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A NONCOMMUTATIVE CONVEXITY IN C^*-BIMODULES

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. We consider a noncommutative convexity in Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodules, called $\mathscr{A}$-$\mathscr{B}$-convexity, as a generalization of C^*-convexity in C^*-algebras. We show that if $\mathcal{X}$ is a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule, then $\mathcal{M}_n(\mathcal{X})$ is a Hilbert $\mathcal{M}_n(\mathscr{A})$-$\mathcal{M}_n(\mathscr{B})$-bimodule and apply it to show that the closed unit ball of every Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule is $\mathscr{A}$-$\mathscr{B}$-convex. Some properties of this kind of convexity and various examples have been given.

1 Introduction and preliminaries

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras. Let $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathscr{A}})$ be a left Hilbert $\mathscr{A}$-module and $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathscr{B}})$ be a right Hilbert $\mathscr{B}$-module satisfying

$$\langle x, y\rangle_{\mathscr{A}} z = x\langle y, z\rangle_{\mathscr{B}} \qquad (x, y, z \in \mathcal{X}).$$

Then $\mathcal{X}$ is called Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. It is known that every C^*-algebra $\mathscr{A}$ is a Hilbert $\mathscr{A}$-$\mathscr{A}$-bimodule via the bimodule structure given by the multiplication in $\mathscr{A}$ and the inner products $\langle a, b\rangle = ab^*$ and $\langle a, b\rangle = a^*b$. Particularity, if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\mathbb{B}(\mathcal{K}, \mathcal{H})$ is the Banach algebra of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$, then $\mathbb{B}(\mathcal{K}, \mathcal{H})$ is a Hilbert $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-bimodule with the following inner products:

$$\langle S, T\rangle_{\mathbb{B}(\mathcal{H})} = ST^*.$$
$$\langle S, T\rangle_{\mathbb{B}(\mathcal{K})} = S^*T.$$

We recall that every Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathcal{X}$ satisfies

$$\langle xb, xb\rangle_{\mathscr{A}} \le \|b\|^2\langle x, x\rangle_{\mathscr{A}}, \quad \langle ax, ax\rangle_{\mathscr{B}} \le \|a\|^2\langle x, x\rangle_{\mathscr{B}}. \tag{1.1}$$
$$\langle xb, y\rangle_{\mathscr{A}} = \langle x, yb^*\rangle_{\mathscr{A}}, \quad \langle ax, y\rangle_{\mathscr{B}} = \langle x, a^*y\rangle_{\mathscr{B}}. \tag{1.2}$$
$$\|axb\| \le \|a\|\|x\|\|b\| \tag{1.3}$$

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for all $a \in \mathscr{A}$, $b \in \mathscr{B}$ and all $x, y \in \mathcal{X}$ (cf. [7, 15]).

For a full description of Hilbert bimodules, see for example [7, 15] and the references therein.

1.1 C^*-convexity

Let $\mathscr{A}$ be a unital C^*-algebra with unit $1_{\mathscr{A}}$. For $a_1, \cdots, a_n \in \mathscr{A}$ with $\sum_{i=1}^n a_i^* a_i = 1_{\mathscr{A}}$, the sum $\sum_{i=1}^n a_i^* x_i a_i$ is called a C^*-convex combination of $\{x_1, \cdots, x_n\} \subseteq \mathscr{A}$, with coefficients $a_1, \cdots, a_n$. A subset $\mathcal{S}$ of $\mathscr{A}$ is called C^*-convex if it is closed under C^*-convex combinations of its elements. It means that

$$\sum_{i=1}^n a_i^* x_i a_i \in \mathcal{S}$$

for all $x_1, \cdots, x_n \in \mathcal{S}$ and all $a_1, \cdots, a_n \in \mathscr{A}$ with $\sum_{i=1}^n a_i^* a_i = 1_{\mathscr{A}}$.

This notion of convexity, called the C^*-convexity, has been introduced by Loebl and Paulsen [10] as a non-commutative generalization of linear convexity. It is known that the sets

(1) $\{T \in \mathbb{B}(\mathcal{H}) : 0 \leq T \leq I_{\mathcal{H}}\}$;

(2) $\{T \in \mathbb{B}(\mathcal{H}); \|T\| \leq M\}$ for a fix scalar $M > 0$;

(3) $\{T \in \mathbb{B}(\mathcal{H}) : \omega(T) \leq r\}$, where $\omega(T)$ is the numerical radius of T

are C^*-convex in the C^*-algebra $\mathbb{B}(\mathcal{H})$ with the identity operator $I_{\mathcal{H}}$. It is evident that the C^*-convexity of a set $\mathcal{S}$ in $\mathscr{A}$, implies its convexity in the usual sense. For if $x, y \in \mathcal{S}$ and $\lambda \in [0, 1]$, then with $a_1 = \sqrt{\lambda} 1_{\mathscr{A}}$ and $a_2 = \sqrt{1-\lambda} 1_{\mathscr{A}}$ we have $a_1^* a_1 + a_2^* a_2 = 1_{\mathscr{A}}$ and

$$\lambda x + (1-\lambda) y = a_1^* x a_1 + a_2^* y a_2 \in \mathcal{S}.$$

But the converse is not true in general. For example, it was shown that [10] if $A \geq 0$, then $[0, A] = \{X \in \mathbb{B}(\mathcal{H}); \quad 0 \leq X \leq A\}$ is convex but not C^*-convex.

Some essential results of convexity theory have been generalized in [3] to C^*-convex sets. Specially, a version of the so-called Hahn-Banach theorem was presented. The operator extension of extreme points, the C^*-extreme points have also been introduced and studied, see [4, 6, 10, 13]. Moreover, Magajna [12, 14] extended the notion of C^*-convexity to operator modules and proved some separation theorems. We refer the reader to [8, 9, 11, 12, 14, 16] for further results concerning C^*-convexity.

In this paper, we consider the notion of $\mathscr{A}$-$\mathscr{B}$-convex sets in Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodules as a generalization of C^*-convex sets in C^*-algebras. We will try to illustrate differences between these notions by giving various examples. Some properties of $\mathscr{A}$-$\mathscr{B}$-convex sets are also presented. In particular, it is shown that the closed unit ball of a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule is $\mathscr{A}$-$\mathscr{B}$-convex.

2 Main results

Throughout this section, suppose that $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras with units $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$, respectively and $\mathbb{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ with the identity operator $I_{\mathcal{H}}$. For given C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{B}(\mathcal{H})$ the notion of "$\mathcal{A}$, $\mathcal{B}$-absolutely convexity" in operator bimodules has been defined and studied in [12]. Similarly, an $\mathscr{A}$-$\mathscr{B}$-convex set in a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule can be defined as follows.

Definition 1. *Let $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. A subset $\mathcal{S}$ of $\mathcal{X}$ is called $\mathscr{A}$-$\mathscr{B}$-convex if*

$$\sum_{i=1}^{n} a_i a_i^* = 1_{\mathscr{A}}, \quad \sum_{i=1}^{n} b_i^* b_i = 1_{\mathscr{B}} \implies \sum_{i=1}^{n} a_i x_i b_i \in \mathcal{S}$$

for all $a_i \in \mathscr{A}$, $b_i \in \mathscr{B}$, $x_i \in \mathcal{S}$ and $n \in \mathbb{N}$.

Remark 2. *Assume that $\mathcal{X}$ is a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule, $\mathcal{S}$ is an $\mathscr{A}$-$\mathscr{B}$-convex subset of $\mathcal{X}$ and $0 \in \mathcal{S}$. Assume that $x_i \in \mathcal{S}$, $a_i \in \mathscr{A}$ and $b_i \in \mathscr{B}$ with $\sum_{i=1}^{k} a_i a_i^* \leq 1_{\mathscr{A}}$ and $\sum_{i=1}^{k} b_i^* b_i \leq 1_{\mathscr{B}}$. Put $c = \sqrt{1_{\mathscr{A}} - \sum_{i=1}^{k} a_i a_i^*}$ and $d = \sqrt{1_{\mathscr{B}} - \sum_{i=1}^{k} b_i^* b_i}$. Then $\sum_{i=1}^{k} a_i a_i^* + cc^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^{k} b_i^* b_i + d^* d = 1_{\mathscr{B}}$. Moreover,*

$$\sum_{i=1}^{k} a_i x_i b_i = \sum_{i=1}^{k} a_i x_i b_i + c0d \in \mathcal{S}.$$

In other words, $\sum_{i=1}^{k} a_i x_i b_i \in \mathcal{S}$ even if $\sum_{i=1}^{k} a_i a_i^* \leq 1_{\mathscr{A}}$ and $\sum_{i=1}^{k} b_i^* b_i \leq 1_{\mathscr{B}}$.

Note that, if r is a positive scalar, then it is easy to see that the set

$$\mathcal{S} := \{T \in \mathbb{B}(\mathcal{H}) : 0 \leq T \leq r\}$$

is C^*-convex, see e.g., [10]. We give some examples in the case of $\mathscr{A}$-$\mathscr{B}$-convexity.

Example 3. *Let Γ be an index set. Define $\mathcal{X}$ to be the set*

$$\mathcal{X} = \left\{ (X_\alpha)_{\alpha\in\Gamma} \;\middle|\; X_\alpha \in \mathbb{B}(\mathcal{H}), \ \sum_{\alpha\in\Gamma} X_\alpha^* X_\alpha \ \text{ converges in } \ \mathbb{B}(\mathcal{H}) \right\}.$$

Define a map $\langle\cdot,\cdot\rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{B}(\mathcal{H})$ by

$$\langle (X_\alpha)_{\alpha\in\Gamma}, (Y_\alpha)_{\alpha\in\Gamma} \rangle = \sum_{\alpha\in\Gamma} X_\alpha^* Y_\alpha.$$

It is not hard to see that $\langle\cdot,\cdot\rangle$ *is well-defined inner product on* $\mathcal{X}$. *Moreover, if* $T\in\mathbb{B}(\mathcal{H})$ *and* $(X_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$, *then*

$$X_\alpha^*T^*TX_\alpha\le\|T\|^2X_\alpha^*X_\alpha.$$

It follows that $\mathcal{X}$ *can be regarded as a* $\mathbb{B}(\mathcal{H})$*-bimodule via the bimodule structure given by*

$$\mathcal{X}\times\mathbb{B}(\mathcal{H})\to\mathcal{X},\qquad (X_\alpha)_{\alpha\in\Gamma}\times T=(X_\alpha T)_{\alpha\in\Gamma}$$

and

$$\mathbb{B}(\mathcal{H})\times\mathcal{X}\to\mathcal{X},\qquad T\times(X_\alpha)_{\alpha\in\Gamma}=(TX_\alpha)_{\alpha\in\Gamma}.$$

Hence, $\mathcal{X}$ *would be a Hilbert* $\mathbb{B}(\mathcal{H})$*-*$\mathbb{B}(\mathcal{H})$*-bimodule.*

Assume that r *is a positive real number. We are going to show that the subset* $\mathcal{S}$ *of* $\mathcal{X}$ *defined by*

$$\mathcal{S}=\{(X_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}\mid 0\le X_\alpha^*X_\alpha\le r,\quad \alpha\in\Gamma\}$$

is $\mathbb{B}(\mathcal{H})$*-*$\mathbb{B}(\mathcal{H})$*-convex.*

Assume that $A_i,B_i\in\mathbb{B}(\mathcal{H})$ *with* $\sum_{i=1}^nA_iA_i^*=I_{\mathcal{H}}=\sum_{i=1}^nB_i^*B_i$. *If*

$$(X_\alpha)_{\alpha\in\Gamma}^i=(X_\alpha^i)_{\alpha\in\Gamma}\in\mathcal{S}\qquad(i=1,\cdots,n),$$

then $0\le\left(X_\alpha^i\right)^*X_\alpha^i\le r$. *Obviously*

$$\left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)^*\left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)\ge0.$$

Moreover, $\left(X_\alpha^i\right)^*X_\alpha^i\le r$ *if and only if* $\frac{1}{\sqrt r}\left(X_\alpha^i\right)^*X_\alpha^i\le\sqrt r$ *if and only if (see e.g., [1, 2, 5])*

$$\begin{pmatrix}\sqrt r & \left(X_\alpha^i\right)^*\\ X_\alpha^i & \sqrt r\end{pmatrix}\ge0,\qquad i=1,\cdots,n.$$

Therefore,

$$\begin{pmatrix}\sqrt r & \left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)^*\\ \sum_{i=1}^nA_iX_\alpha^iB_i & \sqrt r\end{pmatrix}=\sum_{i=1}^n\begin{pmatrix}B_i^* & 0\\ 0 & A_i\end{pmatrix}\begin{pmatrix}\sqrt r & \left(X_\alpha^i\right)^*\\ X_\alpha^i & \sqrt r\end{pmatrix}\begin{pmatrix}B_i & 0\\ 0 & A_i^*\end{pmatrix}\ge0,$$

which implies that $\left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)^*\left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)\le r$. *Hence*

$$\sum_{i=1}^nA_i(X_\alpha)_{\alpha\in\Gamma}^iB_i=\left(\sum_{i=1}^nA_iX_\alpha^iB_i\right)_{\alpha\in\Gamma}\in\mathcal{S},$$

and so $\mathcal{S}$ *is* $\mathbb{B}(\mathcal{H})$*-*$\mathbb{B}(\mathcal{H})$*-convex.*

A similar argument used in Example 3 can be applied to show the following result.

Proposition 4. *Consider $\mathbb{B}(\mathcal{K},\mathcal{H})$ as a Hilbert $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-bimodule. Then for a fixed scalar $r>0$, the set*

$$\mathcal{S} := \{T \in \mathbb{B}(\mathcal{K},\mathcal{H}); \quad 0 \le T^*T \le rI_{\mathcal{K}}\}$$

is $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-convex.

Remark 5. *Let $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. If $\mathcal{S}$ is an $\mathscr{A}$-$\mathscr{B}$-convex subset of $\mathcal{X}$, then it is convex in the usual sense. For if $\lambda_i \in [0,1]$, $(i=1,\dots,n)$, and $\sum_{i=1}^n \lambda_i = 1$, then with $a_i = \sqrt{\lambda_i}1_{\mathscr{A}} \in \mathscr{A}$ and $b_i = \sqrt{\lambda_i}1_{\mathscr{B}} \in \mathscr{B}$ we have*

$$\sum_{i=1}^n a_i a_i^* = \sum_{i=1}^n \lambda_i 1_{\mathscr{A}} = 1_{\mathscr{A}} \quad \text{and} \quad \sum_{i=1}^n b_i^* b_i = \sum_{i=1}^n \lambda_i 1_{\mathscr{B}} = 1_{\mathscr{B}}.$$

Now if $x_i \in \mathcal{S}$ $(i=1,\dots,n)$, then

$$\sum_{i=1}^n \lambda_i x_i = \sum_{i=1}^n a_i x_i b_i \in \mathcal{S},$$

which means that $\mathcal{S}$ is convex.

Remark 6. *Consider the C^*-algebra $\mathscr{A}$ as a Hilbert $\mathscr{A}$-$\mathscr{A}$-bimodule. If a subset $\mathcal{S}$ of $\mathscr{A}$ is $\mathscr{A}$-$\mathscr{A}$-convex, then it is C^*-convex. Assume that $c_1,\dots,c_k \in \mathscr{A}$ with $\sum_{i=1}^k c_i^* c_i = 1_{\mathscr{A}}$. If $x_1,\dots,x_k \in \mathcal{S}$, then the $\mathscr{A}$-$\mathscr{A}$-convexity of $\mathcal{S}$ with $a_i := c_i^*$ and $b_i := c_i$, implies that*

$$\sum_{i=1}^k c_i^* x_i c_i = \sum_{i=1}^k a_i x_i b_i \in \mathcal{S}.$$

Therefore, it seems that the concept of $\mathscr{A}$-$\mathscr{B}$-convexity is stronger than C^*-convexity. The next example reveals this fact.

Example 7. (1) *Consider $\mathcal{M}_2(\mathbb{C})$ as a Hilbert $\mathcal{M}_2(\mathbb{C})$-$\mathcal{M}_2(\mathbb{C})$-bimodule. Let α be a fixed scalar and I be the identity matrix. It is clear that the set $\mathcal{S} = \{\alpha I\}$ is a C^*-convex subset of $\mathcal{M}_2(\mathbb{C})$. However, it is not $\mathcal{M}_2(\mathbb{C})$-$\mathcal{M}_2(\mathbb{C})$-convex. Put*

$$A = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}.$$

Then $AA^* = I = B^*B$, while $A(\alpha I)B = \alpha AB \notin \mathcal{S}$.
(2) Consider $\mathbb{B}(\mathcal{H})$ as a Hilbert $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{H})$-bimodule. The subsets

$$\mathcal{S}_1 = \{T \in \mathbb{B}(\mathcal{H}) : T^* = T\} \quad \text{and} \quad \mathcal{S}_2 = \{T \in \mathbb{B}(\mathcal{H}) : 0 \le T \le I_{\mathcal{H}}\}$$

are C^*-convex subsets of the C^*-algebra $\mathbb{B}(\mathcal{H})$. Let $A, B \in \mathbb{B}(\mathcal{H})$ with $AA^* = I_{\mathcal{H}} = B^*B$ and put $T = I_{\mathcal{H}} \in \mathcal{S}_1 \cap \mathcal{S}_2$. Since $AB = ATB$ is not hermitian at all, we conclude that $AB \notin \mathcal{S}_1$ and $AB \notin \mathcal{S}_2$. It follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ are not $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{H})$-convex.

Example 8. *Let $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. Then the subset*

$$\mathcal{S} := \{x \in \mathcal{X} : \langle x, x\rangle_{\mathscr{A}} \leq r^2 1_{\mathscr{A}}, \text{ for some positive real number } r \neq 1\}$$

of $\mathcal{X}$ is $\mathscr{A}$-$\mathscr{B}$-convex.

Proof. Let $a_i \in \mathscr{A}$ and $b_i \in \mathscr{B}$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^n a_i a_i^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^n b_i^* b_i = 1_{\mathscr{B}}$. We have

$$0 \leq a_i a_i^* \leq \sum_{i=1}^n a_i a_i^* = 1_{\mathscr{A}}, \qquad 0 \leq b_i^* b_i \leq \sum_{i=1}^n b_i^* b_i = 1_{\mathscr{B}}.$$

It follows that $\|b_i\| \leq 1$. If $x_i \in \mathcal{S}$ $(i = 1, \ldots, n)$, then (1.1)implies that

$$\begin{aligned} \langle a_i x_i b_i, a_i x_i b_i\rangle_{\mathscr{A}} &\leq \|b_i\|^2 \langle a_i x_i, a_i x_i\rangle_{\mathscr{A}} \\ &\leq a_i \langle x_i, x_i\rangle_{\mathscr{A}} a_i^* \\ &\leq r^2 a_i a_i^* \\ &\leq r^2 1_{\mathscr{A}}, \quad (1 \leq i \leq n). \end{aligned}$$

Then $a_i x_i b_i \in \mathcal{S}$ for all $i = 1, \ldots, n$. Moreover, if $x, y \in \mathcal{S}$, then there exist positive real numbers $r \neq 1$ and $s \neq 1$ such that $\langle x, x\rangle \leq r^2 1_{\mathscr{A}}$ and $\langle y, y\rangle \leq s^2 1_{\mathscr{A}}$. In a C^*-algebra $\mathscr{A}$ we have

$$(Rea)^2 + (Ima)^2 = \frac{a^*a + aa^*}{2}, \qquad (a \in \mathscr{A}).$$

Therefore

$$0 \leq 2\big(Re\langle y, x\rangle\big)^2 \leq \langle x, y\rangle\langle y, x\rangle + \langle y, x\rangle\langle x, y\rangle.$$

It follows that

$$2\|Re(\langle y, x\rangle)\|^2 \leq \|\langle y, x\rangle\|^2 + \|\langle x, y\rangle\|^2 \leq 2\|x\|^2\|y\|^2 \leq 2r^2 s^2.$$

Hence

$$Re(\langle y, x\rangle) \leq \|Re(\langle y, x\rangle)\| 1_{\mathscr{A}} \leq rs.$$

Consequently

$$\begin{aligned} \langle x + y, x + y\rangle &= \langle x, x\rangle + \langle y, y\rangle + 2Re(\langle y, x\rangle) \\ &\leq (r^2 + s^2 + 2rs) 1_{\mathscr{A}} \\ &= (r + s)^2 1_{\mathscr{A}}. \end{aligned}$$

It follows that $x + y \in \mathcal{S}$ and so $\sum_{i=1}^n a_i x_i b_i \in \mathcal{S}$. □

Many properties of a topological vector space, like locally boundedness, locally compactness and locally convexity come from the structure of the neighborhoods of its origin, the zero vector. In a normed space, the unit ball plays this role. We know that the unit ball of every normed space is convex. More generally, the unit ball of $\mathbb{B}(\mathcal{H})$ is C^*-convex [10]. The next theorems show that more generally, the closed unit ball of every Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule is $\mathscr{A}$-$\mathscr{B}$-convex.

Theorem 9. *Let $\mathscr{A}$ and $\mathscr{B}$ be commutative C^*-algebras and let $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. Then the closed unit ball of $\mathcal{X}$ is $\mathscr{A}$-$\mathscr{B}$-convex.*

Proof. Suppose that $\varphi : \mathscr{A} \to C(T)$ and $\psi : \mathscr{B} \to C(S)$ are the Gelfand representations of $\mathscr{A}$ and $\mathscr{B}$, respectively, where S, T are compact Hausdorff spaces. Let $a_i \in \mathscr{A}$ and $b_i \in \mathscr{B}$ ($i = 1, \cdots, n$) such that

$$\sum_{i=1}^{n} a_i a_i^* = 1_{\mathscr{A}}, \qquad \sum_{i=1}^{n} b_i^* b_i = 1_{\mathscr{B}}.$$

It follows from the Gelfand representation theorem that $\sum_{i=1}^{n} |\varphi(a_i)(t)|^2 = 1$ $(t \in T)$ and $\sum_{i=1}^{n} |\psi(b_i)(s)|^2 = 1$ $(s \in S)$. Let $\mathcal{S} = \{x \in \mathcal{X} : \|x\| \leq 1\}$ and $x_i \in \mathcal{S}$ $(i = 1, \cdots, n)$. Then we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} a_i x_i b_i \right\| &\leq \sum_{i=1}^{n} \|a_i x_i b_i\| \\
&\leq \sum_{i=1}^{n} \|a_i\| \, \|x_i\| \, \|b_i\| \qquad (\text{ by } (1.3)\ ) \\
&\leq \sum_{i=1}^{n} \|a_i\| \, \|b_i\| \\
&= \sum_{i=1}^{n} \|\varphi(a_i)\| \, \|\psi(b_i)\| \qquad (\text{by the Gelfand representation theorem}) \\
&\leq \left(\sum_{i=1}^{n} \|\phi(a_i)\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|\psi(b_i)\|^2 \right)^{\frac{1}{2}} \quad (\text{by the Cauchy-Schwarz inequality}) \\
&\leq \left(\sup_{t \in T} \sum_{i=1}^{n} |\phi(a_i)(t)|^2 \right)^{\frac{1}{2}} \left(\sup_{s \in S} \sum_{i=1}^{n} |\psi(b_i)(s)|^2 \right)^{\frac{1}{2}} = 1.
\end{aligned}$$

Therefore $\mathcal{S}$ is $\mathscr{A}$-$\mathscr{B}$-convex. □

More generally, the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ need not to be commutative. We prove this fact using a different argument.

Theorem 10. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule. If M is a positive scalar, then $\mathcal{S} = \{x \in \mathcal{X}, \quad \|x\| \leq M\}$ is $\mathscr{A}$-$\mathscr{B}$-convex. In particular, the closed unit ball of $\mathcal{X}$ is $\mathscr{A}$-$\mathscr{B}$-convex.*

Proof. Assume that $\mathcal{M}_n(\mathscr{A})$ and $\mathcal{M}_n(\mathscr{B})$ are the matrix C^*-algebras whose elements are $n \times n$ matrices with entries in $\mathscr{A}$ and $\mathscr{B}$, respectively. Put

$$\mathcal{M}_n(\mathcal{X}) = \{[x_{ij}];\ x_{ij} \in \mathcal{X},\ 1 \leq i, j \leq n\}.$$

Then $\mathcal{M}_n(\mathcal{X})$ is a $\mathcal{M}_n(\mathscr{A})$-$\mathcal{M}_n(\mathscr{B})$-bimodule with respect to the following module operations:

$$\begin{aligned}
\centerdot : \mathcal{M}_n(\mathscr{A}) \times \mathcal{M}_n(\mathcal{X}) &\to \mathcal{M}_n(\mathcal{X}) \\
\big([a_{ij}], [x_{ij}]\big) &\mapsto \left[\sum_{k=1}^n a_{ik} x_{kj}\right], \\
\centerdot : \mathcal{M}_n(\mathcal{X}) \times \mathcal{M}_n(\mathscr{B}) &\to \mathcal{M}_n(\mathcal{X}) \\
([x_{ij}], [b_{ij}]) &\mapsto \left[\sum_{k=1}^n x_{ik} b_{kj}\right],
\end{aligned}$$

and the inner products on $\mathcal{M}_n(\mathcal{X})$ defined by

$$\begin{aligned}
\mathcal{M}_n(\mathcal{X}) \times \mathcal{M}_n(\mathcal{X}) &\to \mathcal{M}_n(\mathscr{A}) \ \big(\mathcal{M}_n(\mathscr{B})\big) \\
\big\langle [x_{ij}], [y_{ij}] \big\rangle &\mapsto \left[\sum_{k=1}^n \langle x_{ik}, y_{kj} \rangle_{\mathscr{A}}\right] \ \left(\left[\sum_{k=1}^n \langle x_{ik}, y_{kj} \rangle_{\mathscr{B}}\right]\right).
\end{aligned}$$

Assume that $x_1, \dots, x_n \in \mathcal{S}$. Let $a_i \in \mathscr{A}, b_i \in \mathscr{B}$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^n a_i a_i^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^n b_i^* b_i = 1_{\mathscr{B}}$. Put

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}.$$

Then $A \in \mathcal{M}_n(\mathscr{A}), B \in \mathcal{M}_n(\mathscr{B})$ and $X \in \mathcal{M}_n(\mathcal{X})$. Moreover,

$$|||A||| = |||A^*||| = |||A^*A|||^{\frac{1}{2}} = |||AA^*|||^{\frac{1}{2}}$$

and

$$|||B||| = |||B^*||| = |||B^*B|||^{\frac{1}{2}} = |||BB^*|||^{\frac{1}{2}}$$

and

$$|||X||| = |||\langle X,X\rangle|||^{\frac{1}{2}} = \left|\left|\left|\begin{pmatrix} \|x_1\|^2 & 0 & \dots & 0 \\ 0 & \|x_2\|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \|x_n\|^2 \end{pmatrix}\right|\right|\right|^{\frac{1}{2}} \le M.$$

It follows from using (1.3) in the $\mathcal{M}_n(\mathcal{X})$ that

$$\begin{aligned} \left\|\sum_{i=1}^n a_i x_i b_i\right\| &= \left|\left|\left|\begin{pmatrix} \sum_{i=1}^n a_i x_i b_i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\right|\right|\right| \\ &= |||AXB||| \le |||A|||\cdot|||X|||\cdot|||B||| \\ &\le M|||AA^*|||^{\frac{1}{2}}|||B^*B|||^{\frac{1}{2}} \\ &= \left|\left|\left|\begin{pmatrix} \sum_{i=1}^n a_i a_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\right|\right|\right|^{\frac{1}{2}} \cdot \left|\left|\left|\begin{pmatrix} \sum_{i=1}^n b_i^* b_i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\right|\right|\right|^{\frac{1}{2}} \\ &= \left\|\sum_{i=1}^n a_i a_i^*\right\| \cdot \left\|\sum_{i=1}^n b_i^* b_i\right\| \\ &\le M. \end{aligned}$$

□

Corollary 11. *Consider $\mathbb{B}(\mathcal{K},\mathcal{H})$ as a Hilbert $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-bimodule. If M is a positive scalar, then the set $\mathcal{S} = \{T \in \mathbb{B}(\mathcal{K},\mathcal{H}),\ \ \|T\| \le M\}$ is $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-convex. In particular, the closed unit ball of $\mathbb{B}(\mathcal{K},\mathcal{H})$ is $\mathbb{B}(\mathcal{H})$-$\mathbb{B}(\mathcal{K})$-convex.*

Remark 12. *It should be remarked that our mean by the closed unit ball of $\mathcal{X}$ in Theorem 9 and 10 is the closed unit ball of $\mathcal{X}$ with respect to the norm induced by the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$. In other words, the closed unit ball of a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule with respect to an arbitrary norm need not to be $\mathscr{A}$-$\mathscr{B}$-convex. Too see this, let $\mathcal{M}_n(\mathbb{C})$ be the algebra of all $n\times n$ matrices with complex entries. For $A \in \mathcal{M}_n(\mathbb{C})$, let $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A)$ be the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$. Our mean by the spectral norm $\|\cdot\|_\infty$ is the norm on $\mathcal{M}_n(\mathbb{C})$ defined by $\|A\|_\infty = s_1(A)$, while the trace norm is defined on $\mathcal{M}_n(\mathbb{C})$ by $\|A\|_1 = \mathrm{Tr}(|A|)$. Consider $\mathcal{M}_n(\mathbb{C})$ as a Hilbert $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-bimodule. The closed unit ball of the trace norm, say $\mathcal{B} = \{X \in \mathcal{M}_n(\mathbb{C}) :\ \ \|X\|_1 \le 1\}$ is not $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-convex. Indeed, if*

$$P = X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad Q = Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then P and Q are projections, $P+Q=I$ and $\|PXP\|_1 = \|QYQ\|_1 = 1$. However, $\|PXP+QYQ\|_1 = 2$ and so $PXP+QYQ \notin \mathcal{B}$. This shows that $\mathcal{B}$ is not $\mathcal{M}_2(\mathbb{C})$-$\mathcal{M}_2(\mathbb{C})$-convex.

Note that Theorem 10 guarantees the $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-convexity of the closed unit ball of the spectral norm $\|\cdot\|_\infty$. More generally, the set

$$\mathcal{S} := \left\{X \in \mathcal{M}_n(\mathbb{C}) : \begin{pmatrix} S & X \\ X^* & T \end{pmatrix} \geq 0,\ \exists S,T : 0 \leq S \leq I,\ 0 \leq T \leq I\right\}$$

is $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-convex. Indeed, assume that $A_i, B_i \in \mathcal{M}_n(\mathbb{C})$, $(i = 1, \cdots, k)$ with $\sum_{i=1}^k A_iA_i^* = I = \sum_{i=1}^k B_i^*B_i$. If $X_i \in \mathcal{S}$, $(i = 1, \cdots, k)$, then there exist $S_i, T_i \in \mathcal{M}_n(\mathbb{C})$ with $0 \leq S_i \leq I$ and $0 \leq T_i \leq I$ such that

$$\begin{pmatrix} S_i & X_i \\ X_i^* & T_i \end{pmatrix} \geq 0, \qquad i = 1, \cdots, k.$$

It follows that

$$\begin{bmatrix} \sum_{i=1}^k A_iS_iA_i^* & \sum_{i=1}^k A_iX_iB_i \\ \left(\sum_{i=1}^k A_iX_iB_i\right)^* & \sum_{i=1}^k B_i^*T_iB_i \end{bmatrix} = \sum_{i=1}^k \begin{bmatrix} A_i & 0 \\ 0 & B_i^* \end{bmatrix} \begin{bmatrix} S_i & X_i \\ X_i^* & T_i \end{bmatrix} \begin{bmatrix} A_i^* & 0 \\ 0 & B_i \end{bmatrix} \geq 0.$$

Moreover,

$$0 \leq \sum_{i=1}^k A_iS_iA_i^* \leq \sum_{i=1}^k A_iA_i^* = I \quad \text{and} \quad 0 \leq \sum_{i=1}^k B_i^*T_iB_i \leq \sum_{i=1}^k B_i^*B_i = I,$$

from which we get $\sum_{i=1}^k A_iX_iB_i \in \mathcal{S}$ and so $\mathcal{S}$ is $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-convex. Putting $S = T = I$ and using the fact that that for $X \in \mathcal{M}_n(\mathbb{C})$, $\|X\|_\infty \leq 1$ if and only if $\begin{bmatrix} I & X \\ X^* & I \end{bmatrix} \geq 0$, (see for example [1]) we conclude the $\mathcal{M}_n(\mathbb{C})$-$\mathcal{M}_n(\mathbb{C})$-convexity of

$$\mathcal{S} = \{X \in \mathcal{M}_n(\mathbb{C});\ \ \|X\|_\infty \leq 1\}.$$

Let $\mathcal{X}$ be a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule, $\mathcal{S} \subseteq \mathcal{X}$ and let $\|\cdot\|_{\mathscr{A}}$ and $\|\cdot\|_{\mathscr{B}}$ be the norms on $\mathcal{X}$ induced by $\langle\cdot,\cdot\rangle_{\mathscr{A}}$ and $\langle\cdot,\cdot\rangle_{\mathscr{B}}$, respectively. We mean by $\overline{\mathcal{S}}_{\mathscr{A}}$ and $\overline{\mathcal{S}}_{\mathscr{B}}$ the norm closures of $\mathcal{S}$ in $\mathcal{X}$ with respect to $\|\cdot\|_{\mathscr{A}}$ and $\|\cdot\|_{\mathscr{B}}$, respectively.

Proposition 13. *If $\mathcal{S}$ is $\mathscr{A}$-$\mathscr{B}$-convex, then so are $\overline{\mathcal{S}}_{\mathscr{A}}$ and $\overline{\mathcal{S}}_{\mathscr{B}}$.*

Proof. Let $\mathcal{S}$ be $\mathscr{A}$-$\mathscr{B}$-convex and $x_1, \ldots, x_n \in \overline{\mathcal{S}}_{\mathscr{A}}$. Assume that x_{ik} is a sequence in $\mathcal{S}$ such that $\|x_{ik} - x_i\|_{\mathscr{A}} \to 0$ for $i = 1, \ldots, n$ as $k \to \infty$. If $a_1, \ldots, a_n \in \mathscr{A}$ and

$b_1, \dots, b_n \in \mathscr{B}$ with $\sum_{i=1}^n a_i a_i^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^n b_i^* b_i = 1_{\mathscr{B}}$, then $\sum_{i=1}^n a_i x_{ik} b_i \in \mathcal{S}$, for every $k \in \mathbb{N}$. Moreover, for every $1 \le i \le n$ we have

$$\begin{aligned}\|a_i x_{ik} b_i - a_i x_i b_i\|_{\mathscr{A}}^2 &= \|\langle a_i(x_{ik} - x_i)b_i, a_i(x_{ik} - x_i)b_i\rangle_{\mathscr{A}}\| \\ &\le \|b_i\|_{\mathscr{B}}^2 \|\langle a_i(x_{ik} - x_i), a_i(x_{ik} - x_i)\rangle_{\mathscr{A}}\| \\ &\le a_i \|\langle x_{ik} - x_i, x_{ik} - x_i\rangle_{\mathscr{A}}\| a_i^* \\ &= a_i \|x_{ik} - x_i\|_{\mathscr{A}}^2 a_i^* \to 0.\end{aligned}$$

Therefore,

$$\left\| \sum_{i=1}^n a_i x_{ik} b_i - \sum_{i=1}^n a_i x_i b_i \right\|_{\mathscr{A}} \le \sum_{i=1}^n \|a_i x_{ik} b_i - a_i x_i b_i\|_{\mathscr{A}} \to 0.$$

It follows that $\sum_{i=1}^n a_i x_{ik} b_i \to \sum_{i=1}^n a_i x_i b_i$ as $k \to \infty$ and so $\sum_{i=1}^n a_i x_i b_i \in \overline{\mathcal{S}}_{\mathscr{A}}$. □

For every two element x, y in a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathcal{X}$, we define the $\mathscr{A}$-$\mathscr{B}$-segment connecting x and y by

$$S_{\mathscr{A},\mathscr{B}}(x, y) = \{axb + cyd \,|\; aa^* + cc^* = 1_{\mathscr{A}}, \;\; b^*b + d^*d = 1_{\mathscr{B}}\}.$$

and the $\mathscr{A}$-$\mathscr{B}$-convex segment connecting x and y by

$$CS_{\mathscr{A},\mathscr{B}}(x, y) = \left\{ \sum_{i=1}^n a_i x b_i + \sum_{j=1}^m c_j y d_j \;\middle|\; \sum_{i=1}^n a_i a_i^* + \sum_{j=1}^m c_j c_j^* = 1_{\mathscr{A}}, \;\; \sum_{i=1}^n b_i^* b_i + \sum_{j=1}^m d_j^* d_j = 1_{\mathscr{B}} \right\}.$$

If $\mathscr{A} = \mathscr{B}$, then we denote $S_{\mathscr{A},\mathscr{B}}(x, y)$ and $CS_{\mathscr{A},\mathscr{B}}(x, y)$ by $S_{\mathscr{A}}(x, y)$ and $CS_{\mathscr{A}}(x, y)$, respectively. These concepts are natural generalizations of C^*-segment and C^*-convex segments in C^*-algebras. The $\mathscr{A}$-$\mathscr{B}$-segment connecting x and y, the $S_{\mathscr{A},\mathscr{B}}(x, y)$, is not $\mathscr{A}$-$\mathscr{B}$-convex in general. The next example shows that $S_{\mathscr{A},\mathscr{B}}(x, y)$ is not even convex.

Example 14. *[10] Consider $\mathcal{M}_2(\mathbb{C})$ as a Hilbert $\mathcal{M}_2(\mathbb{C})$-$\mathcal{M}_2(\mathbb{C})$-bimodule. Let $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = 0$. Then every element in the $S_{\mathcal{M}_2(\mathbb{C})}(X, Y)$ is a rank one matrix. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $AA^* = I$ and so $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = AXA^* \in S_{\mathcal{M}_2(\mathbb{C})}(X, Y)$. However, $\lambda T + (1 - \lambda)X = \begin{pmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is not of rank one. It follows that $S_{\mathcal{M}_2(\mathbb{C})}(X, Y)$ is not even convex.*

However, $CS_{\mathscr{A},\mathscr{B}}(x, y)$ is $\mathscr{A}$-$\mathscr{B}$-convex.

Proposition 15. *If $x, y \in \mathcal{X}$, then $CS_{\mathscr{A},\mathscr{B}}(x, y)$ is $\mathscr{A}$-$\mathscr{B}$-convex and contains x and y.*

Proof. Assume that $n = m = 1$, $a_1 = 1_{\mathscr{A}}$, $c_1 = 0$, $b_1 = 1_{\mathscr{B}}$ and $d_1 = 0$. Then

$$x = a_1 x b_1 + c_1 y d_1 \in CS_{\mathscr{A},\mathscr{B}}(x,y).$$

Similarly $y \in CS_{\mathscr{A},\mathscr{B}}(x,y)$. Now assume that $z_1, \ldots, z_n \in CS_{\mathscr{A},\mathscr{B}}(x,y)$. Then

$$z_k = \sum_{i=1}^{n_k} a_{ik} x b_{ik} + \sum_{j=1}^{m_k} c_{jk} y d_{jk} \qquad \forall k = 1, \ldots, n$$

in which $\sum_{i=1}^{n_k} a_{ik}a_{ik}^* + \sum_{j=1}^{m_k} c_{jk}c_{jk}^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^{n_k} b_{ik}^* b_{ik} + \sum_{j=1}^{m_k} d_{jk}^* d_{jk} = 1_{\mathscr{B}}$, for every k. Let $p_1, \ldots, p_n \in \mathscr{A}$ and $q_1, \ldots, q_n \in \mathscr{B}$ with $\sum_{i=1}^{n} p_k p_k^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^{n} q_k^* q_k = 1_{\mathscr{B}}$. We have

$$\begin{aligned}\sum_{k=1}^{n} p_k z_k q_k &= \sum_{k=1}^{n} p_k \left(\sum_{i=1}^{n_k} a_{ik} x b_{ik} + \sum_{j=1}^{m_k} c_{jk} y d_{jk} \right) q_k \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n_k} p_k a_{ik} x b_{ik} q_k + \sum_{k=1}^{n}\sum_{j=1}^{m_k} p_k c_{jk} y d_{jk} q_k \in CS_{\mathscr{A},\mathscr{B}}(x,y),\end{aligned}$$

since

$$\sum_{k=1}^{n}\sum_{i=1}^{n_k} p_k a_{ik} a_{ik}^* p_k^* + \sum_{k=1}^{n}\sum_{j=1}^{m_k} p_k c_{jk} c_{jk}^* p_k^* = \sum_{k=1}^{n} p_k \left(\sum_{i=1}^{n_k} a_{ik}a_{ik}^* + \sum_{j=1}^{m_k} c_{jk}c_{jk}^* \right) p_k^* = 1_{\mathscr{A}}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{n_k} (b_{ik}q_k)^* b_{ik} q_k + \sum_{k=1}^{n}\sum_{j=1}^{m_k} (d_{jk}q_k)^* d_{jk} q_k = \sum_{k=1}^{n} q_k^* \left(\sum_{i=1}^{n_k} b_{ik}^* b_{ik} + \sum_{j=1}^{m_k} d_{jk}^* d_{jk} \right) q_k = 1_{\mathscr{B}}.$$

□

We are going to show that every $\mathscr{A}$-$\mathscr{B}$-convex combination of elements of an $\mathscr{A}$-$\mathscr{B}$-convex set, can be presented as a combination of two terms.

Proposition 16. *Let $\mathcal{S}$ be an $\mathscr{A}$-$\mathscr{B}$-convex subset of the Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathcal{X}$ and let $x_1, \cdots, x_n \in \mathcal{S}$. If $z = \sum_{i=1}^{n} a_i x_i b_i$ with $a_i \in \mathscr{A}$, $b_i \in \mathscr{B}$ and $\sum_{i=1}^{n} a_i a_i^* = 1_{\mathscr{A}}$ and $\sum_{i=1}^{n} b_i^* b_i = 1_{\mathscr{B}}$, then $z = e_1 x f_1 + e_2 y f_2$, for some $x, y \in \mathcal{S}$, $e_1, e_2 \in \mathscr{A}$ and $f_1, f_2 \in \mathscr{B}$ with $e_1 e_1^* + e_2 e_2^* = 1_{\mathscr{A}}$ and $f_1^* f_1 + f_2^* f_2 = 1_{\mathscr{B}}$.*

Proof. Assume that $z = \sum_{i=1}^{n} a_i x_i b_i$. Put $u = \frac{1}{2} a_1 a_1^*$ and $v = \frac{1}{2} b_1^* b_1$ so that u and v are positive invertible elements in $\mathscr{A}$ and $\mathscr{B}$, respectively. Put $c_1 = \frac{1}{\sqrt{2}}(1-u)^{\frac{-1}{2}} a_1$, $d_1 = \frac{1}{\sqrt{2}} b_1 (1-v)^{\frac{-1}{2}}$ and

$$c_i = (1-u)^{\frac{-1}{2}} a_i, \quad d_i = b_i (1-v)^{\frac{-1}{2}} \qquad i = 2, \cdots, n.$$

then $c_i \in \mathscr{A}$, $d_i \in \mathscr{B}$ and

$$\sum_{i=1}^{n} c_i c_i^* = \frac{1}{2}(1-u)^{\frac{-1}{2}} a_1 a_1^* (1-u)^{\frac{-1}{2}} + \sum_{i=2}^{n} (1-u)^{\frac{-1}{2}} a_i a_i^* (1-u)^{\frac{-1}{2}}$$
$$= (1-u)^{\frac{-1}{2}} \left(\frac{1}{2} a_1 a_1^* + \sum_{i=2}^{n} a_i a_i^* \right) (1-u)^{\frac{-1}{2}} = 1_{\mathscr{A}}.$$

Similarly, $\sum_{i=1}^{n} d_i^* d_i = 1_{\mathscr{B}}$. It follows that $y = \sum_{i=1}^{n} c_i x_i d_i \in \mathcal{S}$. But we have

$$z = \sum_{i=1}^{n} a_i x_i b_i = \left(\frac{1}{\sqrt{2}} a_1 \right) x_1 \left(\frac{1}{\sqrt{2}} b_1 \right) + (1-u)^{\frac{1}{2}} y (1-v)^{\frac{1}{2}}$$

in which $x_1, y \in \mathcal{S}$, $\frac{1}{2} a_1 a_1^* + (1-u) = 1_{\mathscr{A}}$ and $\frac{1}{2} b_1^* b_1 + (1-v) = 1_{\mathscr{B}}$. □

Remark 17. *Suppose that $\mathcal{X}$ is a Hilbert $\mathscr{A}$-$\mathscr{B}$-bimodule and $\mathcal{S}$ is an $\mathscr{A}$-$\mathscr{B}$-convex subset of $\mathcal{X}$ and $0 \in \mathcal{S}$. If $x \in \mathcal{S}$ and u and v are unitaries in C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, then trivially $uxv \in \mathcal{S}$. Let $x_1, x_2 \in \mathcal{S}$, $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ with $a_1 a_1^* + a_2 a_2^* = 1_{\mathscr{A}}$ and $b_1^* b_1 + b_2^* b_2 = 1_{\mathscr{B}}$. Assume that $a_i^* = u_i |a_i^*|$ and $b_i = v_i |b_i|$ be the polar decomposition. Then*

$$z = a_1 x_1 b_1 + a_2 x_2 b_2 = |a_1^*| u_1^* x_1 v_1 |b_1| + |a_2^*| u_2^* x_2 v_2 |b_2| = |a_1^*| y_1 |b_1| + |a_2^*| y_2 |b_2|$$

in which, $y_1, y_2 \in \mathcal{S}$ and $|a_1^*|^2 + |a_2^*|^2 = 1_{\mathscr{A}}$ and $|b_1|^2 + |b_2|^2 = 1_{\mathscr{B}}$. It means that z can be presented as a combination with positive coefficients.

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References

[1] R. Bhatia, *Positive Definite Matrices,* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007. MR2284176(2007k:15005). Zbl 1125.15300.

[2] M. Dehghani, S. M. S. Modarres and M. S. Moslehian, *Positive block matrices on Hilbert and Krein C^*-modules,* Surv. Math. Appl. **8** (2013), 23-34 MR3171618.

[3] E.G. Effros and S. Winkler, *Matrix Convexity: Operator Analogues of the Bipolar and Hahn-Banach Theorems*, J. Funct. Anal. **144** (1997), no. 1, 117-152. MR1430718. Zbl 0897.46046.

[4] D.R. Farenick and B.P. Morenz, *C^*-extreme points of some compact C^*-convex sets*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 765-775. MR1139466. Zbl 0782.15017.

[5] T. Furuta, H. Mićić, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space*, Monographs in Inequalities, 1. ELEMENT, Zagreb, 2005. MR3026316. Zbl 1135.47012.

[6] A. Hopenwasser, R.L. Moore, V.I. Paulsen, *C^*-extereme points*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 291-307. MR0613797. Zbl 0471.47024.

[7] T. Kajiwara and Y. Watatani, *Jones index theory by Hilbert C^*-bimodules and K-theory*, Trans. Amer. Math. Soc. **352** (2000), no. 8, 3429-3472. MR1624182. Zbl 0954.46034.

[8] M. Kian, *C^*-convexity of norm unit balls*, J. Math. Anal. Appl. **445** (2017), no. 2, 1417-1427. MR3545251. Zbl 06626211.

[9] M. Kian, *Epigraph of operator functions*, Quaest. Math. **39** (2016), no. 5, 587-594. MR3544211.

[10] R.I. Loebl and V.I. Paulsen, *Some remarks on C^*-convexity*, Linear Algebra Appl. **35** (1981), 63-78. MR0599846. Zbl 0448.46038

[11] P.B. Morenz, *The structure of C^*-convex sets*, Canad. J. Math. **46** (1994), no. 5, 1007-1026. MR1295129. Zbl 0805.47003

[12] B. Magajna, *C^*-convex sets and completely bounded bimodule homomorphisms*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 2, 375-387. MR1750836. Zbl 0970.46041

[13] B. Magajna, *On C^*-extereme points*, Proc. Amer. Math. Soc. **129** (2001), no. 3, 771-780. MR1802000. Zbl 0967.47042

[14] B. Magajna, *C^*-convex sets and completely positive maps*, Integral Equations Operator Theory **85** (2016), no. 1, 37-62. MR3503178. Zbl 06601124

[15] V.M. Manuilov and E. V. Troitsky, *Hilbert C^*-Modules*, Translations of Mathematical Monographs, 226, American Mathematical Society, Providence RI, 2005. MR2125398. Zbl 1074.46001.

[16] C. Webster and S. Winkler, *The Krein-Milman Theorem in operator convexity*, Trans. Amer. Math. Soc. **351** (1999), no. 1, 307-322. MR1615970. Zbl 0908.47042.

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