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ON A BAGLEY-TORVIK FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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Abstract. Existence of solutions for a Bagley-Torvik fractional integro-differential inclusion is investigated in the case when the values of the set-valued map are not convex.

1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([3, 8, 11, 12, 13] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

In some recent papers [1, 7, 10] etc. the attention was focused on a special class of boundary value problems associated to fractional differential equations; namely, Bagley-Torvik type fractional differential equations. Namely, in [10] it is studied a generalized Bagley-Torvik fractional differential equation of the form

$$D_c^\theta x(t) - aD_c^\delta x(t) = f(t, x(t)), \quad x(0) = x_0, \quad x(1) = x_1 \tag{1.1}$$

where $1 \leq \delta < \theta < 2$, $a \in \mathbf{R}$, D_c^q is the Caputo fractional derivative of order q, $f(.,.) : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is a given function and $x_0, x_1 \in \mathbf{R}$. In [7] it is considered the following Bagley-Torvik fractional differential inclusion

$$D_c^\theta x(t) - aD_c^\delta x(t) \in G(t, x(t)), \quad x(0) = h(z), \tag{1.2}$$

with $0 < \delta < \theta \leq 1$, $a \in \mathbf{R}$, $G(.,.) : [0, T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $h(.) : \mathbf{R} \to \mathbf{R}$ is a given function. The paper [1] is devoted to the study of the problem

$$(\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^\alpha)x(t) = f(t, x(t)), \tag{1.3}$$

$$x(0) = 0, x(\xi) = \sum_{i=1}^n j_i x(\eta_i), x(1) = \sum_{i=1}^k \lambda_i \int_{\nu_i}^{\sigma_i} x(s)ds, \tag{1.4}$$

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where $\alpha \in (0,1)$, $f(.,.) : [0,1] \times \mathbf{R} \to \mathbf{R}$, $\delta_i \in \mathbf{R}$, $i = 0,1,2$, $\delta_2 \neq 0$, $j_i \in \mathbf{R}$, $i = \overline{1,n}$, $\lambda_i \in \mathbf{R}$, $i = \overline{1,k}$ and $0 < \xi < \eta_1 < ... < \eta_n < \nu_1 < \sigma_1 < \nu_2 < \sigma_2 < ... < \nu_k < \sigma_k < 1$.

All the papers quoted above provide existence results for the problems studied and these results are obtained using some suitable theorems of fixed point theory.

In the present paper we consider fractional integro-differential inclusions of the form

$$(\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,1]) \tag{1.5}$$

with boundary conditions (1.4), where $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V : C([0,1], \mathbf{R}) \to C([0,1], \mathbf{R})$ is a nonlinear Volterra operator $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

Our goal is to extend the study in [1] to the more general problem (1.5)-(1.4) and to show that Filippov's ideas ([9]) can be suitably adapted in order to obtain the existence of solutions for this problem. Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([9]) consists in proving the existence of a solution starting from a given "quasi" solution. At the same time, the result provides an estimate between the "quasi" solution and the solution obtained.

We note that existence results of the type provided in the present paper exists in the literature ([4, 5, 6] etc.), but their exposure in the framework of problem (1.5)-(1.4) is new.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our results.

2 Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

Let $I = [0,1]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$.

Definition 1. *a) The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by*

$$I^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ *and* $\Gamma(.)$ *is the (Euler's) Gamma function defined by* $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$.

b) The Caputo fractional derivative of order $\alpha>0$ *of a function* $f:[0,\infty)\to\mathbf{R}$ *is defined by*

$$D_c^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{-\alpha+n-1}f^{(n)}(s)ds,$$

where $n=[\alpha]+1$. *It is assumed implicitly that* f *is* n *times differentiable whose* n*-th derivative is absolutely continuous.*

We recall (e.g., [11]) that if $\alpha>0$ and $f\in C(I,\mathbf{R})$ or $f\in L^\infty(I,\mathbf{R})$ then $(\mathrm{D}_c^\alpha I^\alpha f)(t)\equiv f(t)$.

The next technical lemma is proved in [1].

Lemma 2. *Assume that* $\delta_1^2-4\delta_0\delta_2>0$. *For a given* $f(.)\in C(I,\mathbf{R})$, *the unique solution* $x(.)$ *of problem* $(\delta_2 D_c^{\alpha+2}+\delta_1 D_c^{\alpha+1}+\delta_0 D_c^\alpha)x(t)=f(t)$ *a.e.* $([0,T])$ *with boundary conditions (1.4) is given by*

$$x(t)=\frac{1}{\hat\delta}\{\int_0^t\int_0^s(e^{m_2(t-s)}-e^{m_1(t-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u)duds+\rho_1(t)[\int_0^\xi\int_0^s(e^{m_2(\xi-s)}- e^{m_1(\xi-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u)duds-\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s(e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u) duds]+\rho_2(t)[\int_0^1\int_0^s(e^{m_2(1-s)}-e^{m_1(1-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u)duds-\sum_{i=1}^k\lambda_i\int_{\nu_i}^{\sigma_i}\int_0^s (\frac{e^{m_2(\sigma_i-s)}-1}{m_2}-\frac{e^{m_1(\sigma_i-s)}}{m_1})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u)duds]\}, \tag{2.1}$$

where $m_1=\frac{-\delta_1-\sqrt{\delta_1^2-4\delta_0\delta_2}}{2\delta_2}$, $m_2=\frac{-\delta_1+\sqrt{\delta_1^2-4\delta_0\delta_2}}{2\delta_2}$, $\hat\delta=\delta_2(m_2-m_1)$, $\rho_1(t)=\frac{\omega_4a_1(t)-\omega_3a_2(t)}{\mu_1}$, $\rho_2(t)=\frac{\omega_1a_2(t)-\omega_2a_1(t)}{\mu_1}$, $a_1(t)=\frac{\delta_2[m_1(1-e^{m_2t})-m_2(1-e^{m_1t})]}{\delta_0}$, $a_2(t)=\hat\delta(e^{m_2t}-e^{m_1t})$, $\mu_1=\omega_1\omega_4-\omega_2\omega_3\neq0$, $\omega_1=\frac{\delta_2}{\delta_0}[m_2(1-\sum_{i=1}^n j_i-e^{m_1\xi}+\sum_{i=1}^n j_ie^{m_1\eta_i})-m_1(1-\sum_{i=1}^n j_i-e^{m_2\xi}+\sum_{i=1}^n j_ie^{m_2\eta_i})]$, $\omega_2=\hat\delta(e^{m_1\xi}-e^{m_2\xi}-\sum_{i=1}^n j_ie^{m_1\eta_i}+\sum_{i=1}^n j_ie^{m_2\eta_i})$, $\omega_3=\frac{\delta_2}{\delta_0}[m_2(1-e^{m_1}-\sum_{i=1}^k\lambda_i(\sigma_i-\nu_i)+\sum_{i=1}^k\frac{\lambda_i}{m_1}(e^{m_1\sigma_i}-e^{m_1\nu_i})) - m_1(1-e^{m_2}-\sum_{i=1}^k\lambda_i(\sigma_i-\nu_i)+\sum_{i=1}^k\frac{\lambda_i}{m_2}(e^{m_2\sigma_i}-e^{m_2\nu_i}))]$, $\omega_4=\hat\delta[e^{m_1}-e^{m_2}-\sum_{i=1}^k\frac{\lambda_i}{m_1}(e^{m_1\sigma_i}-e^{m_1\nu_i})+\sum_{i=1}^k\frac{\lambda_i}{m_2}(e^{m_2\sigma_i}-e^{m_2\nu_i})]$.

Remark 3. *If we denote*

$$G(t,u)=\frac{1}{\hat\delta}\chi_{[0,t]}(u)\int_u^t(e^{m_2(t-s)}-e^{m_1(t-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{\rho_1(t)}{\hat\delta}\chi_{[0,\xi]}(u)\int_u^\xi(e^{m_2(\xi-s)} -e^{m_1(\xi-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds-\sum_{i=1}^n\frac{j_i\rho_1(t)}{\hat\delta}\chi_{[0,\eta_i]}(u)\int_u^{\eta_i}(e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds +\frac{\rho_2(t)}{\hat\delta}\int_u^1(e^{m_2(1-s)}-e^{m_1(1-s)})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds-\sum_{i=1}^k\frac{\lambda_i\rho_2(t)}{\hat\delta}\chi_{[0,\sigma_i]}(u)\int_{\nu_i}^{\sigma_i}(\frac{e^{m_2(\sigma_i-s)}-1}{m_2}- \frac{e^{m_1(\sigma_i-s)}}{m_1})\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds,$$

where $\chi_S(\cdot)$ *is the characteristic function of the set* S, *then the solution* $x(\cdot)$ *in Lemma 2 may be written as* $x(t)=\int_0^1 G(t,s)f(s)ds$.

Denote $\hat{\rho}_1 = \max_{t\in[0,1]} |\rho_1(t)|$, $\hat{\rho}_2 = \max_{t\in[0,1]} |\rho_2(t)|$, $M_1 = \frac{1}{\hat{\delta}} \max_{t,s\in[0,1]}$ $|e^{m_2(t-s)} - e^{m_1(t-s)}|$, $M_2 = \frac{1}{\hat{\delta}} \max_{t,s\in[0,1]} |\frac{e^{m_2(t-s)}-1}{m_2} - \frac{e^{m_1(t-s)}}{m_1}|$. Using the fact that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ we deduce that for any $t, s \in I$

$$|G(t,s)| \leq \frac{M_1}{\Gamma(\alpha+1)} + \frac{M_1\hat{\rho}_1\xi^\alpha}{\Gamma(\alpha+1)} + \frac{M_1\hat{\rho}_1}{\Gamma(\alpha+1)} \sum_{i=1}^n |j_i|\eta_i^\alpha + \frac{M_1\hat{\rho}_2}{\Gamma(\alpha+1)} + \frac{M_1\hat{\rho}_2}{\Gamma(\alpha+1)} \sum_{i=1}^k |\lambda_i|(\sigma_i^\alpha + \nu_i^\alpha) =: K_0.$$

Definition 4. *A function $x(.) \in C^3(I, \mathbf{R})$ is called a solution of problem (1.5)-(1.4) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ that satisfies $f(t) \in F(t, x(t), V(x)(t))$ a.e. (I) and $x(.)$ is given by (2.1).*

3 The Main Results

First, we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 5. *Consider X a separable Banach space, B is the closed unit ball in X, $G : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $c : I \to X, r : I \to \mathbf{R}_+$ are measurable functions. If*

$$G(t) \cap (c(t) + r(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to G(t) \cap (c(t) + r(t)B)$ has a measurable selection.

In order to prove our results we need the following hypotheses.

Hypothesis H1. i) *$F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I, F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\ x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) *$k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R}$, $(t, s) \to k(t, s, x)$ is measurable.*

iv) $|k(t, s, x) - k(t, s, y)| \leq L(t)|x - y| \quad$ *a.e.* $(t, s) \in I \times I, \quad \forall\, x, y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad K_0 = \int_0^T M(t)dt.$$

Theorem 6. *Assume that Hypothesis H1 is satisfied, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $M_0K_0 < 1$. Let $y(.) \in C^3(I, \mathbf{R})$ be such that $y(0) = 0, y(\xi) = \sum_{i=1}^n j_i y(\eta_i), y(1) = \sum_{i=1}^k \lambda_i \int_{\nu_i}^{\sigma_i}$*

$y(s)ds$ and there exists $p(.) \in L^1(I, \mathbf{R}_+)$ with $d((\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})y(t), F(t, y(t), V(y)(t))) \leq p(t)$ a.e. (I).

Then there exists $x(.)$ a solution of problem (1.5)-(1.4) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{K_0}{1 - K_0 M_0} \int_0^1 p(t)dt. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{(\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})y(t) + p(t)[-1, 1]\} \neq \emptyset \ a.e. \ (I).$$

It follows from Lemma 5 the existence of a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - (\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})y(t)| \leq p(t) \quad a.e. \ (I) \tag{3.2}$$

Define $x_1(t) = \int_0^1 G(t, s) f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq K_0 \int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G(t, s) f_n(s)ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e. \ (I), \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|ds) \quad a.e. \ (I) \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq K_0 (K_0 M_0)^n \int_0^1 p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n - 1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^1 |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \leq$$

$$K_0 \int_0^1 L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} L(s)|x_n(s) - x_{n-1}(s)|ds]dt_1 \leq K_0$$

$$\int_0^1 L(t_1)(1 + \int_0^{t_1} L(s)ds)dt_1.K_0^n M_0^{n-1} \int_0^1 p(t)dt = K_0(K_0 M_0)^n \int_0^1 p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbf{R})$. Therefore, by (3.5), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.
Moreover, one has

$$\begin{aligned}&|x_n(t)-y(t)| \le |x_1(t)-y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t)-x_i(t)| \le \\ &K_0\textstyle\int_0^1 p(t)dt + \sum_{i=1}^{n-1}(K_0\int_0^1 p(t)dt)(K_0M_0)^i = \frac{K_0\int_0^1 p(t)dt}{1-K_0M_0}.\end{aligned} \tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$

$$\begin{aligned}&|f_n(t) - (\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})y(t)| \le \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t)-f_i(t)| \\ &+|f_1(t) - (\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})y(t)| \le L(t)\frac{K_0\int_0^1 p(t)dt}{1-K_0M_0} + p(t).\end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that $x(.)$ is a solution of (1.5)-(1.4). Finally, passing to the limit in (3.6) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \ge 1$ we already constructed $x_n(.) \in C(I,\mathbf{R})$ and $f_n(.) \in L^1(I,\mathbf{R})$, $n = 1, 2, ...N$ satisfying (3.3), (3.5) for $n = 1, 2, ...N$ and (3.4) for $n = 1, 2, ...N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$ $F(t, x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1,1]\} \ne \emptyset$. Lemma 5 yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \le L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.3) with $n = N+1$. Thus $f_{N+1}(.)$ satisfies (3.4) and (3.5) and the proof is complete. □

The assumptions in Theorem 6 are satisfied, in particular, for $y(.) = 0$ and therefore with $p(.) = L(.)$. We obtain the following consequence of Theorem 6.

Corollary 7. *Assume that Hypothesis H1 is satisfied, $\delta_1^2 - 4\delta_0\delta_2 > 0$, $d(0, F(t,0,0) \le L(t)$ a.e. (I) and $K_0M_0 < 1$. Then there exists $x(.)$ a solution of problem (1.5)-(1.4) satisfying for all $t \in I$*

$$|x(t)| \le \frac{K_0}{1-K_0M_0}\int_0^1 L(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 becames

Hypothesis H2. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ *measurable.*

ii) *There exists* $l(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I$, $F(t, .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t, x_1), F(t, x_2)) \leq l(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_0^1 L(t)dt$ and consider the fractional differential inclusion

$$(\delta_2 D_c^{\alpha+2} + \delta_1 D_c^{\alpha+1} + \delta_0 D_c^{\alpha})x(t) \in F(t, x(t)) \quad a.e.\ ([0, 1]), \tag{3.7}$$

Corollary 8. *Assume that Hypothesis H2 is satisfied,* $d(0, F(t, 0) \leq L(t)$ *a.e.* (I), $\delta_1^2 - 4\delta_0\delta_2 > 0$ *and* $K_0L_0 < 1$. *Then there exists* $x(.)$ *a solution of problem (3.7)-(1.4) satisfying for all* $t \in I$

$$|x(t)| \leq \frac{K_0L_0}{1 - K_0L_0}.$$

Remark 9. *If in (3.7) F is single-valued, then a similar result to the one in Corollary 8 may be found in [1]; namely, Theorem 3.4.*

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