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LOCAL CONVERGENCE OF SOME HIGH ORDER ITERATIVE METHODS BASED ON THE DECOMPOSITION TECHNIQUE USING ONLY THE FIRST DERIVATIVE

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Abstract. We present a local convergence analysis of some high order iterative methods based on the decomposition technique using only the first derivative for solving equations in order to approximate a solution of a nonlinear equation. In earlier studies hypotheses on the higher derivatives are used. Thus by using only first derivative, we extended the applicability of these methods. Moreover the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented in this study.

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear function, D is a convex subset of $\mathbb{R}$. Newton-like methods are used for finding solutions of equation (1.1). These methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1, 2, 24, 25, 32].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's require the evaluation of the second derivative F'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint methods [24]. In this paper we study the one-step method defined for each $n =$

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$0, 1, 2, \cdots$, by

$$x_{n+1} = x_n - (F'(x_n) + F(x_n)H(x_n))^{-1}F(x_n), \tag{1.2}$$

the two-step method

$$\begin{aligned} y_n &= x_n - (F'(x_n) + F(x_n)H(x_n))^{-1}F(x_n), \\ x_{n+1} &= y_n - (F'(y_n) + F(y_n)H(y_n))^{-1}F(y_n) \end{aligned} \tag{1.3}$$

and the three-step method

$$\begin{aligned} y_n &= x_n - (F'(x_n) + F(x_n)H(x_n))^{-1}F(x_n), \\ z_n &= y_n - (F'(y_n) + F(y_n)H(y_n))^{-1}F(y_n), \\ x_{n+1} &= z_n - (F'(z_n) + F(z_n)H(z_n))^{-1}F(z_n), \end{aligned} \tag{1.4}$$

where x_0 is an initial point, $G : D \subseteq \mathbb{R} \to \mathbb{R}$ a continuously differentiable function and $H(x) = \frac{G'(x)}{G(x)}$ for each $x \in D$. These methods were derived by using the decomposition technique in [16, 21, 22, 30, 31]. The efficiency, motivation and the advantages of these methods over other competing methods were also given in these references. Several choices of the function G are possible. For example, let $G(x) = e^{-\alpha x}$. Then, methods (1.2)–(1.4) reduce, respectively to the first-step method

$$x_{n+1} = x_n - (F'(x_n) - \alpha F(x_n))^{-1}F(x_n), \tag{1.5}$$

the two-step method

$$\begin{aligned} y_n &= x_n - (F'(x_n) - \alpha F(x_n))^{-1}F(x_n), \\ x_{n+1} &= y_n - (F'(y_n) - \alpha F(y_n))^{-1}F(y_n) \end{aligned} \tag{1.6}$$

and the three-step method

$$\begin{aligned} y_n &= x_n - (F'(x_n) - \alpha F(x_n))^{-1}F(x_n), \\ z_n &= y_n - (F'(y_n) - \alpha F(y_n))^{-1}F(y_n), \\ x_{n+1} &= z_n - (F'(z_n) - \alpha F(z_n))^{-1}F(z_n). \end{aligned} \tag{1.7}$$

For $\alpha = 0$, method (1.5) merges to method obtained by He et al. [16] and Noor et al. [30]. If $\alpha = 0$ in method (1.6), then we obtain the method given by Traub in [32] and Noor et al. in [21, 22, 31]. Moreover, if we set $\alpha = 0, \frac{1}{2}, 1$, we obtain other well known methods for solving nonlinear equations [3, 4, 24, 25, 32]. Furthermore, many other choices of function G are possible. Notice that in particular the eight order of convergence for method (1.4) was shown using Taylor expansions and hypotheses reaching up to the sixth derivative of function F and the third derivative of function

G although only the first derivative of these functions appear in the definition of these methods. Therefore, these hypotheses restrict the applicability of these methods.

As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function f''' is unbounded on D.

Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ [24]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for methods (1.2)-(1.4) in Section 2. The same technique can be used to other methods.

In the present paper we only use hypotheses on the first derivative. This way we expand the applicability of method (1.2).

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (1.2)-(1.4). The numerical examples are presented in the concluding Section 3.

2 Local convergence analysis

We present the local convergence analysis of method (1.4), method (1.3) and method (1.2) in this section, respectively. Let $L_0 > 0, L > 0$ and $M \geq 1$ be given parameters and let $\varphi : [0, \frac{1}{L_0}] \to (0, +\infty)$ be a continuous and nondecreasing function. It is convenient for the local convergence analysis of the method (1.4) that follows to define some functions and parameters. Define function p_1 and h_{p_1} on the interval $[0, \frac{1}{L_0})$ by

$$p_1(t) = (L_0 + M\varphi(t))t,$$

and

$$h_{p_1}(t) = p_1(t) - 1.$$

Then, we have that $h_{p_1}(0) = -1 < 0$ and $h_{p_1}(\frac{1}{L_0}) = \frac{M}{L_0}\varphi(\frac{1}{L_0}) \geq 0$. It follows from the intermediate value theorem that function h_{p_1} has zeros in the interval $(0, \frac{1}{L_0})$.

Denote by r_{p_1} the smallest such zero. Define functions g_1 and h_1 on the interval $[0, r_{p_1})$ by

$$g_1(t) = \frac{1}{2(1-L_0 t)}[1 + \frac{2M^2\varphi(t)}{1-p_1(t)}]t,$$

and

$$h_1(t) = g_1(t) - 1.$$

Then, we have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_1 the smallest zero of functions h_1 on the interval $(0, r_{p_1})$. Define functions p_2 and h_{p_2} on the interval $[0, r_{p_1})$ by

$$p_2(t) = [L_0 + M\varphi(t)]g_1(t)t,$$

and

$$h_{p_2}(t) = p_2(t) - 1.$$

Then, we get that $h_{p_2}(0) = -1 < 0$ and $h_{p_2}(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_{p_2} the smallest zero of functions h_{p_2} on the interval $(0, r_{p_1})$. Define functions g_2 and h_2 on the interval $[0, r_{p_2})$ by

$$g_2(t) = (1 + \frac{M}{1-p_2(t)})g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, r_{p_1})$. Define functions p_3 and h_{p_3} on the interval $[0, r_{p_2})$ by

$$p_3(t) = [L_0 + M\varphi(t)]g_2(t)t$$

$$h_{p_3}(t) = p_3(t) - 1.$$

Then, we have that $h_{p_3}(0) = -1 < 0$ and $h_{p_3}(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_{p_3} the smallest zero of functions h_{p_3} on the interval $(0, r_{p_2})$. Finally, define functions g_3 and h_3 on the interval $[0, r_{p_3})$ by

$$g_3(t) = (1 + \frac{M}{1-p_3(t)})g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, we have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_{p_3}^-$. Denote by r_3 the smallest zero of functions h_3 on the interval $(0, r_{p_3})$. Set

$$r = \min\{r_1, r_2, r_3\}. \tag{2.1}$$

Then, we have that for each $t \in [0, r)$

$$0 \le p_1(t) < 1 \tag{2.2}$$

$$0 \le g_1(t) < 1, \tag{2.3}$$

$$0 \le p_2(t) < 1, \tag{2.4}$$

$$0 \le g_2(t) < 1, \tag{2.5}$$

$$0 \le p_3(t) < 1 \tag{2.6}$$

and

$$0 \le g_3(t) < 1. \tag{2.7}$$

Denote by $U(v, \rho), \bar{U}(v, \rho)$ the open and closed balls in $\mathbb{R}$, respectively, with center $v \in \mathbb{R}$ and of radius $\rho > 0$. Next,we present the local convergence analysis of method (1.4) using the preceding notation.

THEOREM 2.1. *Let $F, G : D \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there exist $x^* \in D$ parameters $L_0 > 0, L > 0, M \geq 1$ and a continuous and nondecreasing function $\varphi : [0, \frac{1}{L_0})$ such that for $H(x) = \frac{G'(x)}{G(x)}$ and for each $x,\ y \in D$*

$$F(x^*) = 0,\ \ F'(x^*) \neq 0,\ G(x) \neq 0,\ \ if\ \ F \neq G, \tag{2.8}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \le L_0|x - x^*|, \tag{2.9}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \le L|x - y|, \tag{2.10}$$

$$|F'(x^*)^{-1}F'(x)| \le M, \tag{2.11}$$

$$|H(x)| \le \varphi(|x - x^*|) \tag{2.12}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.13}$$

where the radius r is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.4) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold

$$|y_n - x^*| \le g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{2.14}$$

$$|z_n - x^*| \le g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| \tag{2.15}$$

and

$$|x_{n+1} - x^*| \le g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.16}$$

where the $''g''$ functions are defined above Theorem 2.1. Furthermore, if there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. We shall show estimates (2.14)–(2.16) using mathematical induction. Using (2.1), (2.9) and the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$ we have that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{2.17}$$

It follows from (2.17) and the Banach Lemma on invertible functions [3, 4, 25, 32] that $F'(x_0)^{-1} \in L(\mathbb{R}, \mathbb{R})$ and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0 r}. \tag{2.18}$$

Next, we shall show that $F'(x_0) + F(x_0)H(x_0) \neq 0$. We can write by (2.12) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.19}$$

Notice that $|x^* + \theta(y_0 - x^*) - x^*| = \theta|y_0 - x^*| < r$. Hence, $x^* + \theta(y_0 - x^*) \in U(x^*, r)$. Then, using (2.11) and (2.19), we get that

$$\begin{aligned} |F'(x^*)^{-1}F(x_0)| &= |\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta| \\ &\leq M|x_0 - x^*|. \end{aligned} \tag{2.20}$$

In view of (2.1), (2.2), (2.17) and (2.20), we have in turn that

$$\begin{aligned} & |F'(x^*)^{-1}[F'(x_0) - F'(x^*) + F(x_0)H(x_0)]| \\ \leq\ & |F'(x^*)^{-1}(F('x_0) - F'(x^*)^{-1}| \\ & +|F'(x^*)^{-1}F(x_0)||H(x_0)| \\ \leq\ & L_0|x_0 - x^*| + M\varphi(|x_0 - x^*|)|x_0 - x^*| \\ =\ & (L_0 + M\varphi(|x_0 - x^*|)) = p_1(|x_0 - x^*|) < 1. \end{aligned} \tag{2.21}$$

It follows from (2.21) that

$$|(F'(x_0) + F(x_0)H(x_0))^{-1}F'(x^*)| \leq \frac{1}{1 - p_1(|x_0 - x^*|)}. \tag{2.22}$$

Hence, y_0 is well defined by the first substep of method (1.4) for $n = 0$, Using first substep of method (1.4) for $n = 0$, (2.1), (2.3), (2.10), (2.18), (2.20) and (2.22) we obtain from

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &+F'(x_0)^{-1}(F'(x_0) + F(x_0)H(x_0) - F'(x_0))(F'(x_0) + F(x_0)H(x_0))^{-1}F(x_0) \end{aligned}$$

in turn that

$$\begin{aligned}
|y_0 - x^*| &\le |F'(x_0)^{-1}F'(x^*)| \\
&\quad \times |F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x^*)]d\theta||x_0 - x^*| \\
&\quad + |F'(x_0)^{-1}F'(x^*)||(F'(x_0) + F(x_0)H(x_0))^{-1}F'(x^*)||F'(x^*)^{-1}F'(x_0)|^2|H(x_0)| \\
&\le \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{M^2\varphi(|x_0 - x^*|)|x_0 - x^*|^2}{(1 - L_0|x_0 - x^*|)(1 - p_1(|x_0 - x^*|))} \\
&= g_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \qquad (2.23)
\end{aligned}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*, r)$. As in (2.22), we show that

$$|(F'(y_0) + F(y_0)H(x_0))^{-1}F'(x^*)| \le \frac{1}{1 - p_2(|x_0 - x^*|)}. \qquad (2.24)$$

Hence, z_0 is well defined by the second substep of method (1.4) for $n = 0$, Using the second substep of method (1.4) for $n = 0$, (2.1), (2.5), (2.23) and (2.24) we get in turn that

$$\begin{aligned}
|z_0 - x^*| &\le |y_0 - x^*| + |(F'(y_0) + F(y_0)H(x_0))^{-1}F'(x^*)||F'(x^*)^{-1}F(y_0)| \\
&\le |y_0 - x^*| + \frac{M|y_0 - x^*|}{1 - p_2(|x_0 - x^*|)} \\
&= [1 + \frac{M}{1 - p_2(|x_0 - x^*|)}|y_0 - x^*| \\
&= [1 + \frac{M}{1 - p_2(|x_0 - x^*|)}g_1(|x_0 - x^*|)|x_0 - x^*| \\
&= g_2(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \qquad (2.25)
\end{aligned}$$

which shows (2.15) for $n = 0$ and $z_0 \in U(x^*, r)$. As in (2.22) (for $z_0 = x_0$), we get that

$$|(F'(z_0) + F(z_0)H(x_0))^{-1}F'(x^*)| \le \frac{1}{1 - p_3(|x_0 - x^*|)}. \qquad (2.26)$$

Hence, x_1 is well defined by the last substep of method (1.4) for $n = 0$. Then, using (2.1), (2.6), (2.7), (2.20) for $z_0 = x_0$, (2.25) and (2.26) we get in turn that

$$\begin{aligned}
|x_1 - x^*| &\le |z_0 - x^*| + |(F'(z_0) + F(z_0)H(z_0))^{-1}F'(x^*)| \\
&\quad \times |F'(x^*)^{-1}F(x_0)| \\
&\le |z_0 - x^*| + \frac{M|z_0 - x^*|}{1 - p_3(|x_0 - x^*|)} \\
&= (1 + \frac{M}{1 - p_3(|x_0 - x^*|}|z_0 - x^*| \\
&= g_3(|x_0 - x^*|)|x_0 - x^*| < M|x_0 - x^*| < r,
\end{aligned}$$

which shows (2.16) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.14)–(2.16). Using the estimate $|x_{k+1} - x^*| < |x_k - x^*| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.9) we get that

$$\begin{aligned} |F'(x^*)^{-1}(Q - F'(x^*))| &\le \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\le \int_0^1 (1-\theta)|x^* - y^*|d\theta \le \frac{L_0}{2}R < 1. \end{aligned} \tag{2.27}$$

It follows from (2.27) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$.

□

REMARK 2.2. 1. *In view of (2.9) and the estimate*

$$\begin{aligned} |F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\ &\le 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \le 1 + L_0|x - x^*| \end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [19]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}\right) / \ln\left(\frac{|x_n-x_{n-1}|}{|x_{n-1}-x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

THEOREM 2.3. *Let $F, G : D \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable functions. Suppose that there exist $x^* \in D$ parameters $L_0 > 0, L > 0, M \geq 1$ and a continuous and nondecreasing functio $\varphi : [0, \frac{1}{L_0})$ such that for $H(x) = \frac{G'(x)}{G(x)}$ and (2.8)–(2.13) hold. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.3) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold*

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{2.28}$$

and

$$|x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.29}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. Simply delete the work for (2.16) in the proof of Theorem 2.1 and define the radius of convergence r by $r = \min\{r_1, r_2\}$.

□

THEOREM 2.4. *Let $F, G : D \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable functions. Suppose that there exist $x^* \in D$ parameters $L_0 > 0, L > 0, M \geq 1$ and a continuous and nondecreasing function $\varphi : [0, \frac{1}{L_0})$ such that for $H(x) = \frac{G'(x)}{G(x)}$ and (2.8)–(2.13) hold. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimate holds*

$$|x_{n+1} - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.30}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. Simply delete the work for (2.15) and (2.16) in the proof of Theorem 2.1 and define the radius of convergence r by $r = r_1$.

□

3 Numerical Examples

We present numerical examples for method (1.4) in this section. We use $G(x) = e^{-\alpha x}$, $\varphi(t) = |\alpha|$ and $\alpha = \frac{1}{2}$ in all the examples below.

EXAMPLE 3.1. *Let $D = [-\infty, +\infty]$. Define function f of D by*

$$f(x) = \sin(x). \tag{3.1}$$

Then we have for $x^* = 0$ that $L_0 = L = M = 1$. The parameters are

$$r_A = 0.6667,\ r_{p_1} = 0.6667,\ r_1 = 0.3772,\ r_{p_2} = 0.4377,$$

$$r_2 = 0.3446,\ r_{p_3} = 0.0591,\ r_3 = 0.3446.$$

EXAMPLE 3.2. *Let $D = [-1, 1]$. Define function f of D by*

$$f(x) = e^x - 1. \tag{3.2}$$

Using (3.2) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = e$. The parameters are

$$r_A = 0.3249,\ r_{p_1} = 0.3679,\ r_1 = 0.1690,\ r_{p_2} = 0.2118,$$

$$r_2 = 0.1326,\ r_{p_3} = 0.0363,\ r_3 = 0.0007.$$

EXAMPLE 3.3. *Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 101.5578008$. The parameters are*

$$r_A = 0.0045,\ r_{p_1} = 0.0068,\ r_1 = 0.0060,\ r_{p_2} = 0.0061,$$

$$r_2 = 0.0001,\ r_{p_3} = 0.00001,\ r_3 = 0.00001.$$

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