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UNIVALENT FUNCTIONS RELATED TO q–ANALOGUE OF GENERALIZED M–SERIES WITH RESPECT TO k–SYMMETRIC POINTS

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Abstract. In this paper, we introduce subclasses of analytic functions by using q–analogue of generalized M–series and k–symmetric points. Some special coefficient inequalities are also discussed.

1 Introduction

The M–series in [8] in given by:

$$ {}_t^\alpha\mathcal{M}_s(z) = \sum_{k=0}^{\infty} \frac{(d_1)_k \cdots (d_t)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{1.1} $$

where $\alpha, z \in \mathbb{C}$, $\mathrm{Re}\{\alpha\} > 0$ and $(d_m)_k$, $(b_m)_k$ are then well-known Pochhammer symbols.

It is easy to see that by the ratio test the series in (1.1) is convergent for all z if $t \leqslant s$.

The extension of both Mittag-Leffler function and generalized hypergeometric function ${}_xF_y$ is called generalized M–series which introduced in [9] as follow:

$$ {}_t^\alpha\mathcal{M}_s^\beta = \sum_{k=0}^{\infty} \frac{(d_1)_k \cdots (d_t)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (z, \alpha, \beta \in \mathbb{C}). \tag{1.2} $$

For more details see [1].

The series in [2] is a convergent series for all z if $t \leqslant s + \mathrm{Re}\{\alpha\}$. Also it is convergence for $|z| < \alpha^\alpha$ if $t = s + \mathrm{Re}\{\alpha\}$.

The q–analogue of Pochhammer symbol is defined by:

$$ (\alpha; q)_n = \prod_{k=0}^{n-1} (1 - \alpha q^k), \qquad (n \in \mathbb{N}), \tag{1.3} $$

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and for $n = 0$ and $q \neq 1$, $(\alpha; q)_0 = 1$. When $n \to \infty$, we shall assume that $|q| < 1$, see [2].

Also q–derivative of a function $f(z)$ is defined by

$$\big(D_q f\big)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad (z \neq 0, \quad q \neq 1), \tag{1.4}$$

and

$$\lim_{q \to 1} D_q f(z) = f'(z). \tag{1.5}$$

By using (1.4), we conclude that:

$$\big(D^n_{\frac{1}{q}} f\big)(x) = q^n \big(D^n_q f\big)\big(\frac{x}{q^n}\big), \tag{1.6}$$

$$D^n_q z^\lambda = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\lambda - n + 1)} z^{\lambda - n}, \qquad \big(\mathrm{Re}\{\lambda\} + 1 \geqslant 0\big). \tag{1.7}$$

Indeed

$$\Gamma_q(z+1) = \frac{1-q^z}{1-z} \Gamma_q(z). \tag{1.8}$$

Also

$$\beta_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} dq(t) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{1.9}$$

where $\mathrm{Re}\{x\} > 0$, $\mathrm{Re}\{y\} > 0$ and $\beta_q(x,y)$ and $\Gamma_q(w)$ are the q–analogue of the beta function and q–gamma function respectively.

Now we consider the q–analogue of generalized M–series as follows:

$$ {}^{\alpha}_{t}\mathcal{M}^{\beta}_{s}(z;q) = \sum_{k=0}^{\infty} \frac{(d_1;q)_k \cdot (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k} \frac{z^k}{\Gamma_q(\alpha k + \beta)}, \tag{1.10}$$

where $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}\{\alpha\} > 0$, $|q| < 1$, $(\gamma; q)_k$ is the q–analogue of Pochhammer symbol and $\Gamma_q(w)$ is the q–gamma function.

By applying the convergent conditions of the well-known Fox-Wright generalized hypergeometric function and generalized H–function, the function ${}^{\alpha}_{t}\mathcal{M}^{\beta}_{s}(z;q)$ is convergent, see [3].

Some special eases of ${}^{\alpha}_{t}\mathcal{M}^{\beta}_{s}(z;q)$ are:

(i) The q–Mittag-Leffler function [5].
(ii) The generalized q–Mittag-Leffler function [10].

(iii) The q–generalized M–series as a special ease of the q–Wright generalized hypergeometric function [6].

Definition 1. *Let $\mathcal{A}$ denote the class of functions $f(z)$ of the type:*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.11}$$

which are analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 2. *the Hadamard product (convolution) for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ belong to $\mathcal{A}$ denoted by $f * g$ in defined as follows:*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k = (g * f)(z). \tag{1.12}$$

Definition 3. *A function $f(z) \in \mathcal{A}$ is in the class $X_n(\theta)$ if*

$$\mathrm{Re}\left\{\frac{z(f * F)' + (f * F)}{H_n(z)}\right\} < \theta, \tag{1.13}$$

where $\theta > 1$, $n \geqslant 1$ is a fixed positive integer,

$$F(z) = \left[1 - \frac{(1-d_1)\cdots(1-d_t)}{(1-b_1)\cdots(1-b_1)\Gamma(\alpha+\beta)}\right]z - \frac{1}{\Gamma_q(\beta)} + {}_t^{\alpha}\mathcal{M}_s^{\beta}(z;q), \tag{1.14}$$

and

$$H_n(z) = \frac{1}{n}\sum_{v=0}^{n-1} E^{-v}(f * F)(E^v z), \qquad \big(E^n = 1, \quad z \in \mathbb{U}\big). \tag{1.15}$$

Further, a function $f(z) \in \mathcal{A}$ is in the class $Y_n(\theta)$, if and only if $zf'(z) \in X_n(\theta)$.

From (1.11), (1.14) and (1.12) with a simple calculation, we get:

$$(f * F)(z) = z + \sum_{k=2}^{\infty} \frac{(d_1;q)_k \cdots (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma_q(\alpha k + \beta)} a_k z^k. \tag{1.16}$$

2 Main results

In this section, we shall obtain some coefficient bounds for functions in $X_n(\theta)$ and $Y_n(\theta)$ and their subclasses positive coefficients.

Note that other subclasses of analytic functions with respect to n-symmetric points have been studied by many authors, see [4, 7] and [11].

Theorem 4. *Let $\theta > 2$. If $f(z) \in \mathcal{A}$ satisfies:*

$$\sum_{k=1}^{\infty}\Bigg[\frac{(nk+2)(d_1;q)_{nk+1}\cdots(d_t;q)_{nk+1}}{(b_1;q)_{nk+1}\cdots(b_s;q)_{nk+1}(q;q)_{nk+1}\Gamma\big(\alpha(nk+1)+\beta\big)} + \left|\frac{(nk+2)(d_1;q)_{nk+1}\cdots(d_t;q)_{nk+1}}{(b_1;q)_{nk+1}\cdots(b_s;q)_{nk+1}(q;q)_{nk+1}\Gamma\big(\alpha(nk+1)+\beta\big)} - 2\theta\right|\Bigg]|a_{nk+1}| + \sum_{\substack{n=2\\ k\neq mk+1}}^{\infty}\frac{2(k+1)(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma(\alpha k+\beta)}|a_k| \leqslant 2(\theta-2), \tag{2.1}$$

then $f \in X_n(\theta)$.

Proof. Suppose that *theta* > 2 and $f(z) \in \mathcal{A}$, it is sufficient to show that:

$$\left|\frac{z(f*F)'+(f*F)}{f_n(z)}\right| < \left|\frac{z(f*F)'+(f*F)}{f_n(z)} - 2\theta\right|, \qquad (z\in\mathbb{U}).$$

By putting

$$W = \left|z(f*F)'+(f*F)\right| - \left|z(f*F)'+(f*F)-2\theta f_n(z)\right|,$$

and

$$H_n(z) = z + \frac{1}{n}\sum_{k=2}^{\infty}\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma(\alpha k+\beta)}\alpha_k\sum_{v=0}^{n-1}E^{v(k-1)}z^n, \tag{2.2}$$

we get

$$W = \left|2z+\sum_{k=2}^{\infty}(k+1)\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma_q(\alpha k+\beta)}a_kz^k\right| - \Bigg|2z+\sum_{k=2}^{\infty}(k+1)\frac{(d_1;q)_k\cdots(d_t;q)_ka_kz^k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k(\alpha k+\beta;q)_k} - 2\theta z - 2\theta\sum_{k=2}^{\infty}\frac{(d_1;q)_k\cdots(d_t;q)_ka_kb_kz^k}{(d_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma_q(\alpha k+\beta)}\Bigg|,$$

where

$$b_k = \frac{1}{n}\sum_{v=0}^{n-1} E^{v(k-1)}, \qquad (E^n = 1). \tag{2.3}$$

Hence for $|z| = r < 1$, we have:

$$\begin{aligned}
W &\leqslant 2r + \sum_{k=2}^{\infty} \frac{(k+1)(d_1;q)_k \cdots (d_t;q)_k a_k r^k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma(\alpha k+\beta)} \\
&- \left\{ 2(\theta-1)r - \sum_{k=2}^{\infty} \left| \frac{(k+1)(d_1;q)_k \cdots (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma(\alpha k+\beta)} - 2\theta b_k \right| |a_k| r^k \right\} \\
&< \left\{ \sum_{k=2}^{\infty} \left[\frac{(k+1)(d_1;q)_k \cdots (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma(\alpha k+\beta)} \right.\right. \\
&+ \left.\left. \left| \frac{(k+1)(d_1;q)_k \cdots (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma(\alpha k+\beta)} - 2\theta b_k \right| \right] |a_k| - 2(\theta-2) \right\} r.
\end{aligned}$$

From (2.3) we know

$$b_k = \begin{cases} 0 \quad , & k \neq mn+1 \\ 1 \quad , & k = mn+1 \end{cases}. \tag{2.4}$$

So we get

$$\begin{aligned}
W &< \left\{ \sum_{k=1}^{\infty} \left[\frac{(nk+2)(d_1;q)_{nk+1} \cdots (d_t;q)_{nk+1}}{(b_1;q)_{nk+1} \cdots (b_s;q)_{nk+1} (q;q)_{nk+1} \Gamma\big(\alpha(nk+1)+\beta\big)} \right.\right. \\
&+ \left. \left| \frac{(nk+2)(d_1;q)_{nk+1} \cdots (d_t;q)_{nk+1}}{(b_1;q)_{nk+1} \cdots (b_s;q)_{nk+1} (q;q)_{nk+1} \Gamma\big(\alpha(nk+1)+\beta\big)} - 2\theta \right| \right] |a_{nk+1}| \\
&+ \left. \sum_{\substack{k=2 \\ k\neq mn+1}}^{\infty} \frac{2(k+1)(d_1;q)_k \cdots (d_t;q)_k}{(b_1;q)_k \cdots (b_s;q)_k (q;q)_k \Gamma(\alpha k+\beta)} |a_k| - 2(\theta-2) \right\} r.
\end{aligned}$$

From (2.1) we know that $W < 0$. Thus we get then required result. □

By definition of $Y_n(\theta)$ we obtain the following corollary.

Corollary 5. *If $\theta > 2$ and $(f * F)(z)$ is defined by (1.16) satisfies*

$$\begin{aligned}
&\sum_{k=0}^{\infty}(nk+2)\Bigg[\frac{(nk+2)(d_1;q)_{nk+1}\cdots(d_t;q)_{nk+1}}{(b_1;q)_{nk+1}\cdots(b_s;q)_{nk+1}(q;q)_{nk+1}\Gamma\big(\alpha(nk+1)+\beta\big)} \\
&+\left|\frac{(nk+2)(d_1;q)_{nk+1}\cdots(d_t;q)_{nk+1}}{(b_1;q)_{nk+1}\cdots(b_s;q)_{nk+1}(q;q)_{nk+1}\Gamma\big(\alpha k+1)+\beta\big)}-2\theta\right|\Bigg]|a_{nk+1}| \qquad (2.5)\\
&+\sum_{\substack{k=2\\ k\neq mk+1}}^{\infty}\frac{2(k+1)^2(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma(\alpha k+\beta)}|a_k| \leqslant 2(\theta-2),
\end{aligned}$$

then $f(z) \in Y_n(\theta)$.

Now, we define two subclasses of $X_n(\theta)$ and $Y_n(\theta)$ as follow:

$$X_n^+(\theta) = \Big\{ f \in X_n(\theta) : \text{ The coefficients of } f * F \text{ are non-negative } \Big\}, \qquad (2.6)$$

$$Y_n^+(\theta) = \Big\{ f \in Y_n(\theta) : \text{ The coefficients of } f * F \text{ are non-negative } \Big\}, \qquad (2.7)$$

Theorem 6. *Let $n \geqslant 2$ and $2 < \theta \leqslant n+1$, then $f(z) \in X_n^+(\theta)$ if and only if*

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(k+1)(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma_q(\alpha k+\beta)}a_k \\
&-\theta\sum_{m=1}^{\infty}\frac{(d_1;q)_{mn+1}\cdots(d_t;q)_{mn+1}a_{mn+1}}{(b_1;q)_{mn+1}\cdots(b_s;q)_{mn+1}(q;q)_{mn+1}\Gamma_q(\alpha(mn+1)+\beta)} \leqslant 2(\theta-2).
\end{aligned} \qquad (2.8)$$

Proof. According to Theorem 4, we need only to prove the necessary. Since $f(z) \in X_n^+(\theta)$, then

$$\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma_q(\alpha k+\beta)}a_k \geqslant 0, \qquad (k \geqslant 2).$$

Also

$$\operatorname{Re}\left\{\frac{z(f*F)'+(f*F)}{H_n(z)}\right\} < \theta,$$

or equivalently

$$\left|\frac{z(f*F)'+(f*F)}{H_n(z)}\right| < \left|\frac{z(f*F)'+(f*F)}{H_n(z)}\right|,$$

or

$$\left|z(f*F)'+(f*F)\right| < \left|z(f*F)'+(f*F)-2\theta H_n(z)\right|.$$

Hence

$$\left|z+\sum_{k=2}^{\infty}\frac{k(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k\Gamma_q(\alpha k+\beta)}a_kz^k\right.$$
$$\left.+z+\sum_{k=2}^{\infty}\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_kz^k\right|$$
$$<\left|z+\sum_{k=2}^{\infty}\frac{k(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_kz^k\right.$$
$$+z+\sum_{k=2}^{\infty}\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_kz^k-2\theta$$
$$\left.-2\theta\sum_{m=1}^{\infty}\frac{(d_1;q)_{mn+1}\cdots(d_t;q)_{mn+1}a_{mn+1}}{(b_1;q)_{mn+1}\cdots(q;q)_{mn+1}\Gamma_q(\alpha(mn+1)+\beta)}z^{mn+1}\right|.$$

Since

$$\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_k\geqslant 0,$$

for $k\geqslant 2$ and $\theta>2$, by setting $z\to 1^-$, we get:

$$2+\sum_{k=2}^{\infty}(k+1)\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_k\leqslant 2\theta-2$$
$$+2\theta\sum_{m=1}^{\infty}\frac{(d_1;q)_{mn+1}\cdots(d_t;q)_{mn+1}}{(b_1;q)_{mn+1}\cdots(q;q)_{mn+1}\Gamma_q(\alpha(mn+1)+\beta)}a_{mn+1}$$
$$-\sum_{k=2}^{\infty}(k+1)\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_k,$$

or

$$\sum_{k=2}^{\infty}(k+1)\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_k$$
$$-\theta\sum_{m=1}^{\infty}\frac{(d_1;q)_{mn+1}\cdots(d_t;q)_{mn+1}a_{mn+1}}{(b_1;q)_{mn+1}\cdots(q;q)_{mn+1}\Gamma_q(\alpha(mn+1)+\beta)}\leqslant 2(\theta-2).$$

□

Similarly, we have the following theorem for the class $Y_n(\theta)$.

Corollary 7. *Let $n\geqslant 2$, $2<\theta\leqslant n+1$ and $f(z)\in\mathcal{A}$, then $f(z)$ is in the class $Y_n(\theta)$ if and only if*

$$\sum_{k=2}^{\infty}(k+1)^2\frac{(d_1;q)_k\cdots(d_t;q)_k}{(b_1;q)_k\cdots(q;q)_k\Gamma_q(\alpha k+\beta)}a_k$$

$$-\theta\sum_{m=1}^{\infty}(mn+1)\frac{(d_1;q)_{mn+1}\cdots(d_t;q)_{mn+1}a_{mn+1}}{(b_1;q)_{mn+1}\cdots(q;q)_{mn+1}\Gamma_q(\alpha(mn+1)+\beta)}\leqslant 2(\theta-2).$$

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