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THE FUNDAMENTAL AND WEAKLY CONTINUOUS PROPERTIES IN COMPLEMENTED TOPOLOGICAL ALGEBRAS

Marina Haralampidou and Konstantinos Tzironis

Abstract. We give conditions so that a certain left complemented algebra turns to be a fundamental one. In the case when the only minimal closed right ideals of a certain complemented algebra $(E, \perp)$ are axial, namely they have the form eE with e a special element and its vector complementor is continuous, then $\perp$ is weakly continuous. Moreover, conditions are supplied so that a left precomplemented locally m-convex algebra turns to be a complemented one.

1 Introduction and preliminaries

In [4], we introduced the notion of a fundamental complemented linear space, through continuous projections. This notion is hereditary, in the sense that, if a certain topological algebra is fundamental, then a concrete subspace is fundamental too [ibid. Theorem 19]. Moreover, for a fundamental complemented linear space, we defined the notion of continuity of the complementor. In some cases, we employ a generalized notion of complementation, that of (left) precomplementation. Relative to this, the continuity of the complementor for a certain fundamental complemented (topological) algebra is inherited to the induced vector complementor of the underlying linear space of a certain right ideal [ibid. Theorem 20]. Our concern here is to face, somehow, the reversed implications. Namely, we investigate under what conditions the properties "fundamental" and "continuous vector complementor" of a substructure of a certain complemented algebra led to the fundamentality of the topological algebra concerned and to a kind of continuity, characterized as weak continuity, of the complementor (Definition 13) in the initial topological algebra (Theorems 7 and 14). The notion of fundamentality, in complemented topological algebras and in complemented linear spaces, was also employed in [5], in connection with the complementarity of subalgebras in topological algebras of

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continuous operators.

All vector spaces and algebras, employed below, are taken over the field $\mathbb{C}$ of complexes. A *topological algebra* is an associative algebra E endowed with a Hausdorff topological vector space topology for which the ring multiplication is separately continuous (see e.g., [6]). If, in particular, the topology is defined by a family $(p_\alpha)_{\alpha\in A}$ of submultiplicative seminorms, then E is named a *locally $m-$convex algebra* and is denoted by $(E,(p_\alpha)_{\alpha\in A})$. We also employ the notation $\overline{S}$ for the (topological) closure of a subset S of a topological algebra E.

The following definition refers to some types of complemented topological vector spaces, which we deal with in the sequel.

By $\mathcal{V}_X$ we denote the set of all closed subspaces of a topological vector space X. For $M,N\in\mathcal{V}_X$ satisfying $X=M\oplus N$, there is a unique (linear mapping) $T=T(M,N)$ on X such that $\operatorname{Im}(T)=M$, $\ker(T)=N$ and $T^2=T$. Note that for a fixed M, there may be more than one such N and for different N, there will be different T. Thus T heavily depends on N.

Definition 1. A topological vector space X is called a *complemented space* if there is a mapping $p:\mathcal{V}_X\to\mathcal{V}_X : M\mapsto M^p$, satisfying the following conditions:

1. *if $M_1,M_2\in\mathcal{V}_X$ with $M_1\subseteq M_2$, then ${M_2}^p\subseteq {M_1}^p$;*

2. *if $M\in\mathcal{V}_X$, then $M^{pp}=M$; and*

3. *if $M\in\mathcal{V}_X$, then $X=M\oplus M^p$.*

The map p is called a *(vector) complementor,* and M^p a *complement* of M.

In what follows, (X,p) stands for a complemented space X with a complementor p.

For a topological algebra E, $\mathcal{L}_l(E)\equiv\mathcal{L}_l$ (resp. $\mathcal{L}_r(E)\equiv\mathcal{L}_r,\mathcal{L}(E)\equiv\mathcal{L}$) stands for the set of all closed left (right, two-sided) ideals of E.

For the next notion, we refer to [3, p. 3723, Definition 2.1]. See also [1].

A topological algebra E is called *left complemented* if there exists a mapping $\perp:\mathcal{L}_l\longrightarrow\mathcal{L}_l : I\mapsto I^\perp$, such that the following hold.

1. If $I\in\mathcal{L}_l$, then $E=I\oplus I^\perp$;

2. if $I,J\in\mathcal{L}_l, I\subseteq J$, then $J^\perp\subseteq I^\perp$;

3. if $I\in\mathcal{L}_l$, then $(I^\perp)^\perp=I$;

where $\perp$ is called a *left complementor* on E and $I^\perp$ is called a *complement* of I. In what follows, we denote by $(E,\perp)$ a left complemented algebra with a left complementor $\perp$.

A topological algebra E is called *left precomplemented*, if for every $I \in \mathcal{L}_l$, there exists $I' \in \mathcal{L}_l$ such that $E = I \oplus I'$. Analogous notions are defined on "the right" and on "both sides" (see [3, p. 3725, Definition 2.7]).

All the results in this paper hold true by interchanging "left" by "right". Throughout the paper, (0) denotes either the zero ideal of an algebra or the zero linear subspace of a linear space.

If for $I \in \mathcal{L}$ in a topological algebra E, the relation $I^2 = (0)$ implies $I = (0)$, then E is called *topologically semiprime*, while E is *topologically simple*, if it has no proper closed two-sided ideals. An idempotent element $0 \neq e = e^2$ is called *minimal*, if the algebra eEe is a division one. On the other hand, if e is a minimal (idempotent) element, then Ee is a minimal left ideal and eE is a minimal right ideal of E (see [7, p. 45, Lemma 2.1.8]). We remind that a *left topological zero divisor* of a topological algebra E is an element $(0 \neq)x \in E$, such that there is a net $(x_\delta)_{\delta\in\Delta}$ in E with $x_\delta \nrightarrow 0$ but $xx_\delta \to 0$.

For the sake of completeness, we refer some notions as in [4, Definitions 1, 2, 3, 5, 17 and 18].

Definition 2. A left precomplemented algebra E is called *fundamental left precomplemented* if for every $I \in \mathcal{L}_l$ there is $I' \in \mathcal{L}_l$ such that $E = I \oplus I'$ and $T(I, I')$ is continuous. A left complemented algebra $(E, \perp)$ is called *fundamental left complemented* if for every $I \in \mathcal{L}_l$, $T(I, I^\perp)$ is continuous.

The corresponding "right" and "two sided" versions could be defined analogously.

Definition 3. Let $(E, \perp)$ be a fundamental left complemented algebra. A net $(I_\delta)_{\delta\in\Delta}$ of minimal closed left ideals of E is said to be $\perp$*-convergent* to $I_0 \in \mathcal{L}_l$, if $T_\delta \longrightarrow T_0$ uniformly on every minimal right ideal of E, where $T_\delta = T_\delta(I_\delta, I_\delta^\perp)$ and $T_0 = T_0(I_0, I_0^\perp)$.

Closedness of minimal left ideals, in the previous definition, is redundant when we consider certain left complemented algebras (see [1, p. 969, Theorem 3.2]).

Definition 4. An element x in a topological algebra E is said to be *axially closed* if the left ideal Ex is minimal closed.

In particular, a subset of E is named *axially closed* if each of its elements is axially closed.

Concerning the previous notion, we note that if $x \in E$ is a primitive idempotent and the (closed) left ideal Ex is a left precomplemented algebra, then, in view of [1, p. 964, Theorem 2.1], the ideal concerned is minimal closed (namely, the element x is axially closed). We remind that, an element of an algebra E is called *primitive*, if it can not be expressed as the sum of two orthogonal idempotents; namely, of some idempotents elements y, z of E with $yz = zy = 0$.

In the same spirit, we introduce the "fundamental" and "continuous complementor" properties in the context of topological complemented linear spaces.

Definition 5. A complemented (topological) linear space (X, p) is called *fundamental complemented* if, for every closed linear subspace S of X, there is a continuous linear mapping $T : X \to X$ such that $T^2 = T$, $\mathrm{Im}(T) = S$ and $\ker(T) = S^p$.

In the rest of the paper, if E is a topological algebra, $[x]$ will stand for the (closed) subspace of E, generated by x.

Definition 6. Let (X, p) be a fundamental complemented linear space. The mapping p is said to be *continuous* if for every net $(x_\delta)_{\delta\in\Delta}$ of elements of X with $\lim_{\delta} x_\delta = x_0 \in X$ and $x_0 \neq 0$, the net $(T_\delta([x_\delta], [x_\delta]^p))_{\delta\in\Delta}$ converges to $T_0([x_0], [x_0]^p)$ uniformly.

2 The properties "fundamental" and "continuity of a complementor" from the partial to the global

In this section, we attempt to carry the fundamental property from a complemented vector subspace X to a certain (one-)sided complemented algebra E that contains X. The importance of topological zero divisors is evident from the results below.

In particular, Theorem 7 is a kind of converse of Theorem 19 in [4, p. 102]. To fix the notation, we recall the following from [4, p. 101, relation (3.9)].

Let $(E, \perp)$ be a topologically simple left complemented locally m-convex algebra and e a minimal element in E. Consider the (minimal closed right) ideal $R = eE$ of E. There exists a map $p : \mathcal{V}_R \to \mathcal{V}_R, S \mapsto S^p$, satisfying *1.*, *2.* and *3.* of Definition 1 (see [4, p. 100, Theorem 14]). In this context, the mapping

$$s : \mathcal{L}_l(E) \to \mathcal{V}_R : I \mapsto s(I) := I \cap R$$

is well defined, and in view of [4, p. 98, Corollary 12, (b)], it is $1 - 1$. Besides, by the proof of [4, p. 100, Theorem 14], $s(\mathcal{L}_l(E)) = \mathcal{V}_R$. Moreover,

$$j : \mathcal{V}_R \to \mathcal{L}_l(E) : S \mapsto j(S) := \overline{ES}$$

is a well defined map, and since $\overline{ES} \cap R = S$, is the inverse of s. Now, we consider the map

$$p : \mathcal{V}_R \to \mathcal{V}_R : S \mapsto S^p := s((j(S))^{\perp}) = (s \circ \perp \circ j)(S). \tag{2.1}$$

The preceding discussion assures that, R, as a linear space, is complemented with an (induced) vector complementor p as in (2.1) (see [4, p. 100, Theorem 14]). Now, we are ready to state the following.

Theorem 7. *Let $(E, \perp)$ be a topologically simple, left complemented locally $m-$convex algebra. Let e be a minimal element in E, which is not a left topological zero divisor and such that the complemented vector space $(R = eE, p)$ is fundamental. Then E is fundamental, as well.*

Proof. Let I be a closed left ideal in E. Then, according to [8, Remark 2.4], there exists a unique linear mapping $T : E \to E$ with $T^2 = T$, $\mathrm{Im}(T) = I$ and $\ker(T) = I^{\perp}$. To show that E is fundamental, it is sufficient to show that T is continuous. We put $S = R \cap I$. Then, $S \in \mathcal{V}_R$ and $S^p = R \cap I^{\perp}$. Let $g : R \to R$ be the unique linear mapping with $g^2 = g$, $\mathrm{Im}(g) = S$ and $\ker(g) = S^p$. Then $T|_R = g$ (see also the proof of Theorem 19 in [4]). Since the complemented vector space (R, p) is fundamental, g is continuous.

Now, take $x \in E$ and a net $(x_\delta)_{\delta\in\Delta}$ in E with $\lim_\delta x_\delta = x$. Since $E = I \oplus I^{\perp}$, there are unique $y \in I$ and $z \in I^{\perp}$ such that $x = y + z$. For every $\delta \in \Delta$, there are unique $y_\delta \in I$, $z_\delta \in I^{\perp}$, with $x_\delta = y_\delta + z_\delta$. Since $T(x_\delta) = y_\delta$, for the continuity of T to x, namely, in order to prove that $\lim_\delta T(x_\delta) = T(x)$, it is enough to show that $\lim_\delta y_\delta = y$. We have $ex = ey + ez$ with $ex \in R$, $ey \in S$ and $ez \in S^p$. For every $\delta \in \Delta$, $ex_\delta = ey_\delta + ez_\delta$ with $ex_\delta \in R$, $ey_\delta \in S$ and $ez_\delta \in S^p$. Moreover, $\lim_\delta(ex_\delta) = ex$. Therefore, $\lim_\delta g(ex_\delta) = g(ex)$. Thus, $\lim_\delta ey_\delta = ey$ and $\lim_\delta(e(y_\delta - y)) = 0$. Since e is not a left topological zero divisor, $\lim_\delta(y_\delta - y) = 0$, and this completes the assertion. □

Now, we are interested in finding conditions that lead to $\perp$-convergence of a certain net $(Ex_\delta)_{\delta\in\Delta}$ (see Proposition 11 and Definition 3), which in turn leads to the weak continuity of a complementor (Theorem 14). For that, we present two lemmas that are useful in the proof of Proposition 11.

For the next, we also refer to [4, Proposition 13].

Lemma 8. *Let E be a topologically simple, locally m-convex algebra. Consider a minimal element $e \in E$ and the (minimal closed right) ideal $R = eE$. If S is a closed subspace of R, then $S = \overline{ES} \cap R$, where ES stands for the left ideal of E, generated by S.*

Proof. Since e is idempotent, every element $x \in R$ has the form $x = ex$. Since $S \subset R$, $S = eS$. Therefore, $S \subseteq ES \subseteq \overline{ES}$, and hence $S \subseteq \overline{ES} \cap R$. Actually, the last relation is an equality. Indeed, take $z \in \overline{ES} \cap R$. Then $z = \lim_\delta z_\delta$ with $(z_\delta)_{\delta\in\Delta}$ a net in ES and $z = ez$. Thus,

$$z = ez = e\lim_\delta z_\delta = \lim_\delta(ez_\delta).$$

But, $ez_\delta \in ES \cap R$ and $z \in \overline{ES \cap R}$. Therefore, $\overline{ES} \cap R \subseteq \overline{ES \cap R}$, and hence,

$$\overline{ES} \cap R = \overline{ES \cap R}. \tag{2.2}$$

It is obvious that $RES \subseteq ES \cap R$. If $x \in ES \cap R$, then $x \in ES$ and $x = ex$. Thus, $x = ex \in RES$. The last argument leads to $RES = ES \cap R$. Since $S = eS$, we get

$$ES \cap R = eEES \subseteq eES = eEeS \simeq \mathbb{C}S \simeq S$$

(see also [6, p. 52, Lemma 3.1 and p. 62, Corollary 5.1] and [2, p. 155, Theorem 3.11]). Thus, $ES \cap R \subseteq S$ and $\overline{ES \cap R} \subseteq S$. So, in view of (2.2), the proof is complete. □

Lemma 9. *Let E be a topologically simple algebra. Consider a minimal element $e \in E$ and the (minimal closed right) ideal $R = eE$. Suppose that e is not a right zero divisor of R. If $x \in R$ with $x \neq 0$, and $S = [x]$ is the (closed) subspace of E, generated by x, then $ES = Ex = Eex$, where ES stands for the left ideal of E, generated by S. Furthermore, the ideal Ex is minimal closed.*

Proof. We first note that since the element e is not a right zero divisor of R and $0 \neq x \in R = eE$, $xe \neq 0$.

We have

$$ES = E[x] = \{\lambda x + ax : \lambda \in \mathbb{C}, a \in E\}.$$

Since $x = ex$, we have $ES = \{yx : y \in E\}$, and $ES = Ex = Eex$. We prove that the left ideal Ex is closed. Take $a \in \overline{Ex}$. Then there is a net $(a_\delta x)_{\delta \in \Delta}$ in Ex with $a_\delta x \underset{\delta}{\to} a$. Since $x = ex$, $a_\delta xe = a_\delta exe \underset{\delta}{\to} ae$. Since the algebra eEe is a division one, and $exe = xe \neq 0$, $(exe)^{-1}$ exists. So, by the separate continuity of multiplication, $a_\delta(exe)(exe)^{-1} \underset{\delta}{\to} ae(exe)^{-1}$ and $a_\delta e \underset{\delta}{\to} ae(exe)^{-1}$. Therefore, $a_\delta ex \underset{\delta}{\to} ae(exe)^{-1}x$ and thus,

$$a_\delta x \underset{\delta}{\to} ae(exe)^{-1}x \in Ex \quad \text{or} \quad a = ae(exe)^{-1}x \in Ex.$$

So, Ex is closed.

Now, we prove that Ex, as a closed left ideal, is minimal. So, let L be a closed left ideal with $(0) \neq L \subseteq Ex$. E, as topologically simple, is topologically semiprime. Thus, $L^2 \neq (0)$ (see [2, p. 149, Theorem 2.1]). Therefore, there are $yex, zex \in L$ such that $yexzex \neq 0$ that yields $exze \neq 0$. Since eEe is a division algebra, there exists $w \in eEe$ such that $wexze = e$. Since $zex \in L$ and L is a left ideal of E, we get $Ezex \subseteq L$. Therefore, $Eex = Ewexzex \subseteq L \subseteq Eex$, that yields $L = Eex = Ex$, proving the minimality of Ex, as asserted. □

Remark 10. *Lemma 9 holds true, if we replace the assumption "e is not a right zero divisor of R" with the stronger condition "R has no nilpotent elements of order 2". Indeed, the only thing we have to prove is that $xe \neq 0$. If $xe = 0$, then $xex = 0$, and since $x = ex$, $x^2 = 0$, so by hypothesis, $x = 0$, that is a contradiction.*

Proposition 11. *Let $(E, \perp)$ be a topologically simple left complemented locally $m-$convex algebra. Consider a minimal element $e \in E$, which is not a left topological zero divisor and it is not a right zero divisor of R. Suppose that the complementor p of the complemented linear space (R, p) (see (2.1)) is continuous. Let $(x_\delta)_{\delta \in \Delta}$ be a net in E, axially closed, with $\lim_\delta x_\delta = x_0$, $x_0 \neq 0$, such that the ideal Ex_0 in E is minimal closed. Then $T_\delta(Ex_\delta, (Ex_\delta)^\perp) \longrightarrow T_0(Ex_0, (Ex_0)^\perp)$ uniformly on the minimal (closed) right ideal R.*

Proof. We first note that the continuity of the complementor p of the complemented topological linear space (R,p), presupposes that (R,p) is fundamental. This is due to the fact that the very definition of the continuity, is given in terms of the existence of continuous mappings on one-dimensional (closed) subspaces (see Definitions 5 and 6). Thus, by Theorem 7, the left complemented algebra $(E,\perp)$ is fundamental.

Since the net $(x_\delta)_{\delta\in\Delta}$ is axially closed, for every $\delta\in\Delta$, the left ideal Ex_δ is minimal closed. We have $Eex_\delta \subseteq Ex_\delta$. Since e is not a left topological zero divisor, it is not left zero divisor either, so, since $x_\delta \neq 0$, we get $ex_\delta \neq 0$ that yields $Eex_\delta \neq (0)$. From Lemma 9, the left ideal Eex_δ of E is closed. So, since Ex_δ is minimal, we get $Eex_\delta = Ex_\delta$. In analogy, we get $Eex_0 = Ex_0$. Denote by $S_\delta = [ex_\delta]$ the linear subspace of E, generated by ex_δ. Then $\overline{ES_\delta} = Eex_\delta$ (see Lemma 9) and $R\cap \overline{ES_\delta} = S_\delta = [ex_\delta]$ (see Lemma 8).

Moreover, $ex_\delta \neq 0$. Similarly, if $S_0 = [ex_0]$, then

$$\overline{ES_0} = Eex_0 \quad \text{and} \quad R\cap \overline{ES_0} = S_0 = [ex_0].$$

Put $T_\delta = T_\delta(Ex_\delta,(Ex_\delta)^\perp), \delta\in\Delta$ and $T_0 = T_0(Ex_0,(Ex_0)^\perp)$, which are the continuous mappings on E with $T_\delta^2 = T_\delta$, $\mathrm{Im}(T_\delta) = Ex_\delta$, $\ker(T_\delta) = (Ex_\delta)^\perp$ and $T_0^2 = T_0$, $\mathrm{Im}(T_0) = Ex_0$, $\ker(T_0) = (Ex_0)^\perp$. Take also the continuous linear mappings $g_\delta = g_\delta([ex_\delta],[ex_\delta]^p)$, $\delta\in\Delta$ and $g_0 = g_0([ex_0],[ex_0]^p)$ on R that satisfy $g_\delta^2 = g_\delta$, $\mathrm{Im}(g_\delta) = [ex_\delta]$, $\ker(g_\delta) = [ex_\delta]^p$, and $g_0^2 = g_0$, $\mathrm{Im}(g_0) = [ex_0]$, $\ker(g_0) = [ex_0]^p$. Then $T_\delta|_R = g_\delta$ for every $\delta\in\Delta$, and $T_0|_R = g_0$ due to the uniqueness of the projections (see [8, p. 265, Remark 2.4]). Since the complementor p is continuous, g_δ converges to g_0 uniformly. Namely, $T_\delta|_R(Ex_\delta,(Ex_\delta)^\perp) \longrightarrow T_0|_R(Ex_0,(Ex_0)^\perp)$ uniformly or $T_\delta \longrightarrow T_0$ uniformly on the minimal (closed) right ideal R. □

Comment 12. *Proposition 11 is the key in obtaining one of our main results, Theorem 14. It is based on a certain element e which is rich in properties and lead to information about certain complemented algebras. For instance, by the minimality of e, the form of the closed subspaces S of the minimal right ideal $R = eE$ is explicitly described via the closed left ideal of R, generated by S (Lemma 8). In the special case, when e is not a right zero divisor on R, the one-dimensional (closed) subspaces give rise to minimal closed ideals (Lemma 9). Moreover, if e is not a left topological zero divisor on E, then $\perp$-convergence on a certain closed left ideal of E is succeeded (Proposition 11) which, in turn, leads to the weak continuity of the complementor (Theorem 14).*

We now introduce the concept of the weak continuity of the complementor $\perp$ in the context which we are interested in.

Definition 13. Let $(E,\perp)$ be a fundamental left complemented algebra. The mapping $\perp$ is said to be *weakly continuous,* if for any convergent axially closed net $(a_\delta)_{\delta\in\Delta}$ with $a_\delta \underset{\delta}{\rightarrow} a_0 \in E, a_0 \neq 0$, and such that Ea_0 is a minimal closed left ideal of

E, the net $(Ea_\delta)_{\delta\in\Delta}$ is $\perp$-convergent to Ea_0. Namely, $T_\delta \longrightarrow T_0$ uniformly on every minimal right ideal of E, where $T_\delta = T_\delta(Ex_\delta, (Ex_\delta)^\perp)$ and $T_0 = T_0(Ex_0, (Ex_0)^\perp)$.

Theorem 14. *Let $(E, \perp)$ be a topologically simple left complemented locally $m-$convex algebra. Suppose that all minimal closed right ideals of E are of the form $R_k = e_kE, k \in K$, with e_k a minimal element of E which is not a left topological zero divisor on E, and it is not a right zero divisor on R_k. Let p_k be the respective vector complementor on $\mathcal{V}_k$ (the set of all closed linear subspaces of R_k; see (2.1)). If, for each $k \in K$, p_k is continuous, then $\perp$ is weakly continuous.*

Proof. Consider $k_0 \in K$. Since the complementor p_{k_0} is continuous, the complemented linear space (R_{k_0}, p_{k_0}) is fundamental. So, since e_{k_0} is not a left topological zero divisor, E turns to be fundamental (see Theorem 7). Let $(x_\delta)_{\delta\in\Delta}$ be an axially closed net in E with $\lim\limits_\delta x_\delta = x_0, x_0 \neq 0$, and Ex_0 a minimal closed left ideal in E. Then, in view of Proposition 11, $T_\delta(Ex_\delta, (Ex_\delta)^\perp) \longrightarrow T_0(Ex_0, (Ex_0)^\perp)$ uniformly on the minimal (closed) right ideal R_k, for every $k \in K$. Provided that every minimal right ideal of E is of the form $R_k = e_kE$, we get the assertion (see also Definition 13). $\square$

In the next result, we give conditions so that a left precomplemented locally m-convex algebra becomes a complemented one. This has to do with the complementarity (as a linear space) of a certain right ideal in the algebra concerned. This result is important not only by its own right. Indeed, as we saw in Theorem 14, the complementarity of a certain minimal right ideal R is a key stone to get weak continuity of the complementor. As a matter of fact, in Corollary 17, we get the equivalence of complementarity in R and E, respectively. Concerning the next proof, by $\mathcal{A}_r(S)$ we denote the right annihilator of an $\emptyset \neq S \subseteq E$. This is a right ideal, which, in particular, is a 2-sided ideal, if S is a right ideal.

Theorem 15. *Let E be a topologically simple left precomplemented locally m-convex algebra. Consider a minimal element $e \in E$ and the (minimal closed right) ideal $R = eE$, which we suppose that, as a linear space, is complemented with complementor p. Then there is defined an $1-1$ and onto mapping $\perp$ on the set of all closed left ideals of E, which reverses the inclusion, is reflexive and satisfies the relation $I \cap I^\perp = (0)$ for all $I \in \mathcal{L}_l(E)$. If, moreover, e is not a left zero divisor, then $E = I \oplus I^\perp$ for all $I \in \mathcal{L}_l(E)$. Namely, E is a left complemented algebra.*

Proof. According to [4, p. 98, Corollary 12, b)], the mapping

$$s : \mathcal{L}_l(E) \to \mathcal{V}_R : I \mapsto s(I) := I \cap R$$

is $1-1$. Moreover, by the proof of Theorem 14 in [ibid. p. 100], s is onto. So, the inverse mapping $s^{-1} : \mathcal{V}_R \to \mathcal{L}_l(E)$ is defined, and since $\overline{ES} \cap R = S$ (see the proof of [4, p. 100, Theorem 14]), we get $s^{-1}(S) := \overline{ES}$. Put $s^{-1} = j$ and consider the mapping

$$\perp : \mathcal{L}_l(E) \to \mathcal{L}_l(E) : I \mapsto I^\perp := j((s(I))^p) = (j \circ p \circ s)(I) = \overline{E(R \cap I)^p}.$$

It is easily seen that $\perp$ is well defined. Moreover, it is $1-1$ and onto, as a composition of $1-1$ and onto maps. Now, take $I, J \in \mathcal{L}_l(E)$ with $I \subseteq J$. We have $R\cap I \subseteq R\cap J$, whence $(R\cap J)^p \subseteq (R\cap I)^p$. Therefore, $\overline{E(R\cap J)^p} \subseteq \overline{E(R\cap I)^p}$, which means $J^\perp \subseteq I^\perp$. Moreover, $\perp$ is reflexive. Indeed, if $I \in \mathcal{L}_l(E)$, then in view of $s\circ j = id|_{\mathcal{V}_R}$, $j\circ s = id|_{\mathcal{L}_l(E)}$ and $p\circ p = id|_{\mathcal{V}_R}$, we have

$$(I^\perp)^\perp = (j\circ p\circ s)(I^\perp) = (j\circ p\circ id|_{\mathcal{V}_R}\circ p\circ s)(I) = (j\circ s)(I) = I.$$

We now prove that, if $I \in \mathcal{L}_l(E)$, then $I\cap I^\perp = (0)$. We have

$$j^{-1}(I^\perp) = j^{-1}((j\circ p\circ s)(I)).$$

Namely, $s(I^\perp) = s((j\circ p\circ s)(I)) = (p\circ s)(I)$. Therefore, $R\cap I^\perp = (R\cap I)^p$. According to [4, p. 99, Proposition 13], $R\cap I = \overline{RI} = RI$ for any closed left ideal of E. Hence, we also have $\overline{RI^\perp} = RI^\perp$. The previous argumentation leads to $(RI)^p = (R\cap I)^p = R\cap I^\perp = \overline{RI^\perp} = RI^\perp$. Namely, $RI^\perp = (RI)^p$.

Put $K = I\cap I^\perp$. Then $K \in \mathcal{L}_l(E)$. We have

$$RK \subseteq RI = R\cap I \text{ and } RK \subseteq RI^\perp = R\cap I^\perp = (R\cap I)^p$$

(see also [4, p. 99, Proposition 13]). Thus, $RK \subseteq (R\cap I)\cap(R\cap I)^p = (0)$, which yields $K \subseteq \mathcal{A}_r(eE)$. We prove that the ideal $\mathcal{A}_r(eE)$ is closed. To this end, take $a \in \overline{\mathcal{A}_r(eE)}$. Then there is a net $(a_\delta)_{\delta\in\Delta}$ in $\mathcal{A}_r(eE)$ with $a_\delta \underset{\delta}{\to} a$. Then, for any $x\in E$, we get $exa = ex\lim\limits_\delta a_\delta = \lim\limits_\delta(exa_\delta) = 0$. Thus, $a \in \mathcal{A}_r(eE)$ that proves the assertion. Since E is topologically simple, either $\mathcal{A}_r(eE) = (0)$ or $\mathcal{A}_r(eE) = E$. The latter can not be true, since $e^2 = e \neq 0$. Therefore, $\mathcal{A}_r(eE) = (0)$, and hence $K = (0)$. Namely, $I\cap I^\perp = (0)$. Let $x\in E$. Then $ex \in R$. By hypothesis, R is complemented. So, since

$$R = (R\cap I)\oplus(R\cap I)^p = (R\cap I)\oplus(R\cap I^\perp)$$

and $R\cap I = RI, R\cap I^\perp = RI^\perp$ (see also [4, p. 99, Proposition 13]), there are $y_1, y_2 \in R, z_1 \in I$ and $z_2 \in I^\perp$, such that $ex = y_1z_1 + y_2z_2$. Since $y_1, y_2 \in R$, $y_1 = ey_1$ and $y_2 = ey_2$. Hence, $ex = ey_1z_1 + ey_2z_2$ and thus, $e(x - y_1z_1 - y_2z_2) = 0$. Therefore, since e is not a left zero divisor, we have $x = y_1z_1 + y_2z_2$, and since $y_1z_1 \in I$ and $y_2z_2 \in I^\perp$, we get $E = I + I^\perp$ that completes the proof. □

Proposition 16. *Let E be a topologically simple locally m-convex algebra. Let $e\in E$ be a minimal element and the (minimal closed right) ideal $R = eE$. Consider the assertions:*

1) *E is a left complemented algebra.*

2) *R is complemented as a linear space.*

Then 1) $\Rightarrow$ 2). If moreover, e is not a left zero divisor and E is a left precomplemented algebra, then 2) $\Rightarrow$ 1).

Proof. 1) $\Rightarrow$ 2). See [4, p. 100, Theorem 14].
2) $\Rightarrow$ 1). Apply Theorem 15. □

Gathering the last result, we get the next.

Corollary 17. *Let E be a topologically simple, left precomplemented locally $m-$convex algebra and e a minimal element in E, which is not a left zero divisor. Consider the (minimal closed right) ideal $R = eE$. Then the following are equivalent.*

1) *E is a left complemented algebra.*
2) *R is complemented as a linear space.*

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Marina Haralampidou
Department of Mathematics,
National and Kapodistrian University of Athens,
Panepistimioupolis, Athens 15784, Greece
e-mail: mharalam@math.uoa.gr

Konstantinos Tzironis
Department of Mathematics,
National and Kapodistrian University of Athens,
Panepistimioupolis, Athens 15784, Greece
e-mail: tzirk@math.uoa.gr

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