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ON THE DISTRIBUTION OF DEDEKIND SUMS

Kurt Girstmair

Abstract. Dedekind sums have applications in quite a number of fields of mathematics. Therefore, their distribution has found considerable interest. This article gives a survey of several aspects of the distribution of these sums. In particular, it highlights results about the values of Dedekind sums, their density and uniform distribution. Further topics include mean values, large and small (absolute) values, and the behaviour of Dedekind sums near quadratic irrationals. The present paper can be considered as a supplement to the survey article [9].

1 Introduction

Let b be a positive integer and $a \in \mathbb{Z}$, $(a, b) = 1$. The classical Dedekind sum $s(a, b)$ is defined by

$$s(a,b) = \sum_{k=1}^{b} ((k/b))((ak/b))$$

where $((\ldots))$ is the "sawtooth function" defined by

$$((t)) = \begin{cases} t - \lfloor t \rfloor - 1/2, & \text{if } t \in \mathbb{R} \setminus \mathbb{Z}; \\ 0, & \text{if } t \in \mathbb{Z}. \end{cases} \tag{1.1}$$

(see, for instance, [35, p. 1]). In many cases it is more convenient to work with

$$S(a,b) = 12s(a,b)$$

instead. We call $S(a, b)$ a *normalized* Dedekind sum.

Dedekind sums have many interesting applications, for instance, in the theory of modular forms (see [1, 9]), in algebraic number theory (class numbers, see [7, 28]), in connection with lattice point problems (see [6, 35]), topology (see [3, 22]) and algebraic geometry (see [40]). Various generalizations of Dedekind sums have been introduced for similar purposes (see [39, 43]). In [24] such a generalization is used for

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the assessment of random number generators. We emphasize that this enumeration is by no means exhaustive (nor are the corresponding references).

The general interest in Dedekind sums justifies the study of the distribution of these rational numbers. Several authors have contributed to this subject (see, for instance, [9, 10, 12, 21, 42, 44]).

The present article is meant as a survey of the distribution of Dedekind sums. But we do not pretend that this survey is complete or adequate in every respect. One may find that much of its focus is on results of the author. We should also like to draw the reader's attention to the survey article [9], most of whose results are *not* rendered here.

The main topics of this article are the following:

- the set of values of Dedekind sums,
- the density of this set and the question of uniform distribution,
- mean values,
- large and small Dedekind sums,
- the behaviour of Dedekind sums near quadratic irrationals.

In addition to proved results, we also present a number of conjectures and open questions.

2 Some basic tools

Given $a, b \in \mathbb{Z}$, $b \geq 1$, $(a, b) = 1$, we call b the *modulus* and a the *argument* of the normalized Dedekind sum $S(a, b)$. From the definition (1.1) one sees

$$S(a + b, b) = S(a, b) \text{ and } S(-a, b) = -S(a, b). \tag{2.1}$$

Accordingly, Dedekind sums $S(a, b)$ are periodic mod b and odd functions in their arguments a. In particular, we obtain all possible values $S(a, b)$ for a fixed modulus b, if we restrict a to the range $0 \leq a < b$.

Probably the most frequently used property of normalized Dedekind sums is the *reciprocity law* (see [35, p. 3]): If a, b are positive integers, then

$$S(a, b) + S(b, a) = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3.$$

This formula allows a recursive computation of $S(a, b)$, basically by means of the Euclidean algorithm. In order to see this, we write $b_{-1} = a$, $b_0 = b$ for $a, b > 0$,

$(a, b) = 1$. The Euclidean algorithm takes the form

$$\begin{aligned} b_{-1} &= c_0 b_0 + b_1, \\ b_0 &= c_1 b_1 + b_2, \\ &\vdots \\ b_{n-2} &= c_{n-1} b_{n-1} + b_n, \\ b_{n-1} &= c_n b_n. \end{aligned}$$

All numbers b_j, c_j are positive integers except possibly c_0, which may be 0. In addition, $b_0 > b_1 > \ldots > b_n = 1$ and $c_n \geq 2$.

Now $S(a, b) = S(b_{-1}, b_0)$, which, due to the periodicity mod b_0, equals $S(b_1, b_0)$ with $b_1 < b_0$. The reciprocity law expresses $S(b_1, b_0)$ in terms of $S(b_0, b_1) = S(b_2, b_1)$ with $b_2 < b_1$, then $S(b_2, b_1)$ in terms of $S(b_1, b_2) = S(b_3, b_2)$ with $b_3 < b_2$, and so on. Finally, we arrive at $S(b_{n-1}, b_n) = 0$. In addition, the regular continued fraction expansion of a/b takes the form $[c_0, c_1, \ldots, c_n]$.

The connection of Dedekind sums with continued fractions becomes most obvious in the *Barkan-Hickerson-Knuth formula* (see [5, 21, 23]). Let a, b be integers, $0 < a < b$, $(a, b) = 1$. Suppose that a/b has the regular continued fraction expansion $[0, c_1, \ldots, c_n]$. Then

$$S(a, b) = \sum_{j=1}^{n} (-1)^{j-1} c_j + \frac{a + a^*}{b} + \begin{cases} -3, & \text{if } n \text{ is odd;} \\ -1, & \text{otherwise.} \end{cases} \tag{2.2}$$

Here a^* is defined by $0 < a^* < b$ and $aa^* \equiv 1 \mod b$. Formula (2.2) is an important tool for the study of Dedekind sums.

Another important tool is the *three-term relation*: Let a, b, c, d be integers, $b, d > 0$, $(a, b) = (c, d) = 1$, $a/b \neq c/d$. Define $q = ad - bc$ and $r = aj - bk$, where $j, k \in \mathbb{Z}$ are such that $-cj + dk = 1$. Observe that $q \neq 0$. Then

$$S(a, b) = S(c, d) + \varepsilon \cdot S(r, |q|) + \frac{b}{dq} + \frac{d}{bq} + \frac{q}{bd} - 3\varepsilon, \tag{2.3}$$

where ε is the sign of q. This formula can be considered as a special case of a three-term relation given in [11].

Some results mentioned here use connections of Dedekind sums with modular forms (see [9]) or values of L-series (see [44]). Here we do not go into details.

3 The values of Dedekind sums

It is known that $bS(a, b)$ is an integer (see [35, p. 27]). Accordingly, $S(a, b)$ is a rational number k/q, $k, q \in \mathbb{Z}$, $q \geq 1$, $(k, q) = 1$, such that q divides the modulus b. So far it is not known which numerators k are possible for a given denominator q of a normalized Dedekind sum (a problem already mentioned in [35, p. 28]). However, it has been shown in [17] that these numerators form *complete* residue classes mod $q(q^2-1)$. In other words, a value k/q does not appear isolated, but all numbers $k/q + r(q^2-1)$, $r \in \mathbb{Z}$, are also values of normalized Dedekind sums. This reduces the search for possible numerators k to finitely many cases for every q.

The numerators k are subject to the following congruence conditions.

(a) If $3 \nmid q$, then $k \equiv 0 \mod 3$.

(b) If $2 \nmid q$, then

$$k \equiv \begin{cases} 2 \mod 4, & \text{if } q \equiv 3 \mod 4; \\ 0 \mod 8, & \text{if } q \text{ is a square}; \\ 0 \mod 4, & \text{otherwise.} \end{cases}$$

In the said paper [17] we conjectured that these necessary conditions, together with $(k,q)=1$, are sufficient for k/q being the value of a normalized Dedekind sum. We verified this conjecture for $q \le 60$ by means of a search procedure based on the finitely many cases just mentioned. Meanwhile this conjecture has been proved for even denominators q and also for odd squares q divisible by 3 or 5 (see [26]). This reference also reports that the conjecture has been verified for $1 \le q \le 200$, which has become possible by a reduction from congruences mod $q(q^2-1)$ to (a different type of) congruences mod q^2-1.

Whereas the value of $S(a,b)$ is subject to the restrictions (a) and (b), no restrictions occur for the *fractional part* of $S(a,b)$. In other words, for every rational number r, $0 \le r < 1$, there are positive integers $a < b$, $(a,b)=1$, such that $S(a,b) \equiv r \mod \mathbb{Z}$ (see [16]).

For each value k/q, $(k,q)=1$, of a normalized Dedekind sum there are infinitely many moduli b such that $S(a,b)=k/q$ for some argument a (see [19]).

Several authors have discussed another type of values, namely, the values of the integers $bS(a,b)$ (see [2, 29, 32, 37, 38]). For instance, it was shown that ± 24, ± 34 and ± 88 do not have the form $bS(a,b)$ for any choice of a and b (see [2]). This list of exceptional values was completed in [37].

In our opinion, however, these results rather concern possible *moduli* b than *values* of $S(a,b)$. We give an example. Suppose that $bS(a,b) = bk/q = 24$. Then [17, Th. 3] shows that k must be either 24 or 12. The case $k=24$ yields $b=q$, and $k=12$ yields $b=2q$. However, the number 24 is *not* of the form $bS(a,b)$. Accordingly, the cases $k=24, b=q$ and $k=12, b=2q$, are impossible. For example, if $k=24$ and $q=5$, we obtain $k/q = S(3,25)$. So $b=25$ is a modulus that yields $S(a,b)=24/5$ for $a=3$. But $b=q=5$ is not a modulus of this kind, since $S(a,5)$ takes only the values 0 and $\pm 12/5$. In the same way $S(1,5)=12/5$. Hence 5 is a possible modulus for $k=12$ and $q=5$, whereas $2q=10$ is not, since $S(a,10) \in \{0, \pm 36/5\}$ for the respective arguments a.

4 Density and uniform distribution

In [21] it has been shown that the set

$$\{(a/b, S(a,b)); a,b \in \mathbb{Z}, b \geq 1, (a,b) = 1\}$$

is dense in the plane $\mathbb{R}^2$. In particular,

$$\{S(a,b); a,b \in \mathbb{Z}, b \geq 1, 0 \leq a < b, (a,b) = 1\}$$

is dense in $\mathbb{R}$. The main tools of [21] are continued fractions, in particular, formula (2.2).

Let $x \in \mathbb{Q}$ and $\varepsilon > 0$ be given. The paper [15] explicitly describes a and b (in terms of x and ε) such that $|S(a,b) - x| < \varepsilon$. Indeed, by (2.1), we may assume $x \geq 0$. Put $l = \lceil 4 + x \rceil$ and write $l - 3 - x$ in the form j/k with positive integers $j, k, (j,k) = 1$. Let m be a positive integer, $m \geq 2/(k\varepsilon) + 1$, such that $mj \equiv 1 \mod k$. Put $t = 2m + (lk - j)(m^2 + 1)$ and $a = mt + 1$, $b = kt(m^2 + 1)$. Then a and b have the desired property.

A different approximation of this kind is given in the article [25].

The following result can also be seen under the aspect of density (see [31]). For a rational number $x = a/b$ with $a, b \in \mathbb{Z}$, $b > 0$, $(a,b) = 1$, put $S(x) = S(a,b)$. Then each line in $\mathbb{R}^2$ with a rational slope $\neq 1$ contains infinitely many points $(x, S(x))$ with $x \in \mathbb{Q}$.

The density of the set of normalized Dedekind sums in the p-adic number field $\mathbb{Q}_p$ has been investigated in [18]. In the case of $p \in \{2,3\}$, normalized Dedekind sums do not approximate p-adic units, so they are not dense in $\mathbb{Q}_p$. For $p \geq 5$, they are dense in $\mathbb{Q}_p$.

A quite different question concerns the *density of the set of moduli* b for a given value k/q of normalized Dedekind sums. In this case we have exhibited a sequence b_n of moduli with b_n growing like Cn^4 (see [19]). So these moduli form a rather sparse subset of the set of positive integers. But in reality the set of *all* possible moduli b for a given value k/q seems to be much denser. Hence this question waits for further investigations.

Results about the uniform distribution of Dedekind sums can be found in [4, 8, 9, 30, 41]. We only mention the following result of [9]. Let f be a real-valued, continuous function on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ with compact support. For $X > 0$, define

$$U_X(f) = \sum_{1 \leq b < X} \sum_{\substack{0 \leq a < b, \\ (a,b)=1}} f(a/b, S(a,b)).$$

Then

$$U_X(f) \sim \frac{X^2}{\log X} \cdot \frac{1}{2\pi^2} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}} f(x,y)dy\,dx \tag{4.1}$$

as X tends to infinity (asymptotic equality). Like other distribution results of [9], this assertion is proved by means of Fourier coefficients of real analytic modular forms. The reader may also look at the diagrams in [9], which illustrate these results.

5 Mean values

Because $S(b-a,b) = -S(a,b)$ (see (2.1)),

$$\sum_{\substack{0\le a<b,\\(a,b)=1}} S(a,b) = 0.$$

Hence only the arithmetic mean of $|S(a,b)|$ for varying arguments a is of interest, provided that a runs through *all* values $0 \le a < b$, $(a,b) = 1$. Here some insight is given by formula (2.2) and mean value results for continued fractions (see [27, 33]). In this way one obtains

$$\frac{1}{\varphi(b)} \sum_{\substack{0\le a<b,\\(a,b)=1}} |S(a,b)| \le \frac{6}{\pi^2}\log^2 b + O(\log b) \tag{5.1}$$

as b tends to infinity (see [20]). The main result of the said paper says

$$\frac{1}{\varphi(b)} \sum_{\substack{0\le a<b,\\(a,b)=1}} |S(a,b)| \ge \frac{3}{\pi^2}\log^2 b + O(\log^2 b/\log\log b)$$

for $b \to \infty$. Computations suggest, however, that (5.1) remains true if "$\le$" is replaced by "=" and the error term on the right hand side by a less good one. This has to do with the fact that the summation process of [20] (which is based on (2.3)) can consider only "large" values $|S(a,b)|$. Hence it would be desirable to show

$$\frac{1}{\varphi(b)} \sum_{\substack{0\le a<b,\\(a,b)=1}} |S(a,b)| \sim \frac{6}{\pi^2}\log^2 b$$

for $b \to \infty$.

The quadratic mean value of $S(a,b)$ for varying arguments a has been determined in [10]. A sharper result of [44] says

$$\frac{1}{\varphi(b)} \sum_{\substack{0\le a<b,\\(a,b)=1}} |S(a,b)|^2 = 5\lambda(b)\cdot b + O\left(\exp\left(\frac{4\log b}{\log\log b}\right)\cdot\frac{b}{\varphi(b)}\right).$$

with

$$\lambda(b) = \frac{\prod_{p^k \,||\, b}((1+1/p)^2 - 1/p^{3k+1})}{\prod_{p\,|\,b}(1+1/p+1/p^2)}.$$

As usual, p runs through the prime divisors of b and $p^k \,||\, b$ means that p^k is the largest power of p dividing b. The quantity $\lambda(b)$ grows at most like $\log\log b$. Hence $\sqrt{b}$ is roughly the order of magnitude of the quadratic mean value of the normalized Dedekind sums $S(a,b)$.

Higher power mean values can be found in [10].

6 Large and small values

First we note that $|S(a,b)| \le S(1,b) < b$ for all arguments a with $(a,b) = 1$ (see [34, Satz 2]). The mean values of the foregoing section give an idea of what "large" and "small" stand for. The quadratic mean value $\approx \sqrt{b}$ of the $S(a,b)$ for varying arguments a suggests that "large" means an order of magnitude $\gg \sqrt{b}$ for $|S(a,b)|$, or, in a somewhat wider sense, $\gg b^\alpha$ for some α, $0 < \alpha \le 1$, if b tends to infinity. On the other hand, the arithmetic mean of the Dedekind sums suggests that in general $|S(a,b)| \ll \log^2 b$, so "small" refers to Dedekind sums of logarithmic size.

The first result describes a subset of the interval $I = [0,b]$ outside which all Dedekind sums are $\le 3\sqrt{b}+5$. To this end let d run through all integers $1 \le d \le \sqrt{b}$, and c through the integers $0 \le c \le d$, $(c,d) = 1$. Define the interval

$$I_{c/d} = \{x \in I; |x - b\cdot c/d| \le \sqrt{b}/d^2\}$$

and the union

$$\mathcal{F} = \bigcup_{1\le d\le\sqrt{b}} \ \bigcup_{\substack{0\le c<d,\\ (c,d)=1}} I_{c/d}.$$

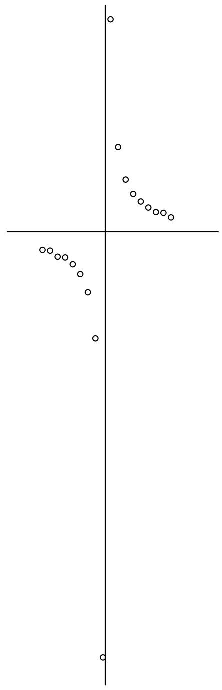
Then for all integers a in $I \setminus \mathcal{F}$ with $(a,b) = 1$, we have $|S(a,b)| \le 3\sqrt{b} + 5$ (see [12]). In other words, the Farey fractions c/d of order $\lfloor\sqrt{b}\rfloor$ determine the intervals $I_{c/d}$ that contain all integers a such that $|S(a,b)|$ is substantially larger than $\sqrt{b}$. Dedekind sums inside and outside $\mathcal{F}$ are illustrated by the diagrams in [12]. The main tools used in the proof of this result are (2.3) and a basic fact about Farey fractions. Note that the number of integers inside $\mathcal{F}$ is $\ll \sqrt{b}\log b$.

The points $(a, S(a,b))$ with a inside the interval $I_{c/d}$ can be described by means of the hyperbola

$$H_{c/d} = \{(x,y); (x - b\cdot c/d)\cdot y = b/d^2\},$$

whose midpoint is $(b\cdot c/d, 0)$. Indeed, for $a \in I_{c/d}$, $a < b\cdot c/d$, $(a, S(a,b))$ lies close to the "negative" branch $\{(x,y) \in H_{c/d}; y < 0\}$ and close to the vertical asymptote $\{(b\cdot c/d, y); y \in \mathbb{R}\}$ of $H_{c/d}$. Conversely, $(a, S(a,b))$ lies close to the "positive" branch $\{(x,y) \in H_{c/d}; y > 0\}$ and close to the same asymptote for $a \in I_{c,d}$, $a > b\cdot c/d$.

In general, the points $(a, S(a,b))$ are less close to the hyperbola if a lies near the endpoints of $I_{c/d}$. These somewhat vague statements have a more precise asymptotic meaning (see [12]).



$b = 6761$:
$(a, S(a,b))$ for a inside $I_{2/3}$,
$S(a,b)$ between ≈ -2251.33 and ≈ 1123.17,
length of $I_{c/d}$: ≈ 18.27

For small values of d the course of the relevant parts of the hyperbola can be seen from the points $(a, S(a,b))$ with $a \in I_{c/d}$ (see the diagram, whose horizontal scale differs from the vertical one). But if $d > \sqrt{2} \cdot b^{1/4}$, the interval $I_{c/d}$ contains at most one integer. Hence for most of the intervals $I_{c/d}$ there is at most one point $(a, S(a,b))$ with $a \in I_{c/d}$ close to the hyperbola $H_{c/d}$.

In order to count large absolute values of Dedekind sums, we consider

$$M_\alpha = \{a; 0 \le a < b; (a,b) = 1, |S(a,b)| > b^\alpha\}$$

for some $\alpha > 0$. One can show

$$|M_\alpha| \ge C_\alpha \varphi(b) \log b / b^\alpha$$

for $1/3 < \alpha < 1$, if $b \to \infty$ (see [13]). This order of magnitude is in some sense best possible. It would be desirable to have an analogous result for $\alpha \le 1/3$, but we only have

$$|M_\alpha| \ge C_\alpha \varphi(b) / b^\alpha$$

for $0 < \alpha \le 1/3$. So the log factor has been lost in this result.

One of the few counting results about *small* values of Dedekind sums that we know is given in [42]. Its proof uses real-analytic modular forms. For $X > 0$, let

$$A(X) = \{(a,b); 0 \le a < b \le X, (a,b) = 1\}.$$

Then for each $\alpha > 0$

$$\lim_{X\to\infty} \frac{|\{(a,b) \in A(X); S(a,b) < \alpha \log b\}|}{|A(X)|} = \frac{1}{\pi} \cdot \arctan\left(\frac{\pi\alpha}{6}\right) + \frac{1}{2}. \tag{6.1}$$

This implies

$$\lim_{X\to\infty} \frac{|\{(a,b) \in A(X); |S(a,b)| < \alpha \log b\}|}{|A(X)|} = \frac{2}{\pi} \cdot \arctan\left(\frac{\pi\alpha}{6}\right). \tag{6.2}$$

From (6.1) one also obtains the following result: Let $M : \mathbb{R} \to \mathbb{R}$ be a function with $M(x) \to \infty$ if $x \to \infty$. Then

$$S(a,b) \le M(b) \log b$$

for almost all a, b, $0 \le a < b$, $(a,b) = 1$ (see the remark following Lemma 1 in [42]).

Computations suggest that a limiting behaviour similar to that of (6.2) takes place for a *fixed* modulus b instead of *all* $b < X$, for instance,

$$\lim_{b\to\infty} \frac{|\{a; 0 \le a < b, (a,b) = 1, |S(a,b)| < \log b\}|}{\varphi(b)} = \frac{2}{\pi} \cdot \arctan\left(\frac{\pi}{6}\right) = 0.307072\ldots$$

But this is far from being proved.

From (4.1) we obtain, in a straightforward manner, the following result. Let $\alpha > 0$. For every $\varepsilon > 0$,

$$\frac{|\{(a,b) \in A(X); |S(a,b)| \le \alpha\}|}{|A(X)|} = \frac{\alpha}{3 \log X} + E(\varepsilon, X)$$

with

$$|E(\varepsilon, X)| \le \varepsilon / \log X,$$

as $X \to \infty$. It would be nice if the E-term could be replaced by a better estimate.

7 Dedekind sums near quadratic irrationals

Let $\alpha \in \mathbb{R}$ be a quadratic irrational. For each $x \in \mathbb{R}$ there are values $S(a,b)$ arbitrarily close to x such that a/b is arbitrarily close to α. This follows from the density of the set $\{(a/b, S(a,b)); a, b \in \mathbb{Z}, b > 0, (a,b) = 1\}$ in $\mathbb{R}^2$ (see Section 4). More interesting is the case when a/b runs through the sequence of *best approximations* of α, i.e., the sequence of *convergents* of the continued fraction expansion of α (see

[36, p. 20 ff.]). In this case it may happen that the Dedekind sums $S(a,b)$ are concentrated near finitely many cluster points (see [14]). We consider only a special case here.

Let α be a quadratic irrational with regular continued fraction expansion $\alpha = [0, \overline{c_1, c_1, \ldots, c_l}]$. So $c_1, \ldots, c_l$ is the repeating block of α. Let the convergents a_k/b_k, $k = 0, 1, 2, \ldots$ be defined as usual, namely,

$$a_{-1} = 1,\ a_0 = c_0,\ b_{-1} = 0,\ b_0 = 1,$$

and

$$a_k = c_k a_{k-1} + a_{k-2},\ b_k = c_k b_{k-1} + b_{k-2}$$

for $k \geq 1$ (observe that $c_{l+1} = c_1, c_{l+2} = c_2$, and so on). The numbers a_k, b_k are integers, $b_k \geq 1$, $(a_k, b_k) = 1$.

Now suppose that l is odd. We put $L = 2l$. Then for each $j \in \{1, 2, \ldots, L\}$, the sequence $S(a_k, b_k)$ converges to

$$\sum_{r=1}^{j} (-1)^{r-1} c_r + \alpha + \begin{cases} 1/\alpha_j - 3, & \text{if } j \text{ is odd;} \\ -1/\alpha_j, & \text{otherwise,} \end{cases}$$

for $k \to \infty$, $k \equiv j \mod L$. Here α_j is the quadratic irrational

$$\alpha_j = [c_j, c_{j-1}, \ldots, c_1, \overline{c_l, c_{l-1}, \ldots, c_1}].$$

This follows from formula (2.2). Accordingly, the sequence $S(a_k, b_k)$ is bounded and has at most L different cluster points when k tends to infinity.

Example. Let $\alpha = [0, \overline{1, 2, 2}] = (\sqrt{85} - 5)/6 = 0.703257\ldots$ In this case the six cluster points are $(\sqrt{85} - 11)/3 = -0.5934851\ldots$, -2/3, $(4\sqrt{85} - 50)/15 = -0.8747881\ldots$, 0, $\sqrt{85}/3 - 3 = 0.0731848\ldots$, $(\sqrt{85} - 5)/15 = 0.2813029\ldots$ The Dedekind sums $S(a_k, b_k)$, $k = 7, \ldots, 12$, are already quite close to these cluster points.

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Kurt Girstmair
Institut für Mathematik,
Universität Innsbruck,
Technikerstr. 13/7
A-6020 Innsbruck, Austria.
e-mail: Kurt.Girstmair@uibk.ac.at

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