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APPROXIMATE SOLUTIONS TO SOME NON-AUTONOMOUS DIFFERENTIAL EQUATIONS FOR GROWTH PHENOMENA

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Abstract. Growth modeling is widely used in various fields of applied sciences. The purpose of this paper is to develop analytic approximate solutions to some non-autonomous differential equations used in population growth. We demonstrate that when the carrying capacity varies with time, an approximate solution to the generalized Turner model and any particular case of this model can be produced without expensive calculations.

1 Introduction

Gompertz and logistic models are among the oldest used models in modeling phenomena arising from real situations. These two models have been introduced in the 18th and 19th century by the mathematicians Gompertz to study the human mortality [2], and by Verhulst to study the population dynamics [34]. Over the last century, these models have been extensively used in other fields of applied sciences to describe and improve the possible relationship between independent and dependent variables in terms of mathematical equations, like in ecology, in sociology, in medicine and other domains of natural and human sciences [8, 9, 11, 21, 23, 24, 31]. Numerous extensions of these models have been developed by mathematicians over the last decades. These extensions have extended the use of mathematical modeling in other fields of applied science. Examples of the most commonly cited ones include, Turner model [13], Richards model [25], Michaelis-Menten (or Morgan) model [20], Bridge (or Weibul) model and several other models [12, 14, 35]. The differential equation that characterizes these standard (growth) models is given in [15, 32, 33, 35]. The main feature of these (standard) growth models is that they have a limit of growth which is presented by an horizontal asymptote. Also, called carrying capacity, this limit, often denoted by k, can be caused by many environmental factors as space, food, or resources [1, 17, 18, 20, 29, 30, 35].

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Models with unchanging carrying capacity k could no longer be used in general to model phenomena arising from life sciences research. According to several authors, many environmental and social factors prevent the carrying capacity to stay unchanged over time [1, 3, 4, 5, 16, 17]. Hence, models with time dependent carrying capacity is of growing need. To overcome this problem a wide varieties of models were provided to model phenomena with varying [27, 28, 29], logistically varying [16, 17], increasing [6, 7, 18, 19] or sinusoidally varying [4, 5, 26] carrying capacity. Several other researchers have devoted much attention to the development of existence and uniqueness theories concerning these models, especially logistic model [3, 4, 5, 10, 22].

Whereas the number of models proposed in the literature keeps growing, the analytic solutions of their corresponding differential equations is not often possible and requires the use of expensive calculations and techniques of numerical analysis. This leads to a growing need to approximate the solutions to their differential equations. The main purpose of this study is to overlap this problem. We provide approximate solutions to Turner differential equation [13] and Richards equation [25] with time dependent coefficients. Some examples are presented to highlight the efficiency and the importance of these approximate solutions for the linear and sinusoidal carrying capacity cases.

2 Preliminaries

Let $I = [t_0, +\infty)$ be an interval such that $t_0 \in \mathbb{R}$ and consider the differential equation

$$\dot{x}(t) = \frac{dx(t)}{dt} = \frac{1}{\alpha}\beta(t)x(t)^{1+\alpha(1-\gamma)}\left[1 - \left(\frac{x(t)}{k(t)}\right)^{\alpha}\right]^{\gamma}, \quad t \geq t_0, \tag{2.1}$$

where $\beta(t), k(t) : I \to (0, +\infty)$ are two continuous functions, and α and γ are two real numbers such that $1 \leq \gamma < 1 + 1/\alpha$. In addition, suppose that the initial conditions satisfy the following inequality

$$0 < x(t_0) = x_0 \leq k_0 = k(t_0). \tag{2.2}$$

In the case where $k(t) = k$ doesn't depend on time, and under the assumption (2.2), the differential equation (2.1), can be solved explicitly using basic integration techniques. Indeed, by performing the transformation

$$z : x(t) \to z(t) = k^{\alpha}x^{-\alpha}(t) - 1, \tag{2.3}$$

(2.1) reduces to

$$\dot{z}(t) = -\alpha k^{\alpha(1-\gamma)}\beta(t)z(t)^{\gamma}, \quad t \geq t_0 \tag{2.4}$$

which has the unique solution given by

$$z(t) = \left(z_0^{1-\gamma} + (\gamma - 1)k^{\alpha(1-\gamma)}\int_{t_0}^{t}\beta(\tau)d\tau\right)^{1/(1-\gamma)}, \tag{2.5}$$

with $z_0 = k^\alpha x_0^{-\alpha} - 1$.

By substituting (2.3) in (2.5), the solution of (2.1) is then given by

$$x(t) = \frac{k}{\left(1 + \left(z_0^{1-\gamma} + (\gamma - 1)k^{\alpha(1-\gamma)} \int_{t_0}^{t} \beta(\tau)d\tau\right)^{1/(1-\gamma)}\right)^{1/\alpha}}. \tag{2.6}$$

We have

- If $x_0 = k$, we have $z_0 = 0$. From (2.4) and (2.5) it follows that $z(t) = 0$. By (2.3) we get $x(t) = k$ for all $t \geq t_0$;
- If $x_0 < k$, we have

$$\lim_{t \to +\infty} x(t) = x_\infty = \begin{cases} k, & if \quad \int_{+\infty}^{t} \beta(\tau)d\tau = +\infty, \\ k_1, & if \quad \int_{+\infty}^{t} \beta(\tau)d\tau = l. \end{cases}$$

  In this case, we have $x(t) < x_\infty$ for all $t \in I$, and the solution $x(t)$ grows up to x_∞.

When k(t) depends on time, (2.1) has no constant solution and the solution (2.4) may crosses $k(t)$. It happens when $\dot{x}(t) = 0$. We have the following Lemma.

Lemma 1. *Let $k : I \to (0, +\infty)$ be continuous and increasing function, and let $x(t)$ be the solution of* (2.1) *passing through the point (t_0, x_0). If for some $t_* \in I$ we have $x(t_*) = k(t_*)$, then $x(t) \leq k(t)$ for all $t \geq t_*$.*

Proof. Let $t_* \in I$ such that $x(t_*) = k(t_*)$. By (2.1) it follows that $x(t_*) = 0$. As $x(t)$ is increasing on I, by continuity of $x(t)$ and $k(t)$ it follows that $x(t) = k(t)$ for all $t > t_*$ provided that $x(t) = k(t) = 0$. If we define s as the maximum of the set of abscissa t such that $x(t) = k(t)$, we have that $x(t) = k(t)$ for all $t_* \leq t \leq s$ and $x(t) < k(t)$ for all $t > s$. Which complete the proof. ∎

An analytic solution of (2.1) is not often possible when the carrying capacity $k(t)$ is time dependent function. It requires the use of expensive calculations and techniques of numerical analysis. In the following sections we give an approximate solution of (2.1) which can be produced without numerical calculations of the solution of (2.1).

3 Main results

When $k(t)$ is a time dependent function, the solution to (2.1) is often not possible and requires the use of expensive calculations and techniques of numerical analysis. The following result produce an approximate solution to (2.1) and any particular case of it.

Theorem 2. *Let $r_\varphi : \mathbb{R} \times I \to (0, +\infty)$ be defined such that*

$$\lim_{\epsilon \to +\infty} \int_{t_0}^{\epsilon} r_\varphi(\tau, t) d\tau = +\infty, \tag{3.1}$$

and

$$\lim_{\epsilon \to +\infty} r_\varphi(\epsilon, t) = \beta(t). \tag{3.2}$$

For small values of ϵ, the function $\varphi(\epsilon, t)$ given by

$$\varphi(\epsilon, t) = \frac{k(t)}{\left(1 + \left((\gamma - 1) \int_{t_0}^{\epsilon} r_\varphi(\tau, t) d\tau k^{\alpha(1-\gamma)}(t) + \left(\left(\frac{k(t)}{x(t)}\right)^{\alpha} - 1\right)^{(1-\gamma)}\right)^{1/(1-\gamma)}\right)^{1/\alpha}} \tag{3.3}$$

is an approximate function of the solution $x(t)$ of (2.1).

Proof. By (3.1) and (3.2), it follows that

$$\lim_{\epsilon \to +\infty} \varphi(\epsilon, t) = k(t), \tag{3.4}$$

and

$$\lim_{\epsilon \to 0} \varphi(\epsilon, t) = \beta(t). \tag{3.5}$$

In addition, we have

$$\frac{d\varphi(\epsilon, t)}{d\epsilon} = \frac{1}{\alpha} r_\varphi(\epsilon, t) \varphi(\epsilon, t)^{1+\alpha(1-\gamma)} \left[1 - \left(\frac{\varphi(\epsilon, t)}{k(t)}\right)^{\alpha}\right]^{\gamma}. \tag{3.6}$$

In (3.6), by taking the limit $\epsilon \to 0$ and by using (3.5) we get

$$\lim_{\epsilon \to 0} \frac{d\varphi(\epsilon, t)}{d\epsilon} = \frac{dx(t)}{dt}. \tag{3.7}$$

Thus, for small values of ϵ, the function $\varphi(\epsilon, t)$ given in (3.3) coincide with the solution $x(t)$ of (2.1). Thus, the proof is completed. ■

In the following section we provide some examples of situations for which equation (2.1) admits an analytic solution.

4 Richards model

If we set $\gamma = 1$, (2.1) reduces to the well-known logistic generalized differential equation given by

$$\dot{x}(t) = \frac{dx(t)}{dt} = \frac{1}{\alpha} \beta(t) x(t) \left(1 - \left(\frac{x(t)}{k(t)}\right)^{\alpha}\right), \quad t \geq t_0, \tag{4.1}$$

which has the explicit solution given by

$$x(t) = \frac{1}{e^{-(1/\alpha)\int_{t_0}^{t}\beta(\tau)d\tau}\left(\int_{t_0}^{t}\frac{\beta(\tau)}{k(\tau)^{\alpha}}e^{\int_{t_0}^{\tau}\beta(u)du}d\tau + 1/x_0^{\alpha}\right)^{1/\alpha}}. \tag{4.2}$$

From (3.3), an approximate solution to this model is given by

$$\varphi(\epsilon, t) = \frac{k(t)}{\left(1 + \left(\left(\frac{k(t)}{x(t)}\right)^{\alpha} - 1\right)e^{-\int_{t_0}^{\epsilon} r_{\varphi}(\tau,t)d\tau}\right)^{1/\alpha}} \tag{4.3}$$

In the case when $\alpha \to 0$, (4.1) reduces to the generalized Gompertz differential equation given by

$$\dot{x}(t) = \frac{dx(t)}{dt} = -\beta(t)x(t)\ln\left(\frac{x(t)}{k(t)}\right), \quad t \geq t_0, \tag{4.4}$$

which has the following solution

$$x(t) = exp\left(\ln(x_0)e^{-\int_{t_0}^{t}\beta(\tau)d\tau} + e^{-\int_{t_0}^{t}\beta(\tau)d\tau}\int_{t_0}^{t}\beta(\tau)e^{\int_{t_0}^{\tau}\beta(u)du}\ln(k(\tau))d\tau\right) \tag{4.5}$$

From (3.3) , an approximate solution to this model is given by

$$\varphi(\epsilon, t) = k(t)exp\left(-\ln(k(t)/x(t))e^{-\int_{t_0}^{\epsilon} r_{\varphi}(\tau,t)d\tau}\right). \tag{4.6}$$

In the case when $\alpha = 1$, (4.1) reduces to the logistic differential equation given by

$$\dot{x}(t) = \beta(t)x(t)\left[1 - \left(\frac{x(t)}{k(t)}\right)\right], \quad t \geq t_0, \tag{4.7}$$

which has the explicit solution given by

$$x(t) = \frac{1}{e^{-\int_{t_0}^{t}\beta(\tau)d\tau}\left(\int_{t_0}^{t}\frac{\beta(\tau)}{k(\tau)}e^{\int_{t_0}^{\tau}\beta(u)du}d\tau + 1/x_0\right)}. \tag{4.8}$$

From (3.3), an approximate solution to this model is given by

$$\varphi(\epsilon, t) = \frac{k(t)}{\left(1 + \left(\left(\frac{k(t)}{x(t)}\right) - 1\right)r^{-\int_{t_0}^{\epsilon} r_{\varphi}(\tau,t)d\tau}\right)} \tag{4.9}$$

In addition, it will be noted that (4.1) and (4.2) reduces to the linear model when $\alpha = -1$.

5 Numerical examples

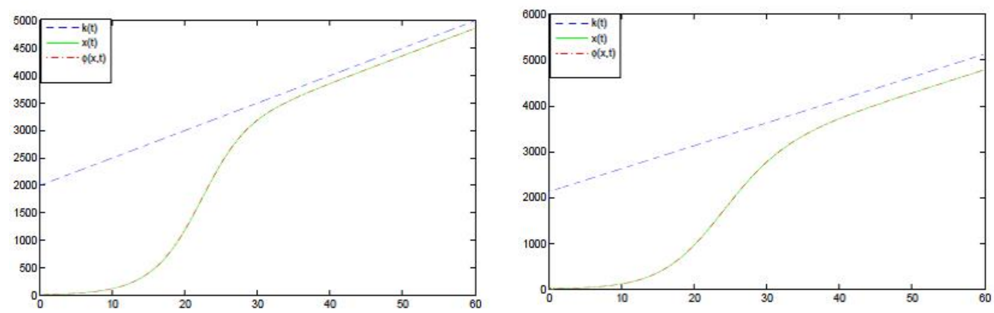
In this section, we consider the differential equation (2.1). For a given initial condition $x_0 = x(t_0) > 0$, we assume the following explicit forms for $\beta(t)$,

$$\beta(t) = \beta_0 > 0.$$

In this particular case, if we set $r_\varphi(\epsilon, t) = \beta_0$, the assumptions (3.1) and (3.2) of Theorem 2. are satisfied and an approximate solution follows immediately from (3.3).

5.1 Linear carrying capacity

Suppose that $k(t) = k_0(1 + ct)$, where k_0 and c are positive real numbers. A graphic representation of the numerical solution $x(t)$ of (2.1), the approximate solution $\varphi(\epsilon, t)$ given in (3.3), and the carrying capacity $k(t)$ are given in Figure 1.



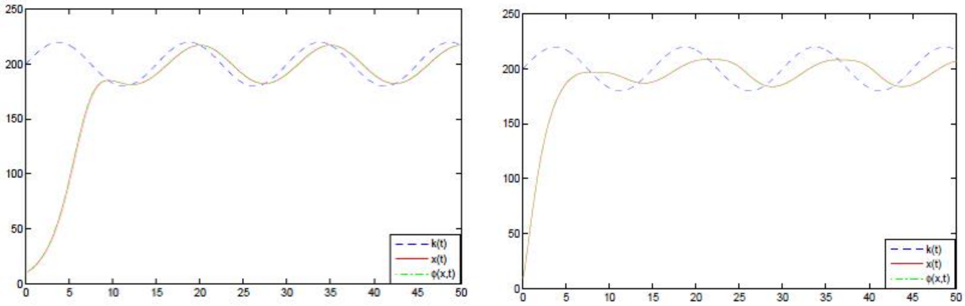
(a) $\alpha = 3$, and $\gamma = 1.25$

(b) $\alpha = 1.5$, and $\gamma = 1$

Figure 1: Representation of the solution $x(t)$ (red solid line), the approximate solution $\varphi(\epsilon, t)$ (green dot-dashed line), and the carrying capacity $k(t)$ (blue long-dashed line) for $x_0 = 10, k_0 = 2000, c = 0.025$, $\epsilon = 0.05$ and $\beta_0 = 0.25$.

5.2 Sinusoidal carrying capacity

Suppose that $k(t) = k_0 + k_1 \sin(\omega t)$, where k_0, k_1 and $\omega = 2\pi/T$ are positive real numbers and define respectively the mean, the amplitude and the frequency of oscillation. A graphic representation of the numerical solution $x(t)$ of (2.1), the approximate solution $\varphi(\epsilon, t)$ given in (3.3), and the carrying capacity $k(t)$ are given in Figure 2.



(a) $\alpha = 1.5$, and $\gamma = 1$ (b) $\alpha = 1$, and $\gamma = 1$

Figure 2: Representation of the solution $x(t)$ (red solid line), the approximate solution $\varphi(\epsilon, t)$ (green dot-dashed line), and the carrying capacity $k(t)$ (blue long-dashed line) for $x_0 = 10, k_0 = 200, k_1 = 20, T = 20$, $\epsilon = 0.05$ and $\beta_0 = 0.5$.

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