Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 14 (2019), 355 – 365

ASPECTS REGARDING THE EXISTENCE OF FIXED POINTS OF THE ITERATES OF STANCU OPERATORS

Amelia Bucur

Abstract. In the papers Iterates of Stancu Operators, via Contraction Principle (2002), respectively Iterates of Bernstein Operators, via Contraction Principle (2004), author I. A. Rus studied the existence of fixed points for Stancu operators $P_{n,\alpha,\beta}$ and Bernstein operators B_n. The aim of this paper is to find conditions for which the Stancu operators $P_{n,\alpha,\beta}$ are contractions on the graph, in order to demonstrate that the contraction principle can be applied for the study of the existence of fixed points for iterates of Stancu operators. The method used for this paper is the spectral method, which was also used in the paper Over-iterates of Bernstein-Stancu operators (2007), authors Gonska, Piţul and Raşa. The study began with finding constant $C \in [0, 1[$ that would satisfy the inequality $\|P^2_{n,\alpha,\beta}(f) - P_{n,\alpha,\beta}(f)\| \leq C\|P_{n,\alpha,\beta}(f) - f\|$, for any $f \in C[0, 1]$. The conclusion is that there are conditions for which the Stancu operators are contractions on the graph, and the methods used for the study of the existence of fixed points of their iterates can also be extended to the study of the existence of fixed points of other linear operators.

1 Introduction

Many authors used the contraction principle to study the existence of fixed points of the iterates of the Bernstein and Stancu operators. Using the same approach, many researchers (for example: [1, 2, 22–24]) obtained results for some other linear and positive operators. Also, many researchers, in recent works, studied the behavior of iterates of some classes of positive linear operators, such as in (for example: [3, 6, 7, 10–14, 19]).

In their papers published in the year 2003, Agratini and Rus [1, 2] studied the convergence of the iterates of discrete linear operators, by applying the contraction principle.

2010 Mathematics Subject Classification: 39B12, 47B37, 40G05, 47H10, 54H25
Keywords: iterate operators; fixed point; Stancu operators

In the year 2002, Rus [22] used the contraction principle to demonstrate that some Stancu operators are, in fact, weakly Picard operators.

In the year 2004, Rus [23] gave another proof to the result obtained by Kelisky and Rivlin, by using the contraction principle.

Rus [24] established some relations between the mixed-extremal point set of $D \subset \mathbb{R}^p$, the fixed point set and the interpolation point set of $A : C(\bar{D}) \to C(\bar{D})$.

The Bernstein operators have been introduced by Bernstein 1912. [6]

Kelisky and Rivlin, in 1967, presented the Bernstein operator B_n with $n \in \mathbb{N}^*$, by the following formula:

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}, \tag{1.1}$$

where $f \in C[0,1], x \in [0,1]$. [19]

We notice that equation (1.1) is very similar to the binomial distribution. Is it known that, by definition, the Bernstein basis polynomials have a similar function to the probability mass function of a binomial distribution from mathematical statistics [7].

In their paper [19] they proved that the Bernstein operators B_n are weakly Picard operators and:

$$\lim_{m \to \infty} B_n^m(f)(x) = f(0) + [f(1) - f(0)]x, \tag{1.2}$$

where B_n^m is the iterate of order m of B_n.

Aleomraninejad, Rezapour, and Shahzad [3] used Reich's concept to extend the results of Kelisky-Rivlin.

Gavrea and Ivan [10, 11] studied the convergence of the iterates of a set of positive linear operators that maintain the affine functions, respectively positive linear operators that maintain constant functions.

In another work published in 2011 by Gavrea and Ivan [12], they studied the convergence of the iterates of a set of positive linear operators that preserve linear functions.

In the years 2006, 2007, the authors [13, 14] introduced a method that allows to determine the degree of approximation towards the first Bernstein operator for the iterates of certain positive linear operators, as well as some continuous type operators.

In the year 1970, Karlin and Ziegler [17] studied the limit behavior of the iterates of positive linear approximation operators.

Gonska and Raşa [15], in the year 2006, extended the results obtained by Karlin and Ziegler and studied the degree of approximation of the iterated Bernstein operators.

Stancu operators $P_{n,\alpha,\beta} : C[0,1] \to C[0,1], \alpha, \beta \in \mathbb{R}, 0 \leq \alpha \leq \beta, n \in \mathbb{N}^*$, are defined by formulas [26]:

$$P_{n,\alpha,\beta}(f)(x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) C_n^k x^k \tag{1.3}$$

Examples of theorems that give conditions of existence for fixed points are the contraction principle and one of its generalizations, the contraction principle for graphs.

The contraction principle (S. Banach (1922) and R. Cacciopoli (1930), [25]).
Let (X,d) be a complete metric space and $f : X \to X$ an α-contraction. Thus:

(i) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$;

(ii) $f^n(x) \to x^*(n \to \infty)$ *for all* $x \in X$;

(iii) $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x))$ *for all* $x \in X$.

The contraction principle for graphs (I. A. Rus (1972) [25], S. Kasahara (1968) [18], T. L. Hicks and B. E. Rhoades (1979) [16]).
Let (X, d) *be a complete metric space,* $f : X \to X$ *and* $\alpha \in [0, 1[$, *so that:*

(a) $d(f^2(x), f(x)) \leq d(x, f(x))$ *for all* $x \in X$;

(b) *operator* f *has a closed graph.*

Then:

(i) $F_f = F_{f^n} \neq O$, *for all* $n \in \mathbb{N}^*$;

(ii) $f^n(x) \to f^\infty(x)(n \to \infty)$ *and for all* $f^\infty(x) \in F_f$ *for all* $x \in X$;

(iii) $d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$ *for all* $x \in X$.

The iterates of the α-contractions have applications in many theoretical and practical fields, such as the fractal theory. Fractal representations can be created in many softwares, such as the graphical programming environment LabVIEW (Laboratory Virtual Instrumentation Engineering Workbench - introduced on the market by the company National Instruments in the year 1986) ([30,31]) (figure 1).

In figure 1, the contractions defined on a square field X endowed with a euclidian norm, from $\mathbb{R}^2$, and are of the following form:

$$f(x, y) = (\alpha x\, cos\theta - \alpha y\, sin\theta, \alpha x\, cos\theta + \alpha y\, sin\theta), \tag{1.4}$$

with $\alpha \in [0, 1[, \theta \in [0, 2\pi]$.

Obviously, the diagrams depend on the rate of convergence.

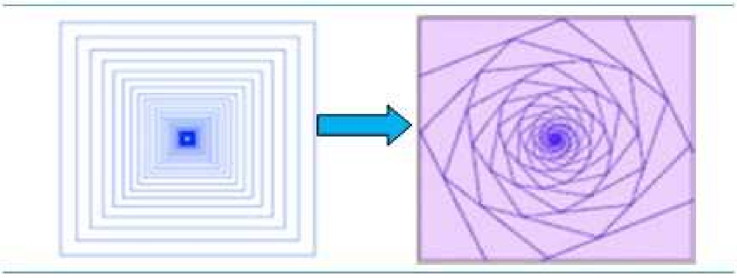

Figure 1: Iterations of an α-contraction on a square (fractal creation) in the programming environment LabVIEW [27]

Problem 1. *Find constant $C \in [0,1[$ that satisfies the inequality:*

$$\|P^2_{n,\alpha,\beta}(f) - P_{n,\alpha,\beta}(f)\| \leq C\|P_{n,\alpha,\beta}(f) - f\|, f \in C[0,1]. \tag{1.5}$$

For $\alpha = 0$, the author found in [22] that:

$$\|P^2_{n,0,\beta}(f) - P_{n,0,\beta}(g)\| \leq \left[1 - \left(1 - \frac{n}{n+\beta}\right)^n\right]\|f-g\|_c = A\|f-g\|_c, \tag{1.6}$$

for all $f, g \in X_\gamma = \left\{f \in C\left[0, \frac{n}{n+\beta}\right] : f(0) = \gamma\right\}, \gamma \in \mathbb{R}$, where $\|\cdot\|_c$ is the Chebyshev's norm.

Observation 2. (a) *If we replace f with $P_{n,0,\beta}(f)$ and g with f, formula (1.6) becomes formula (1.5).*

(b) *By successively applying formula (1.6), we obtain, for $m \in \mathbb{N}^*$:*

$$\|P^m_{n,0,\beta}(f) - P^{m-1}_{n,0,\beta}(f)\| \leq ... \leq A'\|P_{n,0,\beta}(f) - f\|, \quad cu \ \ A' \in [0,1[.$$

For $\alpha = \beta$, the author found in the same paper that:

$$\|P_{n,\alpha,\alpha}(f) - P_{n,\alpha,\alpha}(g)\| \leq [1 - (1 - \frac{n}{n+\beta})^n]\|f-g\|_c = A\|f-g\|_c, \tag{1.7}$$

Observation 3. *If we replace f with $P_{n,\alpha,\alpha}(f)$ and g with f, formula (1.7) becomes formula (1.5).*

In paper [10], the authors use the Bernstein-Stancu operators $S_n^{\langle n,\alpha,\beta\rangle}(f)(x)$, $\alpha > 0, 0 \leq \beta \leq \gamma, f \in C[0,1]$, $e_i(x) = x^i$ and formula:

$$S_n^{\langle n,\alpha,\beta\rangle}(f)(x) = B_n\left(fo\left(\frac{n}{n+\beta}e_1 + \frac{\alpha}{n+\beta}e_0\right)\right)(x) \tag{1.8}$$

and study the iterates of these operators (B_n is the Bernstein operator). In the following sections, we will use relation (1.8) in approaching inequality (1.5).

2 Main Results

Theorem 4 from this section solves Problem 1.

Theorem 4. *Inequality (1.5) is satisfied with constant* $C = \left(1 - \frac{1}{2^{n-1}}\right)\frac{n+\alpha}{n+\beta} \in [0,1[$:

$$\|P^2_{n,\alpha,\beta}(f) - P_{n,\alpha,\beta}(f)\| \leq \left(1 - \frac{1}{2^{n-1}}\right)\frac{n+\alpha}{n+\beta}\|P_{n,\alpha,\beta}(f) - f\|, f \in C[0,1]. \tag{2.1}$$

Proof. We will use the inequality from [23]:

$$\|B_n(f) - B_n(g)\|_c \leq \left(1 - \frac{1}{2^{n-1}}\right)\|f - g\|. \tag{2.2}$$

Then, based on relations (1.8) and (2.2), we have:

$$\begin{aligned}
&\|P^2_{n,\alpha,\beta}(f) - P_{n,\alpha,\beta}(f)\| \leq \\
&\quad \leq \left(1 - \frac{1}{2^{n-1}}\right) \cdot \left\| B_n\left(f \circ \underbrace{\left(\frac{n}{n+\beta}e_1 + \frac{\alpha}{n+\beta}e_0\right)}_{h} \circ \left(\frac{n}{n+\beta}e_1 + \frac{\alpha}{n+\beta}e_0\right) - \right.\right. \\
&\quad \left.\left. - \left(f \circ \left(\frac{n}{n+\beta}e_1 + \frac{\alpha}{n+\beta}e_0\right)\right)\right)\right\| = \\
&\quad = \left(1 - \frac{1}{2^{n-1}}\right) \cdot \|B_n((f \circ h) \circ h - f \circ h)\| \leq \\
&\quad \leq \left(1 - \frac{1}{2^{n-1}}\right) \cdot \|h\| \cdot \|B_n((f \circ h) - f)\| = \\
&\quad = \sup_{x \in [0,1]} |h(x)| \cdot \|B_n((f \circ h) - f)\| = \\
&\quad = \left(1 - \frac{1}{2^{n-1}}\right) \cdot \frac{n+\alpha}{n+\beta} \cdot \|P_{n,\alpha,\beta}(f) - f\|.
\end{aligned}$$

□

Problem 5. *What is* $P^{\infty}_{n,\alpha,\beta}(f)$ *equal to and who is* $\lim_{m\to\infty} P^m_{n,\alpha,\beta}(f)$*?*

In order to solve Problem 5, we will use the following theorem:

Theorem 6. *([12], p.121) If* $n \in \mathbb{N}$ *is fixed and* $\alpha > 0, 0 \leq \beta \leq \gamma$*, then for any* $f \in C[0,1]$*,*

$$\lim_{m\to\infty}\left(S_n^{\langle n,\alpha,\beta\rangle}(f)\right)^m = b_0^{\alpha} e_0 = \left(S_n^{\langle n,\alpha,\beta\rangle}\right)^{\infty}, \tag{2.3}$$

where $b_0^{\alpha} = \sum_{j=0}^{n} d_j^{\alpha} f\left(\frac{j+\beta}{n+\gamma}\right)$*, with* $d_j^{\alpha}, 0 \leq j \leq n$ *independent of* f*.*

Considering Theorem 6, Problem 5 is reduced to having to determine coefficients $d_j^\alpha, 0 \le j \le n$. $P^\infty_{n,\alpha,\beta}(f)$ will be clearly established by relation (2.3), because $P_{n,\alpha,\beta}(f)$ is in fact $S_n^{\langle 0,\alpha,\beta\rangle}(f)$.

As a particular case of Theorem 1, p.119 from paper [12], the eigenvalues for $P_{n,\alpha,\beta}(f) = S_n^{\langle 0,\alpha,\beta\rangle}(f)$ are:

$$\lambda_{n,0} = 1, \lambda_{n,1} = \frac{n}{n+\beta}, \lambda_{n,j} = \frac{n(n-1)(n-j+1)}{(n+\beta)^j}, j = 2, ..., n, \tag{2.4}$$

and the corresponding eigenvectors are

$$q_{n,0} = e_0, q_{n,j}(x) = e_j(x) + a^{(j)}_{n,j-1} e_{j-1}(x) + ... + a^{(j)}_{n,0} e_0(x),\ j = 1, ..., n,$$

with uniquely determined coefficients.

Thus, as previously determined, we can write the following theorem:

Theorem 7. *For all $f \in C[0,1]$,*

$$\lim_{m\to\infty} (S_n^{\langle 0,\alpha,\beta\rangle}(f)) = P^\infty_{n,\alpha,\beta}(f) = b_0 e_0, \tag{2.5}$$

where $b_0 = \sum_{j=0}^n d_j f(\frac{j+\beta}{n+\gamma})$, is a linear combination of values of f.

Proof. Let $f \in C[0,1]$. Then, $P_{n,\alpha,\beta}(f) = S_n^{\langle 0,\alpha,\beta\rangle}(f)$ can be decomposed into a base from $\prod_n$.

Regarding the base composed of eigenvectors, $\{q_{n,0}, q_{n,1}, ..., q_{n,n}\}$ from $\prod_n$, we will write

$$P_{n,\alpha,\beta}(f) = S_n^{\langle 0,\alpha,\beta\rangle}(f) = b_0 q_{n,0}(x) + b_1 q_{n,1}(x) + ... + b_n q_{n,n}(x) \tag{2.6}$$

obviously, with unique coordinates.

For f replaced by an eigenvector, relation (2.6) becomes:

$$S_n^{\langle 0,\alpha,\beta\rangle}(q_{n,j})(x) = \lambda_{n,j} q_{n,j}(x), \tag{2.7}$$

where $\lambda_{n,j}$ are the eigenvalues of $P_{n,\alpha,\beta}$ from relations (2.4).

From (2.6) and (2.7) we obtain that for any $f \in C[0,1]$ and for all $x \in [0,1]$, we obtain:

$$\begin{aligned} P^m_{n,\alpha,\beta}(f)(x) &= P^{m-1}_{n,\alpha,\beta}(P_{n,\alpha,\beta}(f)(x)) = \\ &= P^{m-1}_{n,\alpha,\beta}((b_0 q_{n,0} + b_1 q_{n,1} + ... + b_n q_{n,n})(x)) = \\ &= b_0(\lambda_{n,0})^{m-1} q_{n,0}(x) + b_1(\lambda_{n,1})^{m-1} q_{n,1}(x) + ... + b_n(\lambda_{n,n})^{m-1} q_{n,n}(x). \end{aligned}$$

Because $\lambda_{n,j} \in]0,1[$ for $j = 1, ..., n$, when $m \to \infty$ in (2.6), $\lim_{m\to\infty} P^m_{n,\alpha,\beta}(f)(x) = b_0 e_0(x)$.

We say that $P^\infty f = b_0 e_0$ and demonstrate that

$$b_0 = \sum_{j=0}^{n} d_j f\left(\underbrace{\frac{j+\beta}{n+\gamma}}_{a_j}\right) = \sum_{j=0}^{n} d_j f(a_j),$$

$d_j \in \mathbb{R}$ are independent of f.

We consider two bases of $\prod_n$, one consisting of eigenvectors $\{q_{n,0}, q_{n,1}, ..., q_{n,n}\}$ and the other one consisting of the Stancu fundamental polynomials $\{w_{n,0}, w_{n,1}, ..., w_{n,n}\}$. As defined in [17], the Stancu fundamental polynomials are of the form:

$$w_{n,k}(x;\alpha) = C_n^k \frac{\prod_{\vartheta=0}^{k-1}(x+\vartheta\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)...(1+\overline{n-1}\alpha)} \tag{2.8}$$

and $(\theta_{i,j})_{i,j=\overline{0,n}}$ is the transition matrix from the first to the second base:

$$\begin{cases} w_{n,0} = \theta_{0,0}q_{n,0} + \theta_{1,0}q_{n,1} ... + \theta_{n,0}q_{n,n} \\ \quad\cdots \\ w_{n,n} = \theta_{0,n}q_{n,0} + \theta_{1,n}q_{n,1} ... + \theta_{n,n}q_{n,n} \end{cases}$$

The coordinates of $P_{n,\alpha,\beta}(f)$ in relation to the two bases are:

$$\begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \theta_{0,0} & \theta_{1,0} & \theta_{n,0} \\ & \vdots & \\ \theta_{0,n} & \theta_{1,n} & \theta_{n,n} \end{pmatrix} \begin{pmatrix} f(a_0) \\ \vdots \\ f(a_n). \end{pmatrix}$$

Because $d_0 = \theta_{0,0}, ..., d_n = \theta_{n,n}$ they are independent of f. In conclusion, $d_j, j = 0, ..., n$ can be determined by decomposing $w_{n,j}$ in relation to the base $\{q_{n,0}, q_{n,1}, ..., q_{n,n}\}$.

The symmetry of the applied method consists of the possibility to apply it starting from a certain iteration in order to obtain results for higher order iterations, respectively for lower order iterations. In the case of iterations of linear operators associated to symmetrical matrices, the eigenvalues are determined with specific methods.

□

3 Conclusions

There are conditions for which the Stancu operators are contractions on the graphs, and for the existence of their fixed points to be applied the contraction principle for graphs.

The methods used for the study of the existence of fixed points of their iterates can be extended to the study of the existence of fixed points for other linear operators, such as the Cesàro operators, the generalized Cesàro operators and their iterates, the Kantorovich form of the Stancu operators, Schurer-Stancu operators, Stancu-Durrmeyer operators, Lupaş-Stancu operators, Stancu-King operators, q-Bernstein polynomials, and others.

The discrete Cesàro operator turns any set into an set of averages. Similarly, the integral Cesàro operator is also an averaging operator. In recent years, many researchers studied the iterates of the Cesàro operator (see [9, 28, 29]) on the space of convergent sets, on the space of sets convergent to zero, on the space of summable sets, on the space of functions continuous on the range [0, 1], and on other Banach spaces. The authors obtained results regarding the convergence of the iterates and proved that the Cesàro operator is a contraction on a dense subset of $(C[0,1], B)$, endowed with a certain norm, where B is a Banach space.

Let α, β be two given real parameters that satisfy the conditions $0 \leq \alpha \leq \beta$. Then, the Kantorovich form of the Stancu operators [4], for $f \in C[0,1]$ and $x \in [0,1]$ is:

$$K_m^{(\alpha,\beta)}(f)(x) = (m+\beta+1)\sum_{k=0}^{m} C_m^k x^k (1-x)^{m-k} \sum_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s)ds.$$

Let $p \in N$ be a integer and α, β two given real parameters that satisfy the conditions $0 \leq \alpha \leq \beta$. Then, the Schurer-Stancu form of the Stancu operators [5], for $f \in C[0, 1+p]$ and $x \in [0, 1+p]$ is:

$$\tilde{S}_m^{(\alpha,\beta)}(f)(x) = \sum_{k=0}^{m+p} C_{m+p}^k x^k (1-x)^{m+p-k} f\left(\frac{k+p}{m+p}\right).$$

In the year 2009, Nowak [20] created the q-analogue for the Stancu operators (1.1) for any function $f \in C[0,1], x \in [0,1], q > 0, \alpha \geq 0$ and each $n \in N$ as,

$$B_n^{(q,\alpha)}(f)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\Pi_{\vartheta=0}^{k-1}(x+\alpha[\vartheta]_q)\Pi_{\mu=0}^{n-k-1}(1-q^\mu x+\alpha[\mu]_q)}{\Pi_{\tau=0}^{n-1}(1+\alpha[\vartheta]_q)} f\left(\frac{[k]_q}{[n]_q}\right).$$

Where the q-integer is, for a non-negative integer n,

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1, \end{cases}$$

q-factorial $[n]_{q!}$ are defined as in the following formula:

$$[n]_{q!} = \begin{cases} [1]_q[2]_q[3]_q \ldots [n]_q, & n \geq 1 \\ 1, & n = 0 \end{cases}$$

and the q - binomial coefficient is defined as in the following equality:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_{q!}}{[k]_{q!}[n-k]_{q!}}.$$

The Stancu-Durrmeyer operators, Lupaş-Stancu operators, Stancu-King operators, and the q-Bernstein polynomials are other examples of linear operators.

A new topic and a future direction of study could be the study of fixed points for compositions between such operators with Cesàro-type operators, or for compositions with iterations of Cesàro operators.

References

[1] O. Agratini and I. A. Rus, *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math., **44** (2003), 555–563. MR2025820. Zbl 1096.41015.

[2] O. Agratini and I. A. Rus, *Iterates of some bivariate approximation process via weakly Picard operators*. Nonlinear Analysis Forum, **8** (2003), 159-168. MR2040850. Zbl 1206.41021.

[3] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *Some fixed point results on a metric space with a graph*, Topology and its Applications, **159** (2012), 659-663. MR2868864. Zbl 1237.54042.

[4] D. Bărbosu, *Kantorovich-Stancu Type Operators*, JIPAM. Journal of Inequalities in Pure and Applied Mathematics, **5** (2004), 1-6. MR2084863. Zbl 1054.41010.

[5] D. Bărbosu, *A survey on the approximation properties of Schurer-Stancu operators*, Carpathian .J. Math., **20** (2004), 1-5. MR2138521. Zbl 1089.41017.

[6] S. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Charkow Ges., **13** (2) (1912), 1-2. JFM 43.0301.03.

[7] C.A. Charalambides, *The q-Bernstein basis as a q-binomial distribution*, Journal of Statistical Planning and Inference, **140** (2010), 2184-2190. MR2609477. Zbl 1191.60014.

[8] J. L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: Application à la théorie des moments*. Thése de 3 cycle, Faculté des Sciences de l'Université de Paris, 1967.

[9] F. G. Fontes and F. J. Solís, *Iterating the Cesàro operators*, Proc. Amer. Math. Soc., **136** (2008), 2147-2153. MR2383520. Zbl 1146.47019.

[10] I. Gavrea and M. Ivan, *On the iterates of positive linear operators preserving the affine functions*. J. Math. Anal. Appl., **372** (2010), 366-368. MR2678868. Zbl 1196.41014.

[11] I. Gavrea and M. Ivan, *The iterates of positive linear operators preserving the constants*. Appl. Math. Lett. 2011, **24**, 2068-2071. MR2826127. Zbl 1232.41030.

[12] I. Gavrea and M. Ivan, *On the iterates of positive linear operators*, J. Approximation Theory, **163** (2011), 1076-1079. MR2832743. Zbl 1236.41002.

[13] H. Gonska, D. Kacśo and P. Piţul, *The degree of convergence of over-iterated positive linear operators*, J. Appl. Funct. Anal., **1** (2006), 403-423. MR2220800. Zbl 1099.41011.

[14] H. Gonska, P. Piţul and I. Raşa, *Over-iterates of Bernstein-Stancu operators*, Calcolo, **44** (2007), 117-125. MR2330763. Zbl 1150.41013.

[15] H. Gonska and I. Raşa, *The limiting semigroup of the Bernstein iterates: degree of convergence*, Acta Math. Hungar., **111** (2006), 119-130. MR2188976. Zbl 1121.41004.

[16] T. L. Hicks and B. E. Rhoades, *A Banach type fixed point theorem*, Math. Jap., **24** (1979), 327-330. MR0550217. Zbl 0432.47036.

[17] S. Karlin and Z. Ziegler, *Iteration of positive approximation operators*, J. Approximation Theory, **3** (1970), 310-339. MR0277982. Zbl 0199.44702.

[18] S. Kasahara, *A Remark on the Contraction Principle*, Proc. Japan Acad., **44** (1968), 21-26. MR0230182. Zbl 0169.46002.

[19] R. P. Kelisky and T. V. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math., **21** (1967), 511-520. MR0212457. Zbl 0177.31302.

[20] G. Nowak, *Approximation properties for generalized q-Bernstein polynomials*, J. Math. Anal. Appl., **350** (2009), 50-55. MR2476891. Zbl 1162.33009.

[21] S. Ostrovska, *q-Bernstein polynomials and their iterates*, Journal of Approximation Theory, **123** (2003), 232-255. MR1990098. Zbl 1093.41013.

[22] I. A. Rus, *Iterates of Stancu Operators, via Contraction Principle*, Studia Univ. Babeş-Bolyai Math., **47** (2002), 101-104. MR1993908. Zbl 1249.41020.

[23] I. A. Rus, *Iterates of Bernstein Operators, via Contraction Principle*, J. Math. and Appl., **292** (2004), 259-261. MR2050229. Zbl 1056.41004.

[24] I. A. Rus, *Fixed point and interpolation point set of a positive linear operator on $C(\bar{D})$*, Studia Univ. Babeş-Bolyai Math., **55** (2010), 243-248. MR2785009. Zbl 1240.41069.

[25] I. A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Presa Universitară Clujeană, Romania, 2008. MR2494238. Zbl 1171.54034.

****

[26] D. D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., **13** (1968), 1173-1194. MR0238001. Zbl 0167.05001.

[27] T. Savu, I. Neacşu, Ş. Grigorescu and E. M. Garabet, *Bazele instrumentaţiei virtuale LabView (in romanian)*, Publishing House Atelier didactic, 2002, Bucharest.

[28] A. Szilárd, *Iterates of the multidimensional Cesáro operator*, Carpathian Journal of Mathematics, **28** (2012), 191-198. MR3027244. Zbl 1289.47064.

[29] A. Szilárd and I. A. Rus, *Iterates of Cesàro Operators, via Fixed Point Principle*, Fixed Point Theory, **11** (2010), 171-178. MR2743773. Zbl 1213.47033.

[30] ***, LabVIEW User Manual; April 2003 Edition; National Instruments Corp. Austin, Texas, U.S.A.

[31] http://www.ni.com/ro-ro/shop/labview.html. Accessed on 31 January 2019.

Amelia Bucur

Faculty of Sciences, Department of Mathematics and Informatics,

Lucian Blaga University of Sibiu,

Sibiu, Romania.

e-mail: amelia.bucur@ulbsibiu.ro

License