 39B72, 26D10, 34A34.
Keywords: Retarded integral inequality, Gronwall-Bellman-Bihari, integral inequality.

2 Preliminaries

In what follows, $\mathbb{R}$ denoted the set of real numbers, $I = [0, \infty)$ is the subset of $\mathbb{R}$, $C(I, I)$ denotes the set of all continuous functions from I into I and $C^1(I, I)$ denotes the set of all continuously differentiable functions from I into I. For convenience, we give some lemmas and definitions which will be used in the proof of our main results.

Lemma 1 ([15]). *Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$ then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}, \tag{2.1}$$

for any $k > 0$.

Lemma 2 ([4]). *Let $p(t)$ and $\varphi(t)$ be continuous functions for $t \geq \alpha$, let $z(t)$ be a differentiable function for $t \geq \alpha$, and suppose*

$$\begin{aligned} z'(t) &\leq p(t) + q(t)z(t), \qquad t \geq \alpha, \\ z(\alpha) &\leq z_0. \end{aligned} \tag{2.2}$$

Then, for $t \geq \alpha$,

$$z(t) \leq z_0 \exp\left(\int_\alpha^t q(s)ds \right) + \int_\alpha^t p(s) \exp\left(\int_s^t q(\tau)d\tau \right) ds. \tag{2.3}$$

Definition 3. *A nondecreasing, continuous function $\theta : I \to I$ is said to belong to class $\mathcal{T}$ if it satisfies the following condition*

$$\frac{1}{a}\theta(x) \geq \theta(\frac{x}{a}) \text{ for all } x \geq 0 \text{ and } a \geq 1. \tag{2.4}$$

Example 4. The function $\theta(x) = x^\alpha$, $x \in \mathbb{R}_+, \alpha \geq 1$ is of class $\mathcal{T}$.

Now we state the main results of this work.

3 Main result

In this section, we state and prove some new nonlinear retarded integral inequalities of Gronwall-Bellman-bihari type, which can be used in applications as handy tools, and in the analysis of various problems in the theory of the nonlinear ordinary differential and integral equations.

Theorem 5. *Let $u(t), g(t), h(t), n(t) \in C(I, I)$ and $n(t)$ is nondecreasing. Let $\alpha(t) \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$ and $w : I \to I$ be a differentiable increasing function on $]0, \infty[$ with continuous nonincreasing first derivative w' on $]0, \infty[$. If $u(t)$ satisfies*

$$u^p(t) \leq n(t) + \int_0^t g(s) u^p(s) ds + \int_0^{\alpha(t)} h(s) w(u^q(s)) ds, \tag{3.1}$$

for $p \neq 0$, $p \geq q \geq 0$. Then

$$u(t) \leq \left[n(t) + \int_0^t p_1(s) \exp\left(\int_s^t q_1(\tau) d\tau \right) ds \right]^{\frac{1}{p}}, \forall t \in I, \tag{3.2}$$

where

$$\begin{aligned} p_1(t) &= g(t)n(t) + \alpha'(t)h(\alpha(t))w\left(\frac{q}{p} k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right), \\ q_1(t) &= g(t) + \frac{q}{p} k^{\frac{q-p}{p}} \alpha'(t) h(\alpha(t)) w'\left(\frac{q}{p} k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right), \end{aligned} \tag{3.3}$$

for all $t \in I$.

Proof. Define a function $z(t)$ by :

$$z(t) = \int_0^t g(s) u^p(s) ds + \int_0^{\alpha(t)} h(s) w(u^q(s)) ds \tag{3.4}$$

then, we have

$$u(t) \leq [n(t) + z(t)]^{\frac{1}{p}}, \ z(0) = 0. \tag{3.5}$$

Differentiating $z(t)$ with respect to t, and using (3.5), we get

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))w\left(n(\alpha(t)) + z(\alpha(t))\right)^{\frac{q}{p}}.$$

Taking into-account that $n(t)$ and $z(t)$ are nondecreasing functions, then the above inequality can be expressed as

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))w\left((n(t) + z(t))\right)^{\frac{q}{p}}.$$

By applying Lemma 1, we have

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))w\left(\frac{q}{p} k^{\frac{q-p}{p}} (n(t) + z(t)) + \frac{p-q}{p} k^{\frac{q}{p}} \right). \tag{3.6}$$

Applying the mean value theorem for the function w, then for every $x_1 > y_1 > 0$, there exists $c \in]\, y_1, x_1 [$ such that

$$w(x_1) - w(y_1) = w'(c)(x_1 - y_1) \le w'(y_1)(x_1 - y_1) \ .$$

(3.6) can be rewritten as follows

$$\begin{aligned} z'(t) \le\ & g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t)) \left[w\left(\frac{q}{p}k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right) \right. \\ & \left. + \frac{q}{p}k^{\frac{q-p}{p}} w'\left(\frac{q}{p}k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right) z(t) \right], \end{aligned}$$

then, we get

$$\begin{aligned} z'(t) \le\ & g(t)n(t) + \alpha'(t)h(\alpha(t)) w\left(\frac{q}{p}k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right) \\ & + z(t) \left[g(t) + \frac{q}{p}k^{\frac{q-p}{p}} \alpha'(t)h(\alpha(t)) w'\left(\frac{q}{p}k^{\frac{q-p}{p}} n(t) + \frac{p-q}{p} k^{\frac{q}{p}} \right) \right]. \end{aligned} \tag{3.7}$$

The inequality (3.7) can be expressed as

$$z'(t) \le p_1(t) + q_1(t)z(t),$$

where $p_1(.)$ and $q_1(.)$ are defined as in (3.3).

Using Lemma 2 to the above inequality, we obtain

$$z(t) \le \int_0^t p_1(s) \exp\left(\int_s^t q_1(\tau)d\tau \right) ds \qquad \forall t \in I, \tag{3.8}$$

Using (3.5), we get the required inequality in (3.2). □

Remark 6. If we take $w(t) = t$, inequality (3.1) can be reduced to the inequality (1.1) discussed by Abdeldaim [3] and if we take $w(t) = t, n(t) = u_0$ (positive constant), $\alpha(t) = t$, inequality (3.1) can be reduced to the case discussed by Theorem 3.1 in [2].

Corollary 7. *Let $u(t), g(t), h(t), n(t) \in C(I, I)$ and $n(t)$ is nondecreasing. Let $\alpha(t) \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \le t$, $\alpha(0) = 0$. If $u(t)$ satisfies*

$$u^p(t) \le n(t) + \int_0^t g(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)\arctan(u^q(s))ds,$$

for $p \ne 0$, $p \ge q \ge 0$. Then

$$u(t) \le \left[n(t) + \int_0^t p_1(s) \exp\left(\int_s^t q_1(\tau)d\tau \right) ds \right]^{\frac{1}{p}},$$

where

$$\begin{aligned} p_1(t) &= g(t)n(t) + \alpha'(t)h(\alpha(t)\arctan\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right), \\ q_1(t) &= g(t) + \frac{q}{p}k^{\frac{q-p}{p}}\alpha'(t)h(\alpha(t))\left(\frac{1}{1+\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right)^2}\right), \end{aligned}$$

for all $t \in I$.

Theorem 8. *Assume that* $u(t), g(t), h(t), n(t) \in C(I, I)$ *and* $n(t)$ *is nondecreasing. Let* $\alpha(t) \in C^1(I, I)$ *be a nondecreasing function with* $\alpha(t) \leq t$, $\alpha(0) = 0$. *Let* w_1 *and* $w_2 : I \to I$ *be differentiable increasing functions on* $]0, \infty[$ *with continuous nonincreasing first derivative on* $]0, \infty[$. *If* $u(t)$ *satisfies*

$$u^p(t) \leq n(t) + \int_0^t g(s)w_1(u^p(s))ds + \int_0^{\alpha(t)} h(s)w_2(u^q(s))ds, \quad \forall t \in I, \tag{3.9}$$

for $p \neq 0$, $p \geq q \geq 0$. *Then*

$$u(t) \leq \left[n(t) + \int_0^t p_2(s)\exp\left(\int_s^t q_2(\tau)d\tau\right)ds\right]^{\frac{1}{p}}, \forall t \in I, \tag{3.10}$$

where

$$\begin{aligned} p_2(t) &= g(t)w_1(n(t)) + \alpha'(t)h(\alpha(t))w_2\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right), \qquad (3.11) \\ q_2(t) &= g(t)w_1'(n(t)) + \frac{q}{p}k^{\frac{q-p}{p}}\alpha'(t)h(\alpha(t))w_2'\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right), \end{aligned}$$

for all $t \in I$.

Proof. Define a function $z(t)$ as follows:

$$z(t) = \int_0^t g(s)w_1(u^p(s))ds + \int_0^{\alpha(t)} h(s)w_2(u^q(s))ds \tag{3.12}$$

then, we have

$$u(t) \leq [n(t) + z(t)]^{\frac{1}{p}}, z(0) = 0. \tag{3.13}$$

and

$$z'(t) \leq g(t)w_1(u^p(t)) + \alpha'(t)h(\alpha(t))w_2(u^q(\alpha(t))). \tag{3.14}$$

Using (3.14) and applying Lemma 1, we obtain

$$z'(t) \le g(t)w_1(n(t)+z(t)) + \alpha'(t)h(\alpha(t))w_2\left(\frac{q}{p}k^{\frac{q-p}{p}}(n(t)+z(t)) + \frac{p-q}{p}k^{\frac{q}{p}}\right).$$

Applying the mean value theorem for the functions $w_1$and w_2, we get

$$\begin{aligned} z'(t) \quad \le \quad & g(t)\left[w_1(n(t)) + w_1'(n(t))z(t))\right] + \alpha'(t)h(\alpha(t)) \\ & \times\left[w_2\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right) + \frac{q}{p}k^{\frac{q-p}{p}}w_2'\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right)z(t)\right], \end{aligned} \tag{3.15}$$

then he inequality (3.15) can be rewritten as follows

$$\begin{aligned} z'(t) \quad \le \quad & g(t)w_1(n(t)) + \alpha'(t)h(\alpha(t))w_2\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right) \\ & + z(t)\left[g(t)w_1'(n(t)) + \frac{q}{p}k^{\frac{q-p}{p}}\alpha'(t)h(\alpha(t))w_2'\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right)\right], \end{aligned} \tag{3.16}$$

the inequality (3.16) can be reformulated as

$$z'(t) \le p_2(t) + q_2(t)z(t).$$

Using Lemma 2 to the above inequality, we have

$$z(t) \le \int_0^t p_2(s)\exp\left(\int_s^t q_2(\tau)d\tau\right)ds\ , \tag{3.17}$$

for all $t \in I$, where $p_2(.)$ and $q_2(.)$ are defined as in (3.11). Using (3.13), we get the required inequality in (3.10). □

Theorem 9. *Assume that $u(t), g(t), h(t), n(t) \in C(I, I)$ and $n(t)$ is nondecreasing. Let $\alpha(t) \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \le t$, $\alpha(0) = 0$. Let w_1 and $w_2 : I \to I$ be differentiable increasing functions on $]0, \infty[$ with continuous nonincreasing first derivative on $]0, \infty[$. If $u(t)$ satisfies*

$$u^p(t) \le n(t) + \int_0^t g(s)w_1(u^q(s))ds + \int_0^{\alpha(t)} h(s)w_2(u^r(s))ds, \tag{3.18}$$

for $p \neq 0$, $p \ge q \ge 0$ and $p \ge r \ge 0$. Then

$$u(t) \le \left[n(t) + \int_0^t p_3(s)\exp\left(\int_s^t q_3(\tau)d\tau\right)ds\right]^{\frac{1}{p}}, \forall t \in I, \tag{3.19}$$

where

$$
\begin{aligned}
p_3(t) &= g(t)w_1\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)+\alpha'(t)h(\alpha(t))w_2\left(\frac{r}{p}k^{\frac{r-p}{p}}n(t)+\frac{p-r}{p}k^{\frac{r}{p}}\right),\\
q_3(t) &= \frac{q}{p}k^{\frac{q-p}{p}}g(t)w_1'\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\\
&\quad+\frac{r}{p}k^{\frac{r-p}{p}}\alpha'(t)h(\alpha(t))w_2'\left(\frac{r}{p}k^{\frac{r-p}{p}}n(t)+\frac{p-r}{p}k^{\frac{r}{p}}\right), \qquad (3.20)
\end{aligned}
$$

for all $t\in I$.

Proof. The proof is essentially identical to that of Theorem 8. □

Remark 10. If we have $w_1(t)=t$ and $w_2(t)=t$, $q=p$, inequality (3.18) can be reduced to the inequality (1.1) discussed by Abdeldaim [3].

Corollary 11. *Assume that all assumptions of Theorem 9 hold. Let* $w_1(t)=\arctan t$ *and* $w_2(t)=\ln(1+t)$. *If* $u(t)$ *satisfies*

$$u^p(t)\le n(t)+\int_0^t g(s)\arctan(u^q(s))ds+\int_0^{\alpha(t)} h(s)\ln(1+u^r(s))ds,$$

for $p\neq 0$, $p\ge q\ge 0$, $p\ge r\ge 0$, *then*

$$u(t)\le\left[n(t)+\int_0^t p_3(s)\exp\left(\int_s^t q_3(\tau)d\tau\right)ds\right]^{\frac{1}{p}},$$

where

$$
\begin{aligned}
p_3(t) &= g(t)\arctan\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)+\\
&\quad \alpha'(t)h(\alpha(t))\ln\left(1+\frac{r}{p}k^{\frac{r-p}{p}}n(t)+\frac{p-r}{p}k^{\frac{r}{p}}\right),\\
q_3(t) &= \frac{q}{p}k^{\frac{q-p}{p}}g(t)\frac{1}{1+\left(\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)^2}+\\
&\quad \frac{r}{p}k^{\frac{r-p}{p}}\alpha'(t)h(\alpha(t))\frac{1}{\left(1+\frac{r}{p}k^{\frac{r-p}{p}}n(t)+\frac{p-r}{p}k^{\frac{r}{p}}\right)},
\end{aligned}
$$

for all $t\in I$.

Theorem 12. *Let* $u(t),g(t),h(t),n(t)\in C(I,I)$ *and let* $n(t)$ *be nondecreasing. Let* $\alpha(t)\in C^1(I,I)$ *be a nondecreasing function with* $\alpha(t)\le t,\alpha(0)=0$ *and let* $\mathcal{L},M\in C\left(\mathbb{R}_+^2,\mathbb{R}_+\right)$ *satisfy*

$$0\le\mathcal{L}(t,x)-\mathcal{L}(t,y)\le M(t,y)(x-y),\ x\ge y\ge 0.$$

If $u(t)$ *satifies*

$$u^p(t) \leq n(t) + \int_0^t g(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)\mathcal{L}(s, u^q(s))ds, \tag{3.21}$$

for $p \neq 0$, $p \geq q \geq 0$. *Then*

$$u(t) \leq \left[n(t) + \int_0^t p_4(s) \exp\left(\int_s^t q_4(\tau)d\tau\right) ds\right]^{\frac{1}{p}}, \forall t \in I, \tag{3.22}$$

where

$$\begin{aligned} p_4(t) &= n(t)g(t) + \alpha'(t)h(\alpha(t))\mathcal{L}\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right) \\ q_4(t) &= g(t) + \frac{q}{p}k^{\frac{q-p}{p}}\alpha'(t)h(\alpha(t))M\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right), \end{aligned} \tag{3.23}$$

for all $t \in I$.

Proof. Define a function $z(t)$ by:

$$z(t) = \int_0^t g(s)u^p(s)\, ds + \int_0^{\alpha(t)} h(s)\mathcal{L}(s, u^q(s))ds, \tag{3.24}$$

then,

$$u(t) \leq [n(t) + z(t)]^{\frac{1}{p}}. \tag{3.25}$$

Differentiating $z(t)$, with respect to t and using (3.25), we obtain

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))\mathcal{L}(\alpha(t), (n(t) + z(t))^{\frac{q}{p}}). \tag{3.26}$$

Applying Lemma 1, we get

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))\mathcal{L}\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}(n(t) + z(t)) + \frac{p-q}{p}k^{\frac{q}{p}}\right), \tag{3.27}$$

$$z'(t) \leq g(t)(n(t) + z(t)) + \alpha'(t)h(\alpha(t))L(t), \tag{3.28}$$

where

$$L(t) = \mathcal{L}\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right) + \frac{q}{p}k^{\frac{q-p}{p}}z(t))M\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right),$$

then,

$$\begin{aligned} z'(t) \leq\; & n(t)g(t) + \alpha'(t)h(\alpha(t))\mathcal{L}\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right) \\ & + \left[g(t) + \frac{q}{p}k^{\frac{q-p}{p}}\alpha'(t)h(\alpha(t))M\left(\alpha(t), \frac{q}{p}k^{\frac{q-p}{p}}n(t) + \frac{p-q}{p}k^{\frac{q}{p}}\right)\right] z(t), \end{aligned} \tag{3.29}$$

the inequality (3.29) can be reformulated as

$$z'(t) \leq p_4(t) + q_4(t)z(t),$$

Applying Lemma 2, to the above inequality, we have

$$z(t) \leq \int_0^t p_4(s) \exp\left(\int_s^t q_4(\tau)d\tau\right) ds \ , \tag{3.30}$$

where $p_4(.)$ and $q_4(.)$ are defined as in (3.23). Using (3.25), we get the required inequality in (3.22). □

Remark 13. If we take $\mathcal{L}(t,x) = x$, then inequality (3.21) can be reduced to the inequality (1.1) discussed by Abdeldaim [3].

Theorem 14. *Let $u(t), f(t), g(t) \in C(I,I)$, $\alpha(t) \in C^1(I,I)$ be nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$ and $\mathcal{L}, M \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ satisfy*

$$0 \leq \mathcal{L}\left(t,x\right) - \mathcal{L}\left(t,y\right) \leq M\left(t,y\right)\left(x-y\right), x \geq y \geq 0.$$

If $u(t)$ satisfies

$$u\left(t\right) \leq u_0 + \int_0^{\alpha(t)} f\left(s\right)\left[u^{(2-p)}\left(s\right) + \int_0^s g\left(\tau\right)\mathcal{L}\left(\tau, u^q\left(\tau\right)\right)d\tau\right]^p ds, \tag{3.31}$$

for $0 < p \leq 1$, $0 \leq q < 1$. Then

$$u\left(t\right) \leq u_0 + \int_0^{\alpha(t)} f\left(s\right) q_5\left(\alpha^{-1}\left(s\right)\right) exp\left(p\left(2-p\right)\int_0^s p_5\left(\tau\right)d\tau\right) ds, \tag{3.32}$$

for all $t \in I$, where

$$\begin{cases} p_5\left(t\right) = & f\left(t\right) + \frac{q}{2-p}k^{q-1}g\left(t\right)M\left(t,\left(1-q\right)k^q\right), \\ q_5\left(t\right) = & \left[u_0^{(2-p)} + \int_0^{\alpha(t)} g\left(t\right)\mathcal{L}\left(t,\left(1-q\right)k^q\right) exp\left(-\left(2-p\right)\int_0^s P_5\left(\tau\right)d\tau\right) ds\right]^p. \end{cases} \tag{3.33}$$

Proof. Define a function $z(t)$ by the right hand side of (3.31), then $z\left(0\right) = u_0$ and

$$u\left(t\right) \leq z\left(t\right). \tag{3.34}$$

Differentiating $z\left(t\right)$, with respect to t and using (3.34), we get

$$z'(t) \leq \alpha'\left(t\right) f\left(\alpha\left(t\right)\right)\left[z^{(2-p)}\left(t\right) + \int_0^{\alpha(t)} g\left(s\right)\mathcal{L}\left(s, z^q\left(s\right)\right)ds\right]^p. \tag{3.35}$$

If we take

$$v(t) = z^{(2-p)}(t) + \int_0^{\alpha(t)} g(s)\,\mathcal{L}(s, z^q(s))\,ds,$$

then we have

$$\begin{aligned} z'(t) &\leq \alpha'(t) f(\alpha(t)) v^p(t), \\ v(0) &= u_0^{(2-p)}, \end{aligned} \tag{3.36}$$

$p \leq 1 \Rightarrow 2-p \geq 1$ and $z^{(2-p)}(t) \leq v(t)$, then

$$z(t) \leq v(t) \tag{3.37}$$

Differentiating $v(t)$ with respect to t and using (3.36) and (3.37), we obtain

$$v'(t) \leq (2-p)\,\alpha'(t) f(\alpha(t)) v(t) + \alpha'(t) g(\alpha(t))\,\mathcal{L}(\alpha(t), v^q(t)), \tag{3.38}$$

taking $q = \frac{m}{n}$ with $n > m \geq 0$, $n \neq 0$ and applying Lemma 1, we have

$$v^q(t) \leq q k^{q-1} v(t) + (1-q) k^q,$$

then,

$$\begin{aligned} v'(t) &\leq (2-p)\,\alpha'(t) f(\alpha(t)) v(t) + \alpha'(t) g(\alpha(t)) \\ &\quad \times \left[\mathcal{L}(\alpha(t), (1-q)k^q) + qk^{q-1} M(\alpha(t), (1-q)k^q) v(t)\right] \\ v'(t) &\leq \alpha'(t) g(\alpha(t))\,\mathcal{L}(\alpha(t), (1-q)k^q) \\ &\quad + \left[(2-p)\,\alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t))\, qk^{q-1} M(\alpha(t), (1-q)k^q)\right] v(t), \end{aligned}$$

using Lemma 2 to the above inequality, we get

$$\begin{aligned} v(t) \leq & exp\left((2-p)\int_0^{\alpha(t)} \left(f(s) + \frac{q}{2-p} k^{q-1} g(s) M(s, (1-q)k^q)\right) ds\right) \\ & \times \left[u_0^{(2-p)} + \int_0^{\alpha(t)} g(s)\,\mathcal{L}(s, (1-q)k^q)\right. \\ & \left. exp\left(-(2-p)\int_0^s \left(f(\tau) + \frac{q}{2-p} k^{q-1} g(\tau) M(\tau, (1-q)k^q)\right) d\tau\right) ds\right], \end{aligned}$$

then,

$$v^p(t) \leq exp\left(p(2-p)\int_0^{\alpha(t)} p_5(s)\,ds\right) q_5(t) \tag{3.39}$$

where where $p_5(.)$ and $q_5(.)$ are defined as in (3.33).

From (3.36), we have

$$z'(t) \le \alpha'(t) f(\alpha(t)) q_5(t)\, exp\left(p(2-p)\int_0^{\alpha(t)} p_5(s)\,ds\right).$$

Thus,

$$z(t) \le u_0 + \int_0^{\alpha(t)} f(s) q_5\left(\alpha^{-1}(s)\right) exp\left(p(2-p)\int_0^{s} p_5(\tau)\,d\tau\right) ds.$$

Using (3.34), we get the required inequality in (3.32). □

Remark 15. It is interesting to note that in the special case when $\mathcal{L}(t,x) = x$, then inequality (3.31) can be reduced to to the inequality (1.2) discussed by Abdeldaim [3].

Theorem 16. *Let $u(t), g(t), h(t) \in C(I,I)$, $\alpha(t) \in C^1(I,I)$ be nondecreasing function with $\alpha(0) = 0$, $\alpha(t) \le t$ on I. If $u(t)$ satisfies*

$$\begin{aligned} u(t) \;\le\; & u_0 + \int_0^{\alpha(t)} g(s) u^p(s)\,ds + \\ & + \int_0^{\alpha(t)} g(s)\mathcal{L}(s, u^q(s))\left[u^{2-p}(s) + \int_0^{\alpha(s)} h(\lambda) u(\lambda)\,d\lambda\right]^p ds, \end{aligned} \tag{3.40}$$

for $u_0 > 0$, $0 < p \le 1$, $0 \le q < 1$, and $\mathcal{L}, M \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ satisfy

$$0 \le \mathcal{L}(t,x) - \mathcal{L}(t,y) \le M(t,y)(x-y),\, x \ge y \ge 0.$$

Then,

$$u(t) \le K(t)\exp\{\int_0^{\alpha(t)} \psi(s)\}ds \tag{3.41}$$

where

$$\begin{aligned} \psi(t) &= pk^{p-1} g(s)\left(1 + \frac{q}{p} k^{q-p} M\left(s, (1-q)k^q\right)\theta^p\left(\alpha^{-1}(s)\right)\right) \\ K(t) &= u_0 + \int_0^{\alpha(t)} g(s)\left[(1-p)k^p + \mathcal{L}\left(s,(1-q)k^q\right)\theta^p\left(\alpha^{-1}(s)\right)\right] \\ & \quad \times exp\left\{-\int_0^s \psi(\tau)d\tau\right\} ds \end{aligned} \tag{3.42}$$

$$\theta(t) = \frac{exp\{\int_0^{\alpha(t)} q_6(s)\,ds\}}{C - \int_0^{\alpha(t)} p_6(s)\left(exp\{\int_0^s q_6(\tau)\,d\tau\}\right)ds}, \tag{3.43}$$

and

$$\begin{aligned} p_6(t) &= (2-p)\,qk^{q-1}g(t)\,M(t,(1-q)\,k^q)\,, \\ q_6(t) &= (2-p)\,g(t)\,\{1+\mathcal{L}(t,1-q)\,k^q)\}+h(t)\,, \end{aligned} \tag{3.44}$$

provided that

$$\int_0^{\alpha(t)} p_6(s)\left(exp\{\int_0^s q_6(\tau)\,d\tau\}\right)ds < C.$$

Proof. Define a function $z(t)$ by the right hand side in (3.40), then we have, $z(0) = u_0$ and

$$u(t) \leq z(t)\,. \tag{3.45}$$

Differentiating $z(t)$, with respect to t and using (3.45), we obtain

$$z'(t) \leq \alpha'(t)\,g(\alpha(t))\,z^p(t)+\alpha'(t)\,g(\alpha(t))\,\mathcal{L}(\alpha(t),z^q(t))\left[z^{2-p}(t)+\int_0^{\alpha(t)} h(s)\,z(s)\,ds\right]^p, \tag{3.46}$$

taking $q = \frac{m}{n}$ with $n > m \geq 0$, $n \neq 0$, we have

$$\mathcal{L}(\alpha(t),z^q(t)) \leq \mathcal{L}(\alpha(t),(1-q)\,k^q)+qk^{q-1}M(\alpha(t),(1-q)\,k^q)\,z(t)\,,$$

then

$$\begin{aligned} z'(t) \leq\ & \alpha'(t)\,g(\alpha(t))\,z^p(t)+\alpha'(t)\,g(\alpha(t)) \\ & \times\left[\mathcal{L}(\alpha(t),(1-q)\,k^q)+qk^{q-1}M(\alpha(t),(1-q)\,k^q)\,z(t)\right]v^p(t) \end{aligned} \tag{3.47}$$

where

$$v(t) = z^{2-p}(t)+\int_0^{\alpha(t)} h(s)\,z(s)\,ds,\ \ v(0) = u_0^{(2-p)}, v(t) > 0. \tag{3.48}$$

$$z(t) \leq z^{2-p}(t) \leq v(t)\,. \tag{3.49}$$

Differentiating $v(t)$ with respect to t and using (3.47) and (3.49), we get

$$\begin{aligned} v'(t) \leq\ & (2-p)\,\alpha'(t)\,g(\alpha(t))\,v^p(t)\,v^{1-p}(t)+(2-p)\,\alpha'(t)\,g(\alpha(t))\times \\ & \left[\mathcal{L}(\alpha(t),(1-q)\,k^q)+qk^{q-1}M(\alpha(t),(1-q)\,k^q)\,v(t)\right]v^p(t)\,v^{1-p}(t) \\ & +\alpha'(t)\,h(\alpha(t))\,v(t)\,, \end{aligned} \tag{3.50}$$

$$\begin{aligned} v'(t) \leq\ & (2-p)\,\alpha'(t)\,g(\alpha(t))\,v(t)+(2-p)\,\alpha'(t)\,g(\alpha(t))\,v(t)\times \\ & \left[\mathcal{L}(\alpha(t),(1-q)\,k^q)+qk^{q-1}M(\alpha(t),(1-q)\,k^q)\,v(t)\right]+\alpha'(t)\,h(\alpha(t))\,v(t)\,, \end{aligned}$$

$$z'(t) \leq \left[(2-p)\,\alpha'(t)\,g(\alpha(t))\,(1+\mathcal{L}(\alpha(t),(1-q)\,k^q)) + \alpha'(t)\,h(\alpha(t))\right] v(t) \\ + \left[(2-p)\,\alpha'(t)\,g\left(\alpha(t)\right)qk^{q-1}M(\alpha(t),(1-q)\,k^q)\right] v^2(t)\,.$$

By taking

$$\begin{aligned} A(t) &= (2-p)\,\alpha'(t)\,g(\alpha(t))qk^{q-1}M(\alpha(t),(1-q)\,k^q) \\ \beta(t) &= (2-p)\,\alpha'(t)\,g(\alpha(t))\,(1+\mathcal{L}(\alpha(t),(1-q)\,k^q)) + \alpha'(t)\,h(\alpha(t))\,, \end{aligned}$$

we have

$$\begin{aligned} v'(t) &\leq A(t)\,v^2(t) + \beta(t)\,v(t)\,, \\ v^{-2}(t)\,v'(t) &\leq A(t) + \beta(t)\,v^{-1}(t)\,, \end{aligned}$$

Letting $v^{-1}(t) = y(t)$,we obtain

$$\begin{aligned} -y'(t) &\leq A(t) + \beta(t)\,y(t)\,, \\ y'(t) &= -v'(t)v^{-2}(t)\,, \end{aligned}$$

then, from Lemma 2, we have

$$y(t) \geq u_0^{-(2-p)} exp\{-\int_0^t \beta(s)ds\} - \int_0^t A(s)exp\{-\int_s^t \beta(\tau)d\tau\}ds.$$

Moreover,

$$\begin{aligned} y(t) \geq & u_o^{-(2-p)} exp\{-\int_0^{\alpha(t)} ((2-p)\,g(s)\,\{1+\mathcal{L}(s,(1-q)\,k^q)\} + h(s))\,ds\} \\ & -(2-p)\,qk^{q-1}\int_0^{\alpha(t)} g(s)\,M(s,(1-q)\,k^q) \times \\ & \exp\{-\int_s^{\alpha(t)} ((2-p)\,g(\tau)\,\{1+\mathcal{L}(\tau,(1-q)\,k^q)\} + h(\tau))d\tau\}ds, \end{aligned}$$

$$\begin{aligned} y(t) \geq & exp\{-\int_0^{\alpha(t)} [(2-p)\,g(s)\,(1+\mathcal{L}(s,(1-q)\,k^q)) + h(s)]\,ds\} \\ & \times \left[u_0^{-(2-p)} - (2-p)\,qk^{q-1}\int_0^{\alpha(t)} g(s)\,M(s,(1-q)\,k^q) \right. \\ & \left. \times\, exp\{\int_0^s [(2-p)\,g(\tau)\,(1+\mathcal{L}(\tau,(1-q)\,k^q)) + h(\tau)]\,d\tau\}ds \right], \end{aligned}$$

then,

$$y(t) \geq exp\left(-\int_0^{\alpha(t)} q_6(s)\,ds\right)(u_0^{-(2-p)} - \int_0^{\alpha(t)} p_6(s)\,exp\left(\int_0^s q_6(\tau)\,d\tau\right)ds), \tag{3.51}$$

where $p_6(.)$ and $q_6(.)$ are defined as in (3.44).

$$y(t) \geq \frac{\left(u_0^{-(2-p)} - \int_0^{\alpha(t)} p_6(s)\left(exp\{\int_0^s q_6(\tau)\,d\tau\}\right) ds\right.}{exp\{\int_0^{\alpha(t)} q_6(s)\,ds\}}, \tag{3.52}$$

thus,

$$v(t) \leq \frac{exp\{\int_0^{\alpha(t)} q_6(s)\,ds\}}{u_0^{-(2-p)} - \int_0^{\alpha(t)} p_6(s)\left(exp\{\int_0^s q_6(\tau)\,d\tau\}\right) ds}.$$

Let $u_0^{-(2-p)} = C$ and

$$\int_0^{\alpha(t)} p_6(s)\left(exp\{\int_0^s q_6(\tau)\,d\tau\}\right) ds < C,$$

we easily obtain $v(t) \leq \theta(t)$. Where $\theta(t)$ is well defined as in (3.43).

From (3.47) and taking $p = \frac{m^*}{n^*}$, $n^* \geq m^* > 0$, $n^* \neq 0$, we get

$$\begin{aligned} z'(t) \leq & \alpha'(t)\, g(\alpha(t))\left[(1-p)\,k^p + \mathcal{L}(\alpha(t), (1-q)\,k^q)\,\theta^p(t)\right] \\ & + \alpha'(t)\, g(\alpha(t))\{pk^{p-1} + qk^{q-1} M(\alpha(t), (1-q)\,k^q)\,\theta^p(t)\} z(t). \end{aligned}$$

Applying Lemma 2 to the above inequality, we obtain

$$\begin{aligned} z(t) \leq & exp(\int_0^{\alpha(t)} g(s)\left(pk^{p-1} + qk^{q-1} M(s, (1-q)\,k^q)\,\theta^p\left(\alpha^{-1}(s)\right)\right) ds) \\ & \times \Bigg[u_0 + \int_0^{\alpha(t)} g(s)\left[(1-p)\,k^p + \mathcal{L}(s, (1-q)\,k^q)\,\theta^p\left(\alpha^{-1}(s)\right)\right] \\ & \times exp\{-\int_0^s g(\tau)\left(pk^{p-1} + qk^{q-1} M(\tau, (1-q)\,k^q)\right)\theta^p\left(\alpha^{-1}(\tau)\right) d\tau\} ds\Bigg], \end{aligned}$$

the above inequality can be reformulated as

$$z(t) \leq K(t)\exp(\int_0^{\alpha(t)} \psi(s))ds, \tag{3.53}$$

where $K(.)$ and $\psi(.)$ are defined as in (3.42). From (3.45), we get the desired inequality (3.41). This completes the proof. □

4 Further results

Theorem 17. *Assume $u(t), g(t), h(t), n(t) \in C(I, I)$ and $n(t)$ is nondecreasing. Let $\alpha(t) \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$. Let Φ be a nondecreasing function tbat belongs to class $\mathcal{T}$ (see Definition 3) with $\Phi^{-1}(I) \subset I$ and $\Phi^{-1} \geq 1$.*

If $u(t)$ satisfies

$$\Phi(u^p(t)) \leq n^p(t) + \int_0^t g(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)u^q(s)ds, \tag{4.1}$$

for $p \neq 0$, $p \geq q \geq 0$ (constants), then

$$u(t) \leq U^{\frac{1}{p}}, \text{ where} \tag{4.2}$$

$$U = \max(n^p(t), 1)\ \Phi^{-1}\left(W^{-1}\left(W(1) + \int_0^t g(s)ds + \left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right)\int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}ds\right)\right),$$

and

$$W(1) + \int_0^t g(s)ds + \left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right)\int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}ds \quad \in \quad Dom(W^{-1}),$$

$$W^{-1}\left(W(1) + \int_0^t g(s)ds + \left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right)\int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}ds\right) \quad \in \quad Dom(\Phi^{-1}),$$

where W is the function defined by $W(t) = \int_0^t \frac{ds}{\Phi^{-1}(s)}$, for all $t \in I$.

Proof. Denote $a^p(t) = \max(n^p(t), 1)$. Then (4.1) can be rewritten as :

$$\frac{\Phi(u^p(t))}{a^p(t)} \leq 1 + \int_0^t \frac{g(s)}{a^p(s)}u^p(s)ds + \int_0^{\alpha(t)} \frac{h(s)}{a^p(s)}u^q(s)ds. \tag{4.3}$$

Setting

$$z(t) = \frac{u(t)}{a(t)}. \tag{4.4}$$

Since Φ belongs to class $\mathcal{T}$, one has

$$\Phi(z^p(t)) \leq 1 + \int_0^t g(s)z^p(s)ds + \int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}z^q(s)ds, \quad \forall t \in I. \tag{4.5}$$

Define an auxiliary function $v(t)$ by

$$v(t) = 1 + \int_0^t g(s)z^p(s)ds + \int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}z^q(s)ds, \tag{4.6}$$

from (4.6) and using properties of Φ, we get

$$z(t) \leq \left[\Phi^{-1}(v(t))\right]^{\frac{1}{p}} \text{ and } v(0) = 1, \tag{4.7}$$

Using (4.7), inequality (4.6) becomes

$$v(t) \leq 1 + \int_0^t g(s)\Phi^{-1}(v(s))ds + \int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}\left[\Phi^{-1}(v(s))\right]^{\frac{q}{p}} ds. \tag{4.8}$$

Differentiating $v(t)$ with respect to t and using Lemma 1, we obtain

$$v'(t) \leq g(t)\Phi^{-1}(v(t)) + \alpha'(t)\frac{h(\alpha(t))}{a^{p-q}(\alpha(t))}\left(\frac{q}{p}k^{\frac{q-p}{p}}\Phi^{-1}(v(t)) + \frac{p-q}{p}k^{\frac{q}{p}}\right),$$

then,

$$\frac{v'(t)}{\Phi^{-1}(v(t))} \leq g(t) + \alpha'(t)\frac{h(\alpha(t))}{a^{p-q}(\alpha(t))}\left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right). \tag{4.9}$$

Integrating the last inequality from 0 to t, we get

$$W(v(t)) \leq W(v(0)) + \int_0^t g(s)ds + \left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right)\int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}ds,$$

and then

$$v(t) \leq W^{-1}\left(W(1) + \int_0^t g(s)ds + \left(\frac{q}{p}k^{\frac{q-p}{p}} + \frac{p-q}{p}k^{\frac{q}{p}}\right)\int_0^{\alpha(t)} \frac{h(s)}{a^{p-q}(s)}ds\right).$$

From the above inequality and using (4.7) and (4.4), one obtains the desired inequality (4.2). □

Theorem 18. *Let $u(t), g(t), f(t) \in C(I, I)$ and $n(t)$ be nondecreasing. Let $\alpha(t) \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$ and $\mathcal{L}, m \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ and $\phi : I \to I$ be a continuous and strictly increasing function with $\phi(0) = 0$ such that*

$$0 \leq \mathcal{L}(t, x) - \mathcal{L}(t, y) \leq m(t, y)\phi^{-1}(x - y), \text{ for } t \in \mathbb{R}_+ \; x \geq y \geq 0, \tag{4.10}$$

where ϕ^{-1} is the inverse function of ϕ and

$$\phi^{-1}(xy) \leq \phi^{-1}(x)\phi^{-1}(y). \tag{4.11}$$

If $u(t)$ satisfies

$$u^p(t) \leq n(t) + \phi\left(\int_0^t g(s)\mathcal{L}(s, u^p(s))\,ds + \int_0^{\alpha(t)} h(s)\mathcal{L}(s, u^q(s))\,ds\right), \tag{4.12}$$

for $p \neq 0$, $p \geq q \geq 0$, then

$$u(t) \leq \left(n(t) + \phi\left[\int_0^t p_7(s)\exp\left(\int_s^t q_7(\tau)d\tau\right)ds\right]\right)^{\frac{1}{p}}, \tag{4.13}$$

where

$$\begin{aligned} p_7(t) &= \mathcal{L}(t,n(t))g(t) \\ &\quad +\alpha'(t)\,h(\alpha(t))\,\mathcal{L}\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right) \\ q_7(t) &= g(t)m(t,n(t)) \\ &\quad +m\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\phi^{-1}\left(\frac{p-q}{p}k^{\frac{q}{p}}\right), \end{aligned} \tag{4.14}$$

for all $t\in I$.

Proof. Let

$$z(t)=\int_0^t g(s)\,\mathcal{L}(s,u^p(s))\,ds+\int_0^{\alpha(t)} h(s)\,\mathcal{L}(s,u^q(s))\,ds,\ z(0)=0,$$

and

$$u(t)\le[n(t)+\phi(z(t))]^{\frac{1}{p}}. \tag{4.15}$$

Differentiating $z(t)$, with respect to t and using (4.15), we have

$$z'(t)\le g(t)\mathcal{L}(t,n(t)+\phi(z(t))+\alpha'(t)\,h(\alpha(t))\,\mathcal{L}\left(\alpha(t),(n(t)+\phi(z(t))^{\frac{q}{p}}\right). \tag{4.16}$$

From (4.10) and using Lemma 1, we get

$$\begin{aligned} z'(t) &\le g(t)\left[\mathcal{L}\left(t,n(t)\right)+m(t,n(t))\phi^{-1}(\phi(z(t))\right] \\ &\quad +\alpha'(t)\,h(\alpha(t))\,\mathcal{L}\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}(n(t)+\phi(z(t))+\frac{p-q}{p}k^{\frac{q}{p}}\right), \end{aligned} \tag{4.17}$$

$$\begin{aligned} z'(t) &\le g(t)\left[\mathcal{L}(t,n(t))+m(t,n(t))z(t)\right]+\alpha'(t)\,h(\alpha(t)) \\ &\quad\times\left[\mathcal{L}\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)+\right. \\ &\quad\left.+\,m\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\phi^{-1}(\frac{p-q}{p}k^{\frac{q}{p}}\phi(z(t))\right]. \end{aligned} \tag{4.18}$$

Using (4.11), we obtain

$$\begin{aligned} z'(t) &\le \mathcal{L}(t,n(t))g(t)+\alpha'(t)\,h(\alpha(t))\,\mathcal{L}\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right) \\ &\quad+\left[g(t)m(t,n(t))+m\left(\alpha(t),\frac{q}{p}k^{\frac{q-p}{p}}n(t)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\phi^{-1}\left(\frac{p-q}{p}k^{\frac{q}{p}}\right)\right]z(t), \end{aligned} \tag{4.19}$$

the last inequality can be expressed as

$$z'(t)\le p_7(t)+q_7(t)z(t),$$

where $p_7(.)$ and $q_7(.)$ are defined as in (4.14). Applying Lemma 2 to the above inequality, we have

$$z(t) \leq \int_0^t p_7(s) \exp\left(\int_s^t q_7(\tau) d\tau\right) ds. \tag{4.20}$$

Using (4.15), we get the required inequality in (4.13). □

5 Application

In this section, we present some examples to investigate certain properties of solutions of differential equations and integral equations.

Example 19. Consider the retarded differential equation

$$\begin{array}{c} \frac{du}{dt} = H(t, u(\alpha(t)), K(t, u(\alpha(t)))) \qquad , \forall t \in I \\ u(0) = u_0, \end{array} \tag{5.1}$$

where $K \in C(I \times I, \mathbb{R}), H \in C(I^3, \mathbb{R})$, and u_0 is a positive constant.

Proposition 20. *Assume that the functions H and K in (5.1) satisfy the conditions*

$$|K(t, u(\alpha(t)))| \leq g(\alpha(t)) \; L(t, |u^q(\alpha(t))|), \tag{5.2}$$

$$|H(t, u(\alpha(t)), K(t, u(\alpha(t))))| \leq f(\alpha(t)) \left[\; \left|u^{(2-p)}(\alpha(t))\right| + \int_0^t |K(\tau, u(\alpha(\tau)))| \, d\tau\right]^p, \tag{5.3}$$

where $u(t), f(t), g(t), \alpha(t), L(t, u), p$ and q are defined as in Theorem 14.

If $u(t)$ is a solution of (5.1), then

$$|u(t)| \leq |u_0| + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))} \phi^* \left(\alpha^{-1}(s)\right) exp\left(p\,(2-p) \int_0^s P^*(\tau)\, d\tau\right) ds, \tag{5.4}$$

for all $t \in I$,

$$\begin{cases} P^*(t) = & \frac{f(t)}{\alpha'(\alpha^{-1}(t))} + \frac{q}{2-p} k^{q-1} \frac{g(t)}{\alpha'(\alpha^{-1}(t))} M\left(\alpha^{-1}(t), (1-q)\, k^q\right), \\ \phi^*(t) = & \left[u_0^{(2-p)} + \int_0^{\alpha(t)} \frac{g(t)}{\alpha'(\alpha^{-1}(t))} \mathcal{L}\left(\alpha^{-1}(t), (1-q)\, k^q\right) exp\left(-(2-p) \int_0^s P^*(\tau)\, d\tau\right) ds\right]^p. \end{cases}$$

Proof. Integrating both sides of equation (5.1) from 0 to t, we obtain

$$u(t) = u_0 + \int_0^t H(s, u(\alpha(s)), K(t, u(\alpha(s))))ds. \tag{5.5}$$

Using the conditions (5.2)-(5.3), from (5.5), we get

$$|u(t)| \leq |u_0| + \int_0^t f(\alpha(s)) \left[\left|u^{(2-p)}(\alpha(s))\right| + \int_0^s g(\alpha(\tau))\ L(\tau, |u^q(\alpha(\tau))|)d\tau \right]^p ds.$$

Thus

$$|u(t)| \leq |u_0| + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))} \left[\left|u^{(2-p)}(s)\right| + \int_0^s \frac{g(\tau)}{\alpha'(\alpha^{-1}(\tau))}\ L(\alpha^{-1}(\tau), |u^q(\tau)|)d\tau \right]^p ds, \quad . \tag{5.6}$$

holds for all $t \in I$. Using Theorem 14 in (5.6), we immediately obtain (5.4). This completes the proof. □

Example 21. Consider the following retarded integral equation

$$u(t) = n(t) + \int_0^t G(s, u(s))ds + \int_0^{\alpha(t)} H(s, u(s))ds, \tag{5.7}$$

where $u(t), n(t) \in C(I, I)$ and $n(t)$ is nondecreasing, $\alpha(t) \in C^1(I, I)$ is nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$ and $G, H \in C(I \times I, \mathbb{R})$.

Proposition 22. *Assume that the functions G and H in (5.7) satisfy the conditions*

$$\begin{aligned} |\ G(s, u(s) - G(s, \overline{u}(s))| &\leq g(s)\, |u(s) - \overline{u}(s)|\, , \\ |H(s, u(s)) - H(s, \overline{u}(s))| &\leq h(s) w(|u(s) - \overline{u}(s)|), \end{aligned} \tag{5.8}$$

where g, h and w are defined as in Theorem 5 with $w(0) = 0$ and $u(t)$ is a solution of (5.7). Then (5.7) has at most one solution.

Proof. Let $u(t)$ and $\overline{u}(t)$ be two solutions of (5.7), then

$$\begin{aligned} u(t) - \overline{u}(t) &= \int_0^t G(s, u(s)) - G(s, \overline{u}(s))\ ds \\ &\quad + \int_0^{\alpha(t)} H(s, u(s)) - H(s, \overline{u}(s))\ ds\ . \end{aligned} \tag{5.9}$$

From (5.8) and (5.9), we get

$$\begin{aligned} |u(t) - \overline{u}(t)| &\leq \int_0^t |G(s, u(s)) - G(s, \overline{u}(s))|\, ds + \int_0^{\alpha(t)} |H(s, u(s) - H(s, \overline{u}(s))|\ ds \\ &\leq \int_0^t g(s)\, |u(s) - \overline{u}(s)|\, ds \quad + \int_0^{\alpha(t)} h(s) w(|u(s) - \overline{u}(s)|) ds. \end{aligned} \tag{5.10}$$

Applying Theorem 5 with ($p = q = 1$) to the above inequality, we obtain that $|u(t) - \overline{u}(t)| \leq 0$, which implies $u(t) = \overline{u}(t)$, for $t \in I$, i.e., equation (5.7) has at most one solution. □

Example 23. we consider the following Volterra type retarded integral equation

$$u^2(t) - \int_0^t s^2 \arctan(u(s))ds - \int_0^{\alpha(t)} s\ln(u(s)+1)ds = e^t, \tag{5.11}$$

Corollary 24. *Assume that $u(t) \in C(\mathbb{I}, \mathbb{R})$ and let $\alpha \in C^1(I, I)$ be a nondecreasing function with $\alpha(t) \leq t$, $\alpha(0) = 0$. If $u(t)$ satisfies (5.11), then*

$$u(t) \leq \left[e^t + \int_0^t p(s)\exp\left(\int_s^t q(\tau)d\tau\right)ds\right]^{\frac{1}{2}}, \tag{5.12}$$

where

$$\begin{aligned} p(t) &= t^2 \arctan\left(\frac{1}{2}e^t + \frac{1}{2}\right) + \alpha'(t)\alpha(t)\ln\left(1 + \frac{1}{2}e^t + \frac{1}{2}\right), \\ q(t) &= \frac{1}{2}\frac{t^2}{1+\left(\frac{1}{2}e^t + \frac{1}{2}\right)^2} + \frac{1}{2}\alpha'(t)\alpha(t)\frac{1}{\left(\frac{1}{2}e^t + \frac{3}{2}\right)}, \end{aligned}$$

for all $t \in I$.

Proof. From (5.11), we obtain

$$|u(t)|^2 \leq e^t + \int_0^t s^2 \arctan(|u(s)|\,ds + \int_0^{\alpha(t)} s\ln|u(s)| + 1)ds. \tag{5.13}$$

By application of Theorem 9, and taking $k = 1$, we obtain the inequality (5.12). □

Conclusion 25. *In this paper, some new nonlinear retarded integral inequalities are obtained. They can be seen as generalizations and refinements of many existing results. These inequalities help us in the study of some classes of integral and integro-differential equations.*

Acknowledgement. The authors are very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

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