Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 12 (2017), 229 – 252

CONJECTURE OF TWIN PRIMES (STILL UNSOLVED PROBLEM IN NUMBER THEORY) AN EXPOSITORY ESSAY

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Abstract. The purpose of this paper is to gather as much results of advances, recent and previous works as possible concerning the oldest outstanding still unsolved problem in Number Theory (and the most elusive open problem in prime numbers) called "Twin primes conjecture" (8^{th} problem of *David Hilbert*, stated in 1900) which has eluded many gifted mathematicians. This conjecture has been circulating for decades, even with the progress of contemporary technology that puts the whole world within our reach. So, simple to state, yet so hard to prove. Basic Concepts, many and varied topics regarding the Twin prime conjecture will be cover.

Full text

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Keywords: Twin primes; Brun's constant; Zhang's discovery; Polymath project.

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