Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 15 (2020), 153 – 168

EXISTENCE RESULTS OF SELF-SIMILAR SOLUTIONS TO THE CAPUTO-TYPE'S SPACE-FRACTIONAL HEAT EQUATION

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Abstract. This paper investigates the problem of existence and uniqueness of solutions under the self-similar forms to the space-fractional heat equation. By applying the properties of Banach's fixed point theorems, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type, we establish several results on the existence and uniqueness of self-similar solutions to this equation.

1 Introduction

Many problems and models in physics, chemistry, biology and economics are modeled by partial differential equations (PDEs), we shall give in this work an example of a class of well-known equations, which allow to describe the diffusion phenomena, it's a fractional-order's PDE and known as "space-fractional heat equation", which is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial x^\alpha},\ 1 < \alpha \leq 2, \tag{1.1}$$

with

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \begin{cases} \frac{\partial^n u}{\partial x^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} u(s,t)\, ds & n-1 < \alpha < n \in \mathbb{N}, \end{cases}$$

where $u = u(x,t)$ is a scalar function of space variables $x \in [0, X]$, and time $t \in [t_0, \infty)$, with $X, t_0 > 0$.

Recently, the lie group analysis of this equation has been discussed by Luchko et al. (see [6], [11]), which studied the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = d\frac{\partial^\beta u}{\partial x^\beta},\ x > 0,\ t > 0,\ d > 0,\ \alpha, \beta \geq 0, \tag{1.2}$$

2020 Mathematics Subject Classification: 35R11; 35A01; 34A08; 35C06; 34K37.
Keywords: Space-fractional heat equation; Self-similar solutions; Existence; Uniqueness.
This work was supported by the General Direction of Scientific Research and Technological Development (DGRSTD)- Algeria.

to obtain the partial scale-invariant solutions of this equation.

The method of group analysis of differential equations began with the work of Sophus Lie more than a hundred years ago. Roughly speaking a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. For partial differential equations one can determine special types of solutions, which are invariant under some subgroup of the full symmetry group of the system. These "group-invariant" solutions are found by solving a reduced system of equations having fewer independent variables than the original system.

In [6], this method was used for finding the scale-invariant solutions of the time-fractional diffusion equation ($\beta = 2$ in (1.2)). In the case of equation (1.2), the scale-invariant solutions can be found by solving an ordinary differential equation of fractional order with a new independent variable $\eta = xt^{-\alpha/\beta}$, The derivatives there are the Erdélyi-Kober derivatives (left- and right-hand sided) depending on the parameters α, β of equation (1.2) and on a parameter γ of its scaling group.

The general solution of this differential equation of fractional order is obtained in terms of the generalized Wright function.

Our main goal in this work is to determine the existence, uniqueness and main properties of the solution of the space-fractional PDE (1.1), under the self-similar form which is:

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha}}}\right), \text{ with } (x,t) \in [0,X] \times [t_0,\infty), \tag{1.3}$$

where $X, t_0 > 0$. The "basic profile" f in (1.3), are not known in advance and are to be identified, $\beta \in \mathbb{R}$ is a constant chosen so that the solutions exist.

2 Definitions and preliminary results

In this section we present the necessary definitions from fractional calculus theory. Let $\lambda > 0$, by $C[0,\lambda]$ we denote the Banach space of all continuous functions from $[0,\lambda]$ into $\mathbb{R}$ with the norm:

$$\|y\|_{\infty} = \sup_{0\leq\eta\leq\lambda} |y(\eta)|.$$

We start with the definitions introduced in [10] with a slight modification in the notation.

Definition 1 ([10]). *The left-sided (arbitrary) fractional integral of order $\alpha > 0$ of a continuous function $y : [0,\lambda] \to \mathbb{R}$ is given by:*

$$\mathcal{I}_{0^+}^{\alpha} y(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \xi)^{\alpha-1} y(\xi)\, d\xi, \ \ \eta \in [0,\lambda],$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-s} s^{\alpha-1} ds$, is the Euler gamma function.

Definition 2 (Caputo fractional derivative [10]). *The left-sided Caputo fractional derivative of order $\alpha > 0$ of a function $y : [0, \lambda] \to \mathbb{R}$ is given by:*

$$ {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\eta} (\eta - \xi)^{n-\alpha-1} \frac{d^n y(\xi)}{d\xi^n} d\xi, \ \eta \in [0, \lambda], \ n = [\alpha] + 1, $$

Lemma 3 ([10]). *Assume that ${}^{C}\mathcal{D}_{0+}^{\alpha} y \in C[0, \lambda]$, for all $\alpha > 0$. Then*

$$ \mathcal{I}_{0+}^{\alpha}\, {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) = y(\eta) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} \eta^k, $$

where $n = [\alpha] + 1$.

Remark 4. *For all y, ${}^{C}\mathcal{D}_{0+}^{\alpha} y \in C[0, \lambda]$, where $1 < \alpha \leq 2$, we have:*

$$ \begin{aligned} \mathcal{I}_{0+}^{\alpha-1}\, {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) &= \frac{d}{d\eta} \mathcal{I}_{0+}^{\alpha}\, {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) \\ &= \frac{d}{d\eta} \left[y(\eta) - y(0) - \eta y'(0) \right] \\ &= y'(\eta) - y'(0). \end{aligned} \tag{2.1} $$

Moreover; if $y'(0) = 0$, then

$$ \mathcal{I}_{0+}^{\alpha-1}\, {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) = y'(\eta). \tag{2.2} $$

As $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we have for any $\eta \in [0, \lambda]$,

$$ \begin{aligned} \left| y'(\eta) \right| &= \left| \mathcal{I}_{0+}^{\alpha-1}\, {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) \right| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^{\eta} (\eta-\xi)^{\alpha-2} \left| {}^{C}\mathcal{D}_{0+}^{\alpha} y(\xi) \right| d\xi \\ &\leq \left[-\frac{(\eta-\xi)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \right]_0^{\eta} \left\{ \sup_{0 \leq \eta \leq \lambda} \left| {}^{C}\mathcal{D}_{0+}^{\alpha} y(\eta) \right| \right\} \\ &\leq \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \left\| {}^{C}\mathcal{D}_{0+}^{\alpha} y \right\|_{\infty}. \end{aligned} \tag{2.3} $$

Theorem 5 (Ascoli-Arzelà [1]). *Let E be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{A}$ is relatively compact.*

Definition 6 (Completely continuous [7]). *We say $\mathcal{A} : E \to E$ is completely continuous if for any bounded subset $P \subset E$, the set $\mathcal{A}(P)$ is relatively compact.*

Lemma 7 (Gronwall [9]). *Let $f(\eta)$ and $g(\eta)$ be nonnegative, continuous functions on $0 \leq \eta \leq \lambda$, for which the inequality:*

$$ f(\eta) \leq \mu + \int_0^{\eta} g(\xi) f(\xi) d\xi, \ 0 \leq \eta \leq \lambda, $$

holds, where μ is a nonnegative constant. Then:

$$f(\eta) \leq \mu \exp\left(\int_0^\eta g(\xi)\, d\xi\right), \ 0 \leq \eta \leq \lambda.$$

Theorem 8 (Banach's fixed point [8])**.** *Let P be a non-empty closed subset of a Banach space E, then any contraction mapping $\mathcal{A}: P \to P$ has a unique fixed point.*

Theorem 9 (Schauder's fixed point [8])**.** *Let E be a Banach space, and P be a closed, convex and nonempty subset of E. Let $\mathcal{A}: P \to P$ be a continuous mapping such that $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.*

Theorem 10 (Nonlinear Alternative of Leray-Schauder type [8])**.** *Let E be a Banach space with $P \subset E$ be a closed and convex. Assume V is a relatively open subset of P with $0 \in V$ and $\mathcal{A}: \bar{V} \to P$ is a compact map. Then either,*
(i) *$\mathcal{A}$ has a fixed point in $\bar{V}$; or*
(ii) *there is a point $f \in \partial V$ and $\mu \in (0,1)$ with $f = \mu\mathcal{A}(f)$.*

3 Main results

3.1 Statement of the problem

In this part, we first attempt to find the equivalent approximate to the following problem of the space-fractional heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial x^\alpha}, \ (x,t) \in [0,X] \times [t_0, \infty), & 1 < \alpha \leq 2, \\ u(0,t) = t^\beta U, \ \frac{\partial u}{\partial x}(0,t) = 0, & \beta, U \in \mathbb{R}. \end{cases} \tag{3.1}$$

Under the self-similar form which is:

$$u(x,t) = t^\beta f\left(\frac{x}{t^{\frac{1}{\alpha}}}\right), \ \beta \in \mathbb{R}. \tag{3.2}$$

We should first deduce the equation satisfied by the function f in (3.2) used for the definition of self-similar solutions.

Theorem 11. *Let $\alpha, \beta \in \mathbb{R}$, be such that $1 < \alpha \leq 2$, and $(x,t) \in [0,X] \times [t_0, \infty)$ for some $X, t_0 > 0$. Then the transformation:*

$$u(x,t) = t^\beta f(\eta), \ \text{with } \eta = \frac{x}{t^{\frac{1}{\alpha}}}.$$

Reduces the partial differential equation of space-fractional order (1.1) to the ordinary differential equation of fractional order of the form:

$${}^C\mathcal{D}_{0^+}^\alpha f(\eta) = \beta f(\eta) - \frac{1}{\alpha}\eta f'(\eta), \ \eta \in [0,\lambda],$$

where $\lambda = X t_0^{-\frac{1}{\alpha}}$.

Proof. The fractional equation resulting from the substitution of expression (3.2) in the original PDE (1.1), should be reduced to the standard bilinear functional equation (see [14]). First, for $\eta = \frac{x}{t^{\frac{1}{\alpha}}}$, we get:

$$\frac{\partial u}{\partial t} = t^{\beta-1}\left[\beta f(\eta) - \frac{1}{\alpha}\eta f'(\eta)\right]. \tag{3.3}$$

In another way, we get for $\xi = \frac{s}{t^{\frac{1}{\alpha}}}$, that:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial x^\alpha} &= \frac{t^\beta}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}\frac{d^n}{ds^n} f\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) ds \\
&= \frac{t^{\beta+\frac{1}{\alpha}}}{\Gamma(n-\alpha)}\int_0^\eta \left(x - t^{\frac{1}{\alpha}}\xi\right)^{n-\alpha-1}\frac{d^n}{t^{\frac{n}{\alpha}} d\xi^n} f(\xi)\, d\xi \\
&= \frac{t^{\beta+\frac{1}{\alpha}(1-n)}}{\Gamma(n-\alpha)}\int_0^\eta t^{\frac{1}{\alpha}(n-\alpha-1)}(\eta-\xi)^{n-\alpha-1}\frac{d^n f(\xi)}{d\xi^n} d\xi \\
&= t^{\beta-1}\, {}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta)\,.
\end{aligned} \tag{3.4}$$

If we replace (3.3) and (3.4) in (1.1), we obtain the following equation:

$${}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta) = \beta f(\eta) - \frac{1}{\alpha}\eta f'(\eta)\,,\ \eta \in [0,\lambda]\,,$$

where $\lambda = X t_0^{-\frac{1}{\alpha}}$. The proof is complete. □

3.2 Existence and uniqueness results of the basic profile

According to the preceding part, theorem 11, we go studies this problem:

$${}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta) = \beta f(\eta) - \frac{1}{\alpha}\eta f'(\eta)\,,\ 1 < \alpha \leq 2,\ \eta \in [0,\lambda]\,, \tag{3.5}$$

in which $\beta \in \mathbb{R}$, $\lambda > 0$ are arbitrary real constants. With the conditions:

$$f(0) = U,\ f'(0) = 0. \tag{3.6}$$

In follows, we present some significant lemmas to show the principal theorems, we have:

Lemma 12. *For any $\lambda > 0$, we define:*

$$P = \left\{ f \in C[0,\lambda] \mid f'(0) = 0 \right\}. \tag{3.7}$$

Then $(P, \|\cdot\|_\infty)$ is a Banach space.

Proof. Let $\lambda > 0$. It is clear that the space P with the norm $\|\cdot\|_\infty$ is a subspace of $C[0,\lambda]$ which is a Banach space. It remains to prove that P is a closed subspace in $C[0,\lambda]$.
Let $(f_n)_{n\in\mathbb{N}} \in P$ be a real sequence such that $\lim\limits_{n\to\infty} f_n = f$ in $C[0,\lambda]$. Since

$$\frac{d}{d\eta}\left[f_n(\eta) - f(\eta)\right] = \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta),$$

we get $\frac{d}{d\eta} f_n(\eta) \to \frac{d}{d\eta} f(\eta)$ as $n \to \infty$ for each $\eta \in [0,\lambda]$. Then

$$\lim_{n\to\infty}\left\|\frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta)\right\|_\infty = 0,$$

and for $\eta = 0$, we have also:

$$\lim_{n\to\infty}\left(\frac{df_n}{d\eta}\right)(0) = f'(0) = 0, \text{ then } f \in P.$$

Consequently, P is closed in $C[0,\lambda]$, Hence $(P, \|\cdot\|_\infty)$ is a Banach space. The proof is complete. □

Now, we will define the integral solution of the problem (3.5)-(3.6).

Lemma 13. *Let $\alpha, \beta, \lambda \in \mathbb{R}$, be such that $1 < \alpha \le 2$, and $\lambda > 0$. We give $f, f', {}^C\mathcal{D}^\alpha_{0+} f \in C[0,\lambda]$. Then the problem (3.5)-(3.6) is equivalent to the integral equation:*

$$f(\eta) = U + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}\left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi)\right) d\xi, \ \forall \eta \in [0,\lambda]. \tag{3.8}$$

Proof. Let $\alpha, \beta, \lambda \in \mathbb{R}$, be such that $1 < \alpha \le 2$, and $\lambda > 0$. We may apply lemma 3 to reduce the fractional equation (3.5) to an equivalent fractional integral equation. By applying $\mathcal{I}^\alpha_{0+}$ to equation (3.5) we obtain:

$$\mathcal{I}^\alpha_{0+}\, {}^C\mathcal{D}^\alpha_{0+} f(\eta) = \mathcal{I}^\alpha_{0+}\left(\beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)\right). \tag{3.9}$$

From lemma 3, we find easily:

$$\mathcal{I}^\alpha_{0+}\, {}^C\mathcal{D}^\alpha_{0+} f(\eta) = f(\eta) - f(0) - \eta f'(0).$$

Then, the fractional integral equation (3.9), gives:

$$f(\eta) = \mathcal{I}^\alpha_{0+}\left(\beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)\right) + f(0) + \eta f'(0). \tag{3.10}$$

If we use the condition (3.6) in equation (3.10), the problem (3.5)-(3.6) is equivalent to:

$$f(\eta) = U + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\xi)^{\alpha-1}\left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi)\right) d\xi. \tag{3.11}$$

The proof is complete. □

Lemma 14. *Let $\mathcal{A}: P \to C[0,\lambda]$ be an integral operator, which is defined by:*

$$\mathcal{A}f(\eta) = U + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi)\right) d\xi, \tag{3.12}$$

equipped with the standard norm:

$$\|\mathcal{A}f\|_\infty = \sup_{0\le\eta\le\lambda} |\mathcal{A}f(\eta)|.$$

Then $\mathcal{A}(P) \subset P$. In which, P is the Banach space defined by (3.7)

Proof. Let $f \in P$, be such that ${}^C\mathcal{D}_{0+}^\alpha f(\eta) = \beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)$. From (3.12), we have:

$$\begin{aligned} \frac{d}{d\eta}\mathcal{A}f(\eta) &= \frac{d}{d\eta}\left[U + \mathcal{I}_{0+}^\alpha \left(\beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)\right)\right] \\ &= \mathcal{I}_{0+}^{\alpha-1}\left(\beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)\right) \\ &= \mathcal{I}_{0+}^{\alpha-1}\, {}^C\mathcal{D}_{0+}^\alpha f(\eta). \end{aligned}$$

If we use (2.1) and (2.2) from remark 4 we have:

$$\frac{d}{d\eta}\mathcal{A}f(\eta) = \mathcal{I}_{0+}^{\alpha-1}\, {}^C\mathcal{D}_{0+}^\alpha f(\eta) = f'(\eta).$$

Thus $\frac{d}{d\eta}\mathcal{A}f(0) = f'(0) = 0$. Consequently $\mathcal{A}(P) \subset P$. The proof is complete. □

Now, we will prove our first existence result for the problem (3.5)-(3.6) which is based on Banach's fixed point theorem.

Theorem 15. *Let $\alpha, \lambda, \beta \in \mathbb{R}$, be such that $1 < \alpha \le 2$, $\lambda \in \left(0, \Gamma^{\frac{1}{\alpha}}(\alpha+1)\right)$. If*

$$\frac{\lambda^\alpha |\beta|}{\Gamma(\alpha+1) - \lambda^\alpha} < 1. \tag{3.13}$$

Then the problem (3.5)-(3.6) admits a unique solution on $[0,\lambda]$.

Proof. To begin the proof, we will transform the problem (3.5)-(3.6) into a fixed point problem. Define the operator $\mathcal{A}: P \to P$ by:

$$\mathcal{A}f(\eta) = U + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-\xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi)\right) d\xi. \tag{3.14}$$

Because the problem (3.5)-(3.6) is equivalent to the fractional integral equation (3.14), the fixed points of $\mathcal{A}$ are solutions of the problem (3.5)-(3.6).
Let $f, g \in P$, be such that:

$${}^C\mathcal{D}_{0+}^\alpha f(\eta) = \beta f(\eta) - \frac{\eta}{\alpha} f'(\eta),\ {}^C\mathcal{D}_{0+}^\alpha g(\eta) = \beta g(\eta) - \frac{\eta}{\alpha} g'(\eta).$$

Which implies that:

$$\mathcal{A}f\left(\eta\right)-\mathcal{A}g\left(\eta\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta-\xi\right)^{\alpha-1}\left(\beta\left(f\left(\xi\right)-g\left(\xi\right)\right)-\frac{\xi}{\alpha}\left(f'\left(\xi\right)-g'\left(\xi\right)\right)\right)d\xi.$$

Also

$$\left|\mathcal{A}f\left(\eta\right)-\mathcal{A}g\left(\eta\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta-\xi\right)^{\alpha-1}\left|{}^{C}\mathcal{D}_{0+}^{\alpha}f\left(\xi\right)-{}^{C}\mathcal{D}_{0+}^{\alpha}g\left(\xi\right)\right|d\xi. \tag{3.15}$$

For all $\eta\in\left[0,\lambda\right]$, we have:

$$\begin{aligned}\left|{}^{C}\mathcal{D}_{0+}^{\alpha}f\left(\eta\right)-{}^{C}\mathcal{D}_{0+}^{\alpha}g\left(\eta\right)\right| &= \left|\beta\left(f\left(\eta\right)-g\left(\eta\right)\right)-\frac{\eta}{\alpha}\left(f'\left(\eta\right)-g'\left(\eta\right)\right)\right|\\ &\leq \left|\beta\right|\left|f\left(\eta\right)-g\left(\eta\right)\right|+\frac{\lambda}{\alpha}\left|f'\left(\eta\right)-g'\left(\eta\right)\right|.\end{aligned}$$

By using (2.3) from remark 4, we have:

$$\left\|{}^{C}\mathcal{D}_{0+}^{\alpha}f-{}^{C}\mathcal{D}_{0+}^{\alpha}g\right\|_{\infty}\leq\left|\beta\right|\left\|f-g\right\|_{\infty}+\frac{\lambda^{\alpha}}{\Gamma\left(\alpha+1\right)}\left\|{}^{C}\mathcal{D}_{0+}^{\alpha}f-{}^{C}\mathcal{D}_{0+}^{\alpha}g\right\|_{\infty}.$$

As $\Gamma\left(\alpha+1\right)-\lambda^{\alpha}>0$, we have:

$$\left\|{}^{C}\mathcal{D}_{0+}^{\alpha}f-{}^{C}\mathcal{D}_{0+}^{\alpha}g\right\|_{\infty}\leq\frac{\left|\beta\right|\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha+1\right)-\lambda^{\alpha}}\left\|f-g\right\|_{\infty}.$$

From (3.15) we find:

$$\left\|\mathcal{A}f-\mathcal{A}g\right\|_{\infty}\leq\frac{\lambda^{\alpha}\left|\beta\right|}{\Gamma\left(\alpha+1\right)-\lambda^{\alpha}}\left\|f-g\right\|_{\infty}$$

This implies that by (3.13), $\mathcal{A}$ is a contraction operator.
As a consequence of theorem 8, using Banach's contraction principle [8], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (3.5)-(3.6) on $\left[0,\lambda\right]$. The proof is complete. □

Theorem 16. *Let $\lambda>0$, $\beta\in\mathbb{R}$, and $1<\alpha\leq2$. If we put*

$$\frac{\lambda^{\alpha}\left(\left|\beta\right|+1\right)}{\Gamma\left(\alpha+1\right)}<1. \tag{3.16}$$

Then the problem (3.5)-(3.6) has at least one solution on $\left[0,\lambda\right]$.

Proof. In the previous theorem 15, we already transform the problem (3.5)-(3.6) into a fixed point problem

$$\mathcal{A}f\left(\eta\right)=U+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{\eta}\left(\eta-\xi\right)^{\alpha-1}\left(\beta f\left(\xi\right)-\frac{\xi}{\alpha}f'\left(\xi\right)\right)d\xi.$$

We demonstrate that $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem 9. This could be proved through three steps:

Step 1: $\mathcal{A}$ is a continuous operator.

Let $(f_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty} f_n = f$ in P. Then for each $\eta\in[0,\lambda]$,

$$\begin{aligned}\left|\mathcal{A}f_n(\eta)-\mathcal{A}f(\eta)\right| &\le \int_0^\eta \frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}\times\\ &\left|\beta\left(f_n(\xi)-f(\xi)\right)-\frac{\xi\left(f_n'(\xi)-f'(\xi)\right)}{\alpha}\right|d\xi,\end{aligned}\tag{3.17}$$

where

$$^{C}\mathcal{D}^{\alpha}_{0+}f_n(\eta)=\beta f_n(\eta)-\frac{\eta}{\alpha}f_n'(\eta)\,,\ \text{and}\ ^{C}\mathcal{D}^{\alpha}_{0+}f(\eta)=\beta f(\eta)-\frac{\eta}{\alpha}f'(\eta)\,.$$

As a consequence of (H2), we find easily $^{C}\mathcal{D}^{\alpha}_{0+}f_n\to{}^{C}\mathcal{D}^{\alpha}_{0+}f$ in P. In fact, we have:

$$\begin{aligned}\left|^{C}\mathcal{D}^{\alpha}_{0+}f_n(\eta)-{}^{C}\mathcal{D}^{\alpha}_{0+}f(\eta)\right| &= \left|\beta\left(f_n(\eta)-f(\eta)\right)-\frac{\eta}{\alpha}\left(f_n'(\eta)-f'(\eta)\right)\right|\\ &\le |\beta|\left|f_n(\eta)-f(\eta)\right|+\frac{\lambda}{\alpha}\left|f_n'(\eta)-f'(\eta)\right|.\end{aligned}$$

By using (2.3) from remark 4, we have:

$$\left\|^{C}\mathcal{D}^{\alpha}_{0+}f_n-{}^{C}\mathcal{D}^{\alpha}_{0+}f\right\|_\infty\le|\beta|\left\|f_n-f\right\|_\infty+\frac{\lambda^\alpha}{\Gamma(\alpha+1)}\left\|^{C}\mathcal{D}^{\alpha}_{0+}f_n-{}^{C}\mathcal{D}^{\alpha}_{0+}f\right\|_\infty.$$

As $\Gamma(\alpha+1)-\lambda^\alpha>\lambda^\alpha|\beta|>0$, thus:

$$\left\|^{C}\mathcal{D}^{\alpha}_{0+}f_n-{}^{C}\mathcal{D}^{\alpha}_{0+}f\right\|_\infty\le\frac{|\beta|\,\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\lambda^\alpha}\left\|f_n-f\right\|_\infty.$$

Since $f_n\to f$, then we get $^{C}\mathcal{D}^{\alpha}_{0+}f_n\to{}^{C}\mathcal{D}^{\alpha}_{0+}f$ as $n\to\infty$ for each $\eta\in[0,\lambda]$. Now let $K_1>0$, be such that for each $\eta\in[0,\lambda]$, we have:

$$\left|^{C}\mathcal{D}^{\alpha}_{0+}f_n(\eta)\right|\le K_1,\ \ \left|^{C}\mathcal{D}^{\alpha}_{0+}f(\eta)\right|\le K_1.$$

Then, we have:

$$\begin{aligned}\left|\mathcal{A}f_n(\eta)-\mathcal{A}f(\eta)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\times\\ &\left|\beta\left(f_n(\xi)-f(\xi)\right)-\frac{\xi}{\alpha}\left(f_n'(\xi)-f'(\xi)\right)\right|d\xi\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\left|^{C}\mathcal{D}^{\alpha}_{0+}f_n(\xi)-{}^{C}\mathcal{D}^{\alpha}_{0+}f(\xi)\right|d\xi\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\left[\left|^{C}\mathcal{D}^{\alpha}_{0+}f_n(\xi)\right|+\left|^{C}\mathcal{D}^{\alpha}_{0+}f(\xi)\right|\right]d\xi\\ &\le \frac{2K_1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}d\xi.\end{aligned}$$

For each $\eta \in [0,\lambda]$, the function $\xi \to \frac{2K_1}{\Gamma(\alpha)}(\eta-\xi)^{\alpha-1}$ is integrable on $[0,\eta]$, then the Lebesgue dominated convergence theorem and (3.17) imply that:

$$|\mathcal{A}f_n(\eta)-\mathcal{A}f(\eta)| \to 0 \text{ as } n\to\infty,$$

and hence:

$$\lim_{n\to\infty}\|\mathcal{A}f_n-\mathcal{A}f\|_\infty = 0.$$

Consequently, $\mathcal{A}$ is continuous.

Step 2: According to (3.16), we put the positive real

$$r \geq \left(1+\frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+1)}\right)|U|,$$

and define:

$$P_r = \{f\in P : \|f\|_\infty \leq r\}.$$

It is clear that P_r is a bounded, closed and convex subset of P.
Let $f\in P_r$, and $\mathcal{A}: P_r \to P$ be the integral operator defined in (3.14), then $\mathcal{A}(P_r)\subset P_r$.
In fact, by using (2.3) from lemma 4, we have for each $\eta\in[0,\lambda]$:

$$\left|{}^C\mathcal{D}^\alpha_{0+}f(\eta)\right| = \left|\beta f(\eta)-\frac{\eta}{\alpha}f'(\eta)\right| \leq |\beta||f(\eta)|+\frac{\lambda^\alpha}{\Gamma(\alpha+1)}\left\|{}^C\mathcal{D}^\alpha_{0+}f\right\|_\infty.$$

Then

$$\left\|{}^C\mathcal{D}^\alpha_{0+}f\right\|_\infty \leq \frac{|\beta|\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\lambda^\alpha}r \tag{3.18}$$

Thus

$$\begin{aligned}
|\mathcal{A}f(\eta)| &\leq |U|+\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\left|\beta f(\xi)-\frac{\xi}{\alpha}f'(\xi)\right|d\xi\\
&\leq \frac{|U|\left(1+\frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+1)}\right)}{1+\frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+1)}}+\frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha}r\\
&\leq \frac{(\Gamma(\alpha+1)-\lambda^\alpha(|\beta|+1))r}{\Gamma(\alpha+1)-\lambda^\alpha}+\frac{\lambda^\alpha|\beta|}{\Gamma(\alpha+1)-\lambda^\alpha}r\\
&\leq r.
\end{aligned}$$

Then $\mathcal{A}(P_r)\subset P_r$.

Step 3: $\mathcal{A}(P_r)$ is relatively compact.

Let $\eta_1, \eta_2 \in [0, \lambda]$, $\eta_1 < \eta_2$, and $f \in P_r$. Then

$$\begin{aligned}
|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - \xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) d\xi \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - \xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) d\xi \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} \left| \left((\eta_2 - \xi)^{\alpha-1} - (\eta_1 - \xi)^{\alpha-1} \right) \times \right. \\
&\quad \left. \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) \right| d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha-1} \left| \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) \right| d\xi \\
&\leq \frac{1}{\Gamma(\alpha)} \times \frac{|\beta| \Gamma(\alpha+1) r}{\Gamma(\alpha+1) - \lambda^\alpha} \left[\int_0^{\eta_1} \left| (\eta_2 - \xi)^{\alpha-1} - \right. \right. \\
&\quad \left. \left. (\eta_1 - \xi)^{\alpha-1} \right| d\xi + \int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha-1} d\xi \right]. \qquad (3.19)
\end{aligned}$$

We have:

$$(\eta_2 - \xi)^{\alpha-1} - (\eta_1 - \xi)^{\alpha-1} = -\frac{1}{\alpha} \frac{d}{d\xi} \left[(\eta_2 - \xi)^\alpha - (\eta_1 - \xi)^\alpha \right]$$

then

$$\int_0^{\eta_1} \left| (\eta_2 - \xi)^{\alpha-1} - (\eta_1 - \xi)^{\alpha-1} \right| d\xi \leq \frac{1}{\alpha} \left[(\eta_2 - \eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha) \right]$$

we have also

$$\int_{\eta_1}^{\eta_2} (\eta_2 - \xi)^{\alpha-1} d\xi = -\frac{1}{\alpha} \left[(\eta_2 - \xi)^\alpha \right]_{\eta_1}^{\eta_2} \leq \frac{1}{\alpha} (\eta_2 - \eta_1)^\alpha .$$

Then (3.19) gives

$$|\mathcal{A}f(\eta_2) - \mathcal{A}f(\eta_1)| \leq \frac{|\beta| r}{\Gamma(\alpha+1) - \lambda^\alpha} \left(2(\eta_2 - \eta_1)^\alpha + (\eta_2^\alpha - \eta_1^\alpha) \right).$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero.
As a consequence of steps 1 to 3 together, and by means of the Ascoli-Arzelà theorem 5, we deduce that $\mathcal{A} : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 9. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (3.5)-(3.6) on $[0, \lambda]$.

The proof is complete. □

Our next existence result is based on the nonlinear alternative of Leray-Schauder type.

Theorem 17. *Let* $\alpha, \lambda, \beta \in \mathbb{R}$, *be such that* $1 < \alpha \leq 2$, $\lambda \in \left(0, \Gamma^{\frac{1}{\alpha}}(\alpha+1)\right)$. *Then the problem (3.5)-(3.6) has at least one solution on* $[0, \lambda]$.

Proof. Let $\alpha, \lambda, \beta \in \mathbb{R}$, be such that $1 < \alpha \leq 2$, $\lambda \in \left(0, \Gamma^{\frac{1}{\alpha}}(\alpha+1)\right)$.
We shall show that the operator $\mathcal{A}$ defined in (3.14), satisfies the assumption of Leray-Schauder fixed point theorem 10. The proof will be given in several steps.

Step 1: Clearly $\mathcal{A}$ is continuous.

Step 2: $\mathcal{A}$ maps bounded sets into bounded sets in P.
Indeed, it is enough to show that for any $\omega > 0$ there exist a positive constant ℓ such that for each $f \in B_\omega = \{f \in P : \|f\|_\infty \leq \omega\}$, we have $\|\mathcal{A}f\|_\infty \leq \ell$.
For $f \in B_\omega$, we have, for each $\eta \in [0, \lambda]$,

$$|\mathcal{A}f(\eta)| \leq |U| + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} \left| \beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right| d\xi. \tag{3.20}$$

Similarly (3.18), for each $\eta \in [0, \lambda]$, we have:

$$\left| \beta f(\eta) - \frac{\eta}{\alpha} f'(\eta) \right| \leq \frac{|\beta| \Gamma(\alpha+1)}{\Gamma(\alpha+1) - \lambda^\alpha} \omega.$$

Thus (3.20) implies that:

$$\|\mathcal{A}f\|_\infty \leq |U| + \frac{\lambda^\alpha |\beta|}{\Gamma(\alpha+1) - \lambda^\alpha} \omega = \ell.$$

Step 3: Clearly, $\mathcal{A}$ maps bounded sets into equicontinuous sets of P.
We conclude that $\mathcal{A} : P \to P$ is continuous and completely continuous.

Step 4: A priori bounds.
We now show there exists an open set $V \subset P$ with $f \neq \mu \mathcal{A}(f)$ for $\mu \in (0, 1)$ and $f \in \partial V$.
Let $f \in P$ and $f = \mu \mathcal{A}(f)$ for some $0 < \mu < 1$. Thus for each $\eta \in [0, \lambda]$, we have:

$$f(\eta) \leq \mu U + \frac{\mu}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) d\xi.$$

For all solution $f \in P$, of the problem (3.5)-(3.6), we have:

$$\begin{aligned} |f(\eta)| &= \left| U + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} \left(\beta f(\xi) - \frac{\xi}{\alpha} f'(\xi) \right) d\xi \right| \\ &\leq |U| + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} \left| {}^C\mathcal{D}_{0+}^\alpha f(\xi) \right| d\xi. \end{aligned}$$

Then for each $\eta \in [0, \lambda]$, we have:

$$\begin{aligned}\left|{}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta)\right| &= \left|\beta f(\eta) - \frac{\eta}{\alpha} f'(\eta)\right| \leq |\beta| |f(\eta)| + \frac{\lambda}{\alpha} |f'(\eta)| \\ &\leq |\beta| |f(\eta)| + \frac{\lambda^{\alpha}}{\Gamma(\alpha+1)} \sup_{0\leq\eta\leq\lambda} \left|{}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta)\right|.\end{aligned}$$

Then

$$\sup_{0\leq\eta\leq\lambda} \left|{}^{C}\mathcal{D}_{0+}^{\alpha} f(\eta)\right| \leq \frac{|\beta| \Gamma(\alpha+1)}{\Gamma(\alpha+1) - \lambda^{\alpha}} \sup_{0\leq\eta\leq\lambda} |f(\eta)|.$$

Hence

$$\sup_{0\leq\eta\leq\lambda} |f(\eta)| \leq |U| + \frac{|\beta| \Gamma(\alpha+1)}{\Gamma(\alpha)(\Gamma(\alpha+1) - \lambda^{\alpha})} \int_0^{\eta} (\eta - \xi)^{\alpha-1} \left\{\sup_{0\leq\xi\leq\lambda} |f(\xi)|\right\} d\xi.$$

After the Gronwall lemma [9], we have:

$$\sup_{0\leq\eta\leq\lambda} |f(\eta)| \leq |U| \exp\left(\frac{|\beta| \Gamma(\alpha+1)}{\Gamma(\alpha)(\Gamma(\alpha+1) - \lambda^{\alpha})} \int_0^{\eta} (\eta - \xi)^{\alpha-1} d\xi\right).$$

Thus

$$\|f\|_{\infty} \leq |U| \exp\left(\frac{\lambda^{\alpha} |\beta|}{\Gamma(\alpha+1) - \lambda^{\alpha}}\right) = M_2.$$

Let

$$V = \{f \in P : \|f\|_{\infty} < M_2 + 1\}.$$

By choosing of V, there is no $f \in \partial V$, such that $f = \mu \mathcal{A}(f)$, for $\mu \in (0, 1)$. As a consequence of Leray-Schauder's theorem 10, $\mathcal{A}$ has a fixed point f in V which is a solution to (3.5)-(3.6).

The proof is complete. □

3.3 Existence results of solutions to the original problem

In this section, we prove the existence and uniqueness of solutions of the the following problem of the space-fractional heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^{\alpha} u}{\partial x^{\alpha}}, & (x,t) \in [0, X] \times [t_0, \infty), \quad 1 < \alpha \leq 2, \\ u(0,t) = t^{\beta} U, \ \frac{\partial u}{\partial x}(0,t) = 0, & \beta, U \in \mathbb{R}. \end{cases} \tag{3.21}$$

Under the self-similar form:

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha}}}\right), \text{ with } (x,t) \in [0, X] \times [t_0, \infty). \tag{3.22}$$

Theorem 18. *Let* $\alpha, \beta, t_0, X \in \mathbb{R}$, *be such that* $1 < \alpha \leq 2$, $t_0 > 0$ *and* $X \in (0, t_0\Gamma(\alpha+1))^{\frac{1}{\alpha}}$. *If*

$$\frac{X^\alpha |\beta|}{t_0 (t_0\Gamma(\alpha+1) - X^\alpha)} < 1. \tag{3.23}$$

Then, for $f \in P$, *the problem (3.21) admits a unique solution in the self-similar form (3.22).*

Proof. The transformation (3.22) reduces the space-fractional heat equation (3.21) to the ordinary differential equation of fractional order of the form:

$${}^{C}\mathcal{D}^{\alpha}_{0^+} f(\eta) = \beta f(\eta) - \frac{\eta}{\alpha} f'(\eta), \; 0 < \eta < \lambda, \tag{3.24}$$

where

$$\lambda = X t_0^{-\frac{1}{\alpha}}, \text{ with } X \in (0, t_0\Gamma(\alpha+1))^{\frac{1}{\alpha}}, \; t_0 > 0, \; 1 < \alpha < 2.$$

With the conditions:

$$f(0) = U, \; f'(0) = 0. \tag{3.25}$$

Let $f \in P$ be a continuous function. By using (3.22), the condition (3.23), is equivalent to (3.13), which is

$$|\beta| < \frac{\rho^\alpha \Gamma(\alpha+1)}{\xi \lambda^{\alpha\rho}}. \tag{3.26}$$

We already proved in theorem 15, the existence and uniqueness of a solution of the problem (3.24)-(3.25) provided that (3.26) holds true. Consequently, if (3.23) holds for any $(x,t) \in [0,X] \times [t_0, \infty)$, then there exists a unique solution of the problem of the space-fractional heat equation (3.21) under the self-similar form (3.22). The proof is complete. □

Theorem 19. *Let* $\alpha, \beta, t_0, X \in \mathbb{R}$, *be such that* $1 < \alpha \leq 2$ *and* $X, t_0 > 0$. *If*

$$\frac{\lambda^\alpha (|\beta| + 1)}{\Gamma(\alpha+1)} < 1. \tag{3.27}$$

Then, for $f \in P_r$, *the problem (3.21) has at least one solution in the self-similar form (3.22).*

Proof. Based on theorem 16, we use the same steps through which we proved theorem 18 to prove the existence of a self-similar solution to the problem (3.21) provided that the condition (3.27) holds true. The proof is complete. □

Theorem 20. *Let* $\alpha, \beta, t_0, X \in \mathbb{R}$, *be such that* $1 < \alpha \leq 2$, $t_0 > 0$, *and* $X \in (0, t_0\Gamma(\alpha+1))^{\frac{1}{\alpha}}$. *Then, for* $f \in P$, *the problem (3.21) has at least one solution in the self-similar form (3.22).*

Proof. Based on theorem 17, we use the same steps through which we proved theorem 18 to prove that the problem (3.21) has at least one solution in the self-similar form (3.22). The proof is complete. □

4 Conclusion

In this paper, we have discussed the existence and uniqueness of solutions for a class of space-fractional diffusion equation, known as space-fractional heat equation with mixed conditions under the self-similar form. For that we used the Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type, and the differential operators used are the Caputo operators.

Acknowledgement. The authors are deeply grateful to the reviewers and editors for their insightful comments that have helped to improve the quality of this research work that is supported by the General Direction of Scientific Research and Technological Development (DGRSTD)- Algeria.

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