Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 10 (2015), 1 – 21

QUANTAM LIÉNARD II EQUATION AND JACOBI'S LAST MULTIPLIER

A. Ghose Choudhury and Partha Guha

Abstract. In this survey the role of Jacobi's last multiplier in mechanical systems with a position dependent mass is unveiled. In particular, we map the Liénard II equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ to a position dependent mass system. The quantization of the Liénard II equation is then carried out using the point canonical transformation method together with the von Roos ordering technique. Finally we show how their eigenfunctions and eigenspectrum can be obtained in terms of associated Laguerre and exceptional Laguerre functions. By employing the exceptional Jacobi polynomials we construct three exactly solvable potentials giving rise to bound-state solutions of the Schrödinger equation.

1 Introduction

In recent times there has been a lot of interest in PT-symmetric Hamiltonians owing to the fact that they provide examples of non-hermitian Hamiltonians with a real spectrum. There has also been a surfeit of interest in quantization of dissipative dynamical systems. In this context we consider the equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$, in which the overdot denotes differentiation with respect to time t, and will refer to it as the Liénard II equation [40] since it involves a quadratic damping term in contrast to the usual Liénard equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$.

The Liénard II equation arises naturally whenever the mass is position dependent [2]. Indeed assuming the linear momentum, $p = m(x)\dot{x}$, it follows that

$$\frac{dp}{dt} = m(x)\ddot{x} + m'(x)\dot{x}^2.$$

Consequently if we set the force to be proportional to $m(x)$, i.e., take $\mathcal{F}(x) = -m(x)g(x)$ one obtains from Newton's second law the equation of motion, $\ddot{x} + m'(x)/m(x)\dot{x}^2 + g(x) = 0$. This is obviously a special case of the Liénard II

2010 Mathematics Subject Classification: 58A15; 58A30.
Keywords: Liénard II equation; position-dependent mass; Jacobi last multiplier; Schrödinger equation; exceptional Laguerre equation; exceptional Jacobi polynomial.

equation being considered with $f(x) = m'(x)/m(x)$. Besides the above from a more mechanical perspective such equations with a quadratic damping term often result from the movement of an object through a fluid medium as with an automobile pushing through air or a boat through water. For a practical application one may note that such quadratic damping assumes significance in the design of a cam, which is an eccentric-shaped device usually used to convert the rotational motion of a shaft into a translation especially at high speeds or in case of very dense fluids [31].

In our previous work [15] on the Liénard II equation we have shown that employing the notion of Jacobi's last multiplier (JLM) [22, 23, 32, 46] one can recast the Liénard II equation into the Euler-Lagrange form so that using the usual Legendre transformation we can easily construct the Hamiltonian of the system. An interesting feature displayed by such a Hamiltonian was the presence of a kinetic energy term of the form, $p^2/2M(x)$, where $M(x)$ represents the JLM of the equation. In view of this dependence of the kinetic term on the JLM it is reasonable to interpret physically the JLM as a kind of effective mass of the corresponding dynamical system and regard the Hamiltonian as one involving a position-dependent mass (PDM). The presence of such PDM terms have in fact appeared in several nonlinear oscillators [8, 28] and in the PT-symmetric cubic anharmonic oscillator [30]. In recent years the study of the exact solutions of the position-dependent mass Schrödinger equation (PDMSE) (see for example [3, 4, 10, 20, 25, 29]) using the method of point canonical transformations [6, 26, 11, 24] or supersymmetric quantum mechanics [45] has gained a certain degree of importance owing to their relevance in diverse areas of physics ranging from quantum dots [42], quantum liquids [1], metal clusters [35], compositionally graded crystals [16] etc. and thus provides sufficient motivation for the study of the Liénard II equation from a quantum mechanical perspective.

The quantization of such PDM Hamiltonians is, however, beset with a number of problems foremost among which is the issue of ordering. In the coordinate representation the situation was resolved by Von Roos [43] who developed a novel scheme which we have put to use here. As for the issue of requiring PT-symmetry it is easy to show that the Liénard II equation respects this symmetry provided the functions $f(x)$ and $g(x)$ are odd.

Bhattacharjie and Sudarshan in [6] introduced a method for determining classes of potentials appearing in the Schrödinger equation whose solutions corresponded to the classical orthogonal polynomials. These polynomials are characterized as the polynomial solutions of a Sturm-Liouville problem in connection with the celebrated theorem of Bochner [7]. In recent years, one of the most enchanting developments in quantum mechanics has been the construction of new class of exactly solvable potentials associated with the appearance of a new family of exceptional orthogonal polynomials [17, 18, 19]. Gomez-Ullate *et al* [18] extended Bochner's result by dropping the assumption that the first element of the orthogonal polynomial sequence be a constant. In other words, contrary to families of classical orthogonal polynomials which start with a constant, the families of the exceptional orthogonal polynomials begin with some polynomial of degree greater than or equal to one, but still form complete, orthogonal sets with respect to some positive-definite measure.

In our case this provides a useful technique for determining suitable forms of the functions $f(x)$ and $g(x)$ appearing in the Liénard II equation for which one can solve the Schrödinger equation using point canonical transformation method [6, 26] and determine the eigenspectrum. The recent discovery of the exceptional classes of X_1-polynomials [17] for the Laguerre and Jacobi polynomial class further enlarges the class of potentials and in turn the classes of functions $f(x)$ and $g(x)$ which may be tackled. For instance the exceptional X_1 Laguerre or Jacobi type polynomials were shown to be the eigenfunctions of the rationally extended radial oscillator or Scarf I potentials by using the point canonical transformation method in [36] and by the methods of supersymmetric quantum mechanics in [38]. Construction of two distinct families of Laguerre and Jacobi type X_m exceptional orthogonal polynomials as eigenfunctions of infinitely many shape invariant potentials by deforming the radial oscillator, the hyperbolic (or trigonometric) Poschl Teller potentials and hyperbolic (or trigonometric) Scarf potentials has been done in [33, 34, 29]. Recently Quesne [36] has constructed certain exactly solvable potentials giving rise to bound-state solutions to the Schrödinger equation, which are new and can be written in terms of the Jacobi-type X_1 exceptional orthogonal polynomials.

Furthermore, exactly solvable potentials and their corresponding solutions in terms of the exceptional classes of X_1-polynomials [17] for the Laguerre and Jacobi polynomial class provide us with a useful technique for determining suitable forms of the functions $f(x)$ and $g(x)$ appearing in the Liénard II equation for which one can solve the Schrödinger equation and determine the eigenspectrum using the point canonical transformation method [6, 26].

Recently an exact quantization of a PT symmetric (reversible) Liénard type one dimensional nonlinear oscillator both semiclassically and quantum mechanically has been carried out in [39].

The paper is organized as follows. In Section 2 we introduce the notion of Jacobi's last multiplier and use it to deduce the Hamiltonian of the Liénard II equation. In Section 3 we consider the Schrödinger equation in the coordinate representation and by combining the approaches of Von Roos and Bhattacharjie and Sudarshan identify certain potentials which give rise to solutions belonging to well known classical second-order linear ODEs including the recently discovered exceptional Laguerre equation. In Section 4 we derive new potentials using the X_1- Jacobi equation and compute their associated eigenfunctions and eigenvalues.

2 JLM and the Hamiltonian of the Liénard II equation

Given a second-order ordinary differential equation (ODE)

$$\ddot{x} = F(x, \dot{x}) \tag{2.1}$$

we define the Jacobi last multiplier M as a solution of the following ODE

$$\frac{d \log M}{dt} + \frac{\partial F(x, \dot{x})}{\partial \dot{x}} = 0. \tag{2.2}$$

Assuming (2.1) to be derivable from the Euler-Lagrange equation one can show that the JLM is related to the Lagrangian by the following equation

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{2.3}$$

In case of the Liénard II equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2.4}$$

one can show that the solution of the JLM is given by

$$M(x) = e^{2F(x)}, \quad F(x) := \int^x f(s)ds. \tag{2.5}$$

Furthermore it follows from (2.3) that its Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2} e^{2F(x)} \dot{x}^2 - V(x), \tag{2.6}$$

where the potential term

$$V(x) = \int^x e^{2F(s)} g(s)ds. \tag{2.7}$$

Clearly the conjugate momentum

$$p := \frac{\partial L}{\partial \dot{x}} = \dot{x} e^{2F(x)} \text{ implies } \dot{x} = p e^{-2F(x)}, \tag{2.8}$$

so that the final expression for the Hamiltonian is

$$H = \frac{p^2}{2M(x)} + \int^x M(s)g(s)ds, \tag{2.9}$$

where we have purposely written it in terms of the last multiplier $M(x)$ to highlight its role as a position dependent mass term.

Remark on PT-symmetry Liénard II Consider the following transformation

$$(x,t) \longrightarrow (\tilde{x},\tilde{t}), \quad \tilde{x} = -x, \quad \tilde{t} = -t, \tag{2.10}$$

under which the equation (2.4) may be seen to be invariant provided

$$f(-\tilde{x}) = -f(\tilde{x}) \ \text{ and } \ g(-\tilde{x}) = -g(\tilde{x}). \tag{2.11}$$

This shows that the functions $f(x)$ and $g(x)$ must be odd in order to ensure that (2.4) is PT-symmetric. In view of (2.5) we conclude that the last multiplier $M(x)$ is necessarily an even function so that the potential term in (2.9) is even since the integrand is an odd function. The kinetic energy term is also clearly invariant under the transformation (2.10) thereby ensuring that Hamiltonian in (2.9) is PT-symmetric.

3 Quantization of Liénard II equation

Using the von Roos decomposition for position dependent mass (PDM) we write the Hamiltonian (2.9) as follows:

$$H(\hat{x},\hat{p}) = \frac{1}{4}[M^{\alpha}(\hat{x})\hat{p}M^{\beta}(\hat{x})\hat{p}M^{\gamma}(\hat{x}) + M^{\gamma}(\hat{x})\hat{p}M^{\beta}(\hat{x})\hat{p}M^{\alpha}(\hat{x})] + V(\hat{x}). \tag{3.1}$$

Here the parameters α, β and γ are required to satisfy the condition

$$\alpha + \beta + \gamma = -1 \tag{3.2}$$

in order to ensure dimensional correctness of the PDM term while the potential term is given by (2.7). Then in the coordinate representation with $\hat{p} = -i\hbar d/dx$ the Schrödinger equation

$$H\psi = E\psi \tag{3.3}$$

implies

$$(E-V(x))\psi(x) = -\frac{\hbar^2}{2M(x)}\left[\psi'' - \frac{M'}{M}\psi' + \frac{\beta+1}{2}\left(2\frac{M'^2}{M^2} - \frac{M''}{M}\right)\psi + \alpha(\alpha+\beta+1)\frac{M'^2}{M^2}\psi\right], \tag{3.4}$$

where the $'$ denotes differentiation with respect to the argument x. Using (2.3) the last equation has the appearance

$$-\frac{2}{\hbar^2}(E-V(x))\psi(x)e^{2F(x)} = \psi''(x)-2f(x)\psi'(x)+[(\beta+1)(2f^2(x)-f'(x))+4\alpha(\alpha+\beta+1)f^2(x)]\psi(x). \tag{3.5}$$

Next using point canonical transformation we assume that the wave function $\psi(x)$ is of the form

$$\psi(x) = w(x)G(u(x)), \tag{3.6}$$

such that G satisfies the second-order ODE

$$\frac{d^2G}{du^2} + Q(u)\frac{dG}{du} + R(u)G(u) = 0. \tag{3.7}$$

Substituting (3.6) in (3.5) and comparing with (3.7) leads to the identifications

$$Q(u) = \frac{u''}{u'^2} + \frac{2w' - 2fw}{wu'}, \tag{3.8}$$

and

$$u'^2R(u) = \frac{2}{\hbar^2}(E-V)e^{2F(x)} + \frac{w'' - 2fw'}{w} + (\beta+1)(2f^2 - f') + 4\alpha(\alpha+\beta+1)f^2. \tag{3.9}$$

From (3.8) it follows that

$$w(x) = (u')^{-1/2}e^{F(x)}e^{\frac{1}{2}\int Q(u)du}. \tag{3.10}$$

A simple way of finding the unknown function $u(x)$ has been first proposed by Bhattacharjie and Sudarshan [6]. A particular choice of the special function $G(u)$ provides the complete functional forms of the first two unknowns $Q(u)$ and $R(u)$. A specific choice of the special function $G(u)$ and a clever choice of $u(x)$ make the Schrödinger equation an exactly solvable potential $V(x)$. Using the expression for $w(x)$ from (3.10) to simplify (3.9) we finally arrive at

$$\frac{2}{\hbar^2}(E-V)e^{2F(x)} = \frac{u'''}{2u'} - \frac{3}{4}\left(\frac{u''}{u'}\right)^2 + u'^2\left[R(u) - \frac{1}{2}Q'(u) - \frac{Q^2}{4}\right] + \beta f' - (2\beta+1+4\alpha(\alpha+\beta+1))f^2. \tag{3.11}$$

This equation is central to our present analysis. It is clear that the choice of the second-order ODE in (3.7) must be such that its coefficients $R(u)$ and $G(u)$, which appear in (3.11), together with their argument $u = u(x)$ cause the right hand side to be have a term proportional to $e^{2F(x)}$; whose coefficient can then to identified with the energy eigenvalue occurring on the left. The remaining terms, depending on the variable x, can then be said to represent the potential function $V(x)$. In general this expression involves both the functions $f(x)$ and $g(x)$ since the latter occurs explicitly in the definition of the potential function (2.7). For an appropriate choice of $u(x)$ additional terms proportional to f^2 and f' may be generated from its derivatives so that the expression for the potential resulting from (3.11) can be simplified by making suitable choices of the parameters α, β and γ so as to ensure that the coefficients of f^2 and f' vanish. Note that the choice of the parameters α, β and γ, which are often called the ambiguity parameters, is not unique. Several possibilities have been explored in the literature such as those of Gora and Williams $(\beta = \gamma = 0, \alpha = -1)$, BenDaniel and Duke $(\alpha = \gamma = 0, \beta = -1)$, Zhu and Kroemer $(\alpha = \gamma = -1/2, \beta = 0)$, Li and Kuhn $(\beta = \gamma = -1/2, \alpha = 0)$ and Bastard $(\alpha =$

$-1, \beta = \gamma = 0$) [21, 5, 47, 27]. In our case however the choice of these parameters is dictated more by our endeavour to map the Liénard II equation to an exactly solvable quantum mechanical problem, in particular to an exactly solvable potential. Moreover as our motivation is based on exploiting an appropriate linear ordinary differential equation for this purpose the values of the ambiguity parameters follow as a consequence of this requirement instead of being assigned *ab intio*.

3.1 The linear harmonic oscillator and the Jacobi last multiplier

In order to illustrate these possibilities we consider first of all the case when (3.7) is taken to be the Hermite differential equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2ny = 0, \quad n = 0, 1, 2, ... \tag{3.12}$$

so that we may identify

$$Q(u) = -2u, \quad R(u) = 2n, \quad n \in \mathbb{N}_0. \tag{3.13}$$

Furthermore demanding that

$$u' = \frac{1}{\sqrt{\hbar}} e^{F(x)} \quad \text{which implies} \quad u(x) = \frac{1}{\sqrt{\hbar}} \int^x e^{F(s)} ds, \tag{3.14}$$

we find that

$$\frac{2}{\hbar^2}(E-V)e^{2F(x)} = \frac{e^{2F(x)}}{\hbar}[2n+1-u^2]+(\beta+\frac{1}{2})f'-(2\beta+\frac{5}{4}+4\alpha(\alpha+\beta+1))f^2. \tag{3.15}$$

Choosing the coefficients of f' and f^2 to be zero yields the following values of the ambiguity parameters:

$$\alpha = -\frac{1}{4}, \quad \beta = -\frac{1}{2}, \quad \gamma = -\frac{1}{4}. \tag{3.16}$$

Next equating the constant terms in (3.16) implies that the energy eigenvalues are given by

$$E_n = \left(n + \frac{1}{2}\right) \hbar, \quad n = 0, 1, 2, ... \tag{3.17}$$

It remains to obtain the expression for the potential which is given by

$$V(x) = \frac{\hbar}{2} u^2 = \frac{1}{2} \left(\int^x \sqrt{M(s)} ds \right)^2. \tag{3.18}$$

Each integrable function $M(x) \geq 0$ defines a point canonical transformation from the variable x onto a new variable u by the formula

$$u = q(x) := \int \sqrt{M(x)} dx. \tag{3.19}$$

The inverse transformation is also well defined. Thus one essentially has a quadratic potential in terms of the new coordinate as was expected for a harmonic oscillator potential. The choice of $M(x) = \lambda {u'}^2$ has been used in various places (for example, [4, 20]) where λ is a constant parameter.

The new coordinate there naturally fixes the nature of the function $g(x)$ because from (2.7) we have

$$V(x) = \int M(x)g(x)dx = \frac{1}{2}(\int \sqrt{M(x)}dx)^2$$

which determine

$$g(x) = \frac{1}{\sqrt{M(x)}} \int \sqrt{M(x)}dx, \tag{3.20}$$

entirely in terms of the JLM or PDM term $M(x)$.

We conclude this section by noting that if the parameters are such that the coefficients of f' and f^2 do not vanish independently then there exists an alternative possibility in which the hitherto arbitrary function $f(x)$ is assumed to be such that

$$(\beta + \frac{1}{2})f' - (2\beta + \frac{5}{4} + 4\alpha(\alpha + \beta + 1))f^2 = 0$$

This is a first-order Riccati equation and its solution is given by

$$f(x) = \frac{\lambda}{(x+\delta)}, \quad \text{where } \lambda = -\frac{\beta + \frac{1}{2}}{2\beta + \frac{5}{4} + 4\alpha(\alpha + \beta + 1)}, \tag{3.21}$$

δ being the constant of integration. Since the JLM, $M(x) = e^{2F(x)}$, we have from (2.5) and (3.21) $M(x) = (x + \delta)^{2\lambda}$. We list below in Table I the explicit values of $M(x)$ for some of the rational values of the ambiguity parameter existing in the literature:

Table I: List of values of the ambiguity parameters
and the associated form of the JLM.

Model	Ambiguity parameters.	λ	JLM $M(x)$.
Gora and Williams	$\alpha = -1, \beta = \gamma = 0$	-2/5	$(x+\delta)^{-4/5}$
Daniel and Duke	$\alpha = \gamma = 0,\ \beta = -1$	$-2/3$	$(x+\delta)^{-4/3}$
Li and Kuhn	$\beta = \gamma = -1/2,\ \alpha = 0$	0	1
Zhu and Kromer	$\alpha = \gamma = -1/2,\ \beta = 0$	-2	$(x+\delta)^{-4}$

3.2 Associated Laguerre Equation

The associated Laguerre equation is given by

$$x\frac{d^2L_n^\mu}{dx^2}+(\mu+1-x)\frac{dL_n^\mu}{dx}+nL_n^\mu=0,\quad \mu>-1,\ n=0,1,2,\ldots. \tag{3.22}$$

The associated Laguerre polynomials are orthogonal over $[0,\infty)$ with respect to the measure with weighting function $x^\mu e^{-x}$

$$\int_0^\infty x^\alpha e^{-x}L_n^{(\mu)}(x)L_m^{(\mu)}(x)dx=\frac{\Gamma(n+\mu+1)}{n!}\delta_{n,m},$$

A comparison of (3.22) with (3.7) reveals that in this case

$$Q(u)=\frac{\mu+1-u}{u},\quad Q'(u)=-\frac{\mu+1}{u^2}\ \text{ and }\ R(u)=\frac{n}{u}. \tag{3.23}$$

Furthermore from (3.11) it is found that

$$\frac{2}{\hbar^2}(E-V)e^{2F(x)}=\frac{u'''}{2u'}-\frac{3}{4}\left(\frac{u''}{u'}\right)^2+u'^2\left[\frac{2n+1+\mu}{2u}-\frac{1}{4}+\frac{(1+\mu)(1-\mu)}{4u^2}\right]+K(\alpha,\beta,f,f'), \tag{3.24}$$

where $K(\alpha,\beta,f,f')=\beta f'(x)-(2\beta+1+4\alpha(\alpha+\beta+1))f^2(x)$. Suppose we choose now $u(x)$ and E such that

$$\frac{2}{\hbar^2}Ee^{2F(x)}=(2n+1+\mu)\frac{u'^2}{2u}$$

with

$$E_{n,\mu}=\left(n+\frac{1+\mu}{2}\right)\hbar,\quad n=0,1,2,\ldots. \tag{3.25}$$

This requires that

$$\frac{u'^2}{2u}=\frac{e^{2F(x)}}{\hbar}\quad\Rightarrow\quad u(x)=\frac{1}{2\hbar}\left(\int^x e^{F(s)}ds\right)^2,\quad 0<u(x)<\infty, \tag{3.26}$$

and therefore serves to define the function (diffeomorphism) $u(x)$. The corresponding wave function is

$$\psi_n^\mu(x)=\text{const.}e^{F(x)/2}\left(\int e^{F(x)}dx\right)^{\mu+\frac{1}{2}}e^{-\frac{1}{4\hbar}(\int e^{F(x)}dx)^2}L_n^\mu(u(x)). \tag{3.27}$$

Now from (3.24)-(3.26) it follows that

$$\frac{2}{\hbar^2}V(x)=\frac{1}{4\hbar^2}\left(\int^x e^{F(s)}ds\right)^2+\left[\frac{3}{4}-(1+\mu)(1-\mu)\right]\frac{1}{\left(\int^x e^{F(s)}ds\right)^2}, \tag{3.28}$$

upon demanding

$$\beta f' - (2\beta + 1 + 4\alpha(\alpha + \beta + 1))f^2 + \frac{1}{2}f' - \frac{1}{4}f^2 = 0. \tag{3.29}$$

As before the last equation may be dealt with in two ways: one can demand the coefficients of f^2 and f' to vanish, which leads to [12]

$$\beta = -\frac{1}{2}, \quad \alpha = -\frac{1}{4} = \gamma, \tag{3.30}$$

in view of the constraint $\alpha + \beta + \gamma = 1$ or else we could keep α and β arbitrary and fix the function $f(x)$ as a solution of (3.29) as already explained in subsection 3.1. The former procedure however has the advantage of allowing us to choose the form of the function $f(x)$ while at the same time fixing the structure of the Van Roos Hamiltonian (3.1).

3.2.1 Reduction to the isotonic oscillator

The spectrum of the isotonic oscillator was shown by Goldman and Krivchenkov [16, 44] to be isomorphous to the linear harmonic oscillator, in the sense that it consists of an infinite set of equispaced energy levels. The reduction to the isotonic oscillator follows upon setting $f(x) = 0$ which naturally implies $F(x) = 0$. Using this in (3.28) we see that

$$\frac{2}{\hbar^2}V(x) = \frac{x^2}{4\hbar^2} + \frac{(\mu + \frac{1}{2})(\mu - \frac{1}{2})}{x^2}.$$

Setting $\mu = l + 1/2$ then leads to the isotonic potential [9]

$$V(x) = \frac{x^2}{8} + \frac{l(l+1)\hbar^2}{2x^2}, \tag{3.31}$$

with the energy eigenvalue being given by

$$2E_{n,l} = (2n + l + \frac{3}{2})\hbar, \quad n = 0, 1, 2, ... \tag{3.32}$$

The corresponding wave function is

$$\psi_n^{(l)}(x) = C_n^{(l)} x^{l+1} e^{-\frac{x^2}{4\hbar}} L_n^l(u), \quad u(x) = \frac{x^2}{2\hbar} \tag{3.33}$$

where $C_n^{(l)}$ is the normalization factor. The results contained in (3.31)-(3.33) are essentially the same as those derived in [37] upon scaling $V(x)$ and $E_{n,l}$ by a factor of $1/2$ and setting the oscillator frequency to be unity.

3.3 Exceptional Laguerre Equation

The exceptional X_1-Laguerre differential equation is given by [37, 36, 34]

$$-xy'' + \left(\frac{x-a}{x+a}\right)\left[(a+x+1)y' - y\right] - (n-1)y = 0, \quad n = 1,2,3..., \tag{3.34}$$

with $a > 0$ being real. Its polynomial solutions, $\hat{L}_n^{(a)}(x)$, are called the exceptional X_1-Laguerre polynomials and are orthogonal with respect to the rational weight $\hat{w}(x,a) = e^{-x}x^a/(x+a)^2$ with normalization given by

$$\int_0^\infty \frac{e^{-x}x^a}{(x+a)^2}\hat{L}_n^{(a)}(x)\hat{L}_m^{(a)}(x)dx = \frac{(a+n)\Gamma(a+n-1)}{(n-1)!}\delta_{nm}. \tag{3.35}$$

Comparison with (3.7) shows that, in case of the exceptional Laguerre polynomials, the functions $Q(u)$ and $R(u)$ are

$$Q(u) = -\frac{(u-a)(u+a+1)}{u(u+a)}, \quad R(u) = \frac{1}{u}\left(\frac{u-a}{u+a} + n - 1\right). \tag{3.36}$$

With these forms one obtains the following simplified expression for the quantity Z defined below

$$Z := R(u) - \frac{1}{2}Q'(u) - \frac{Q^2}{4} = \left[\frac{2an + a^2 - a + 2}{2au} - \frac{1}{a(u+a)} - \frac{a^2-1}{4u^2} - \frac{2}{(u+a)^2} - \frac{1}{4}\right],$$

so that (3.11) now becomes

$$\frac{2}{\hbar^2}(E-V)e^{2F(x)} = \frac{u'''}{2u'} - \frac{3}{4}\left(\frac{u''}{u'}\right)^2 + u'^2 Z + (\beta+1)f' - (2\beta + 1 + 4\alpha(\alpha+\beta+1))f^2. \tag{3.37}$$

Setting

$$\frac{u'^2}{u} = \frac{2}{\hbar}e^{2F(x)}, \quad \text{we obtain} \quad u(x) = \frac{1}{2\hbar}\left(\int^x e^{F(s)}ds\right)^2. \tag{3.38}$$

With this form of $u(x)$ we find upon equating the constant term on either side of (3.37) that the eigenvalues are

$$E_{n,a} = \frac{2na + 2 + a^2 - a}{2a}\hbar, \quad n = 1,2,3,... \tag{3.39}$$

Consequently the expression for the potential becomes

$$V(x) = -\frac{\hbar^2}{2}e^{-2F(x)}\left[\frac{u'''}{2u'} - \frac{3}{4}\left(\frac{u''}{u'}\right)^2 - \frac{u'^2}{a(u+a)} - \frac{a^2-1}{4}\left(\frac{u'}{u}\right)^2 - \frac{2u'^2}{(u+a)^2} - \frac{u'^2}{4}\right]$$
$$-\frac{\hbar^2}{2}e^{-2F(x)}[\beta f' - (2\beta + 1 + 4\alpha(\alpha+\beta+1))f^2]. \tag{3.40}$$

Using (3.39) the simplified expression for the potential is then given by

$$V(x) = \left[\frac{(a-\frac{1}{2})(a+\frac{1}{2})\hbar^2}{2\left(\int^x e^{F(s)}ds\right)^2} + \frac{1}{8}\left(\int^x e^{F(s)}ds\right)^2 + \frac{\hbar^2}{a\hbar}\frac{\left(\int^x e^{F(s)}ds\right)^2 + 6a\hbar}{\left(\left(\int^x e^{F(s)}ds\right)^2 + 2a\hbar\right)^2} - \frac{\hbar^2}{2}e^{-2F(x)}S\right] \tag{3.41}$$

with

$$S(\alpha,\beta,f',f) = (\beta+\frac{1}{2})f' - (2\beta+\frac{5}{4}+4\alpha(\alpha+\beta+1))f^2,$$

so that setting the coefficients of f' and f^2 to be zero we find once again the following values of the ambiguity parameters: $\beta = -1/2$ and $\alpha = \gamma = -1/4$.
Consequently the Von Roos Hamiltonian has the following structure

$$H(\hat{x},\hat{p}) = \frac{1}{2}\left[M^{\alpha}(\hat{x})\hat{p}M^{\beta}(\hat{x})\hat{p}M^{\alpha}(\hat{x})\right] + V(\hat{x}) \tag{3.42}$$

with the potential given by

$$V(x) = \left[\frac{(a-\frac{1}{2})(a+\frac{1}{2})\hbar^2}{2\left(\int^x e^{F(s)}ds\right)^2} + \frac{1}{8}\left(\int^x e^{F(s)}ds\right)^2 + \frac{\hbar^2}{a\hbar}\frac{\left(\int^x e^{F(s)}ds\right)^2 + 6a\hbar}{\left(\left(\int^x e^{F(s)}ds\right)^2 + 2a\hbar\right)^2}\right]. \tag{3.43}$$

It is interesting to note that this potential inherits the isotonic character as evidenced from the first two terms, which arose previously as a result of our use of the associated Laguerre differential equation (see eqn. (3.28)); the remaining rational term seems to be the extra contribution of the X_1-Laguerre differential equation.

The general form of the wave function now reads

$$\psi_n(x) = \text{const.}(u')^{-1/2}e^{F(x)}e^{u/2}(u+a)u^{-(a+1)/2}\hat{L}_n^{(a)}(u), \tag{3.44}$$

where we have made use of (3.6), (3.10) and (3.36) for the X_1-Laguerre differential equation with $u(x)$ being defined by (3.38).
Finally let us consider an example in which we assume that the function [41]

$$e^{F(x)} = \sqrt{M(x)} = \frac{\nu + x^2}{1+x^2},$$

with ν being a constant. It follows that

$$\int^x e^{F(s)}ds = (\nu-1)\arctan x + x$$

and as a consequence the potential has the form

$$V(x) = \left[\frac{(a-\frac{1}{2})(a+\frac{1}{2})\hbar^2}{2\left((\nu-1)\arctan x + x\right)^2} + \frac{1}{8}\left((\nu-1)\arctan x + x\right)^2 + \frac{\hbar^2}{a\hbar}\frac{\left((\nu-1)\arctan x + x\right)^2 + 6a\hbar}{\left(\left((\nu-1)\arctan x + x\right)^2 + 2a\hbar\right)^2}\right]$$

4 The exceptional Jacobi equation

The X_1-Jacobi polynomials $\hat{P}_n^{(a,b)}(x)$, with $n = 1, 2, ...$ and $a, b > -1$ $(a \neq b)$ are the solutions of the second-order ODE

$$\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

with

$$Q(x) = -\frac{(b+a+2)x-(b-a)}{1-x^2} - \frac{2(b-a)}{(b-a)x-(b+a)}, \tag{4.1}$$

$$R(x) = -\frac{(b-a)x-(n-1)(n+b+a)}{1-x^2} - \frac{(b-a)^2}{(b-a)x-(b+a)}. \tag{4.2}$$

In order to make use of (3.11) we note that

$$X := R(u) - \frac{1}{2}Q' - \frac{1}{4}Q^2(u) = \frac{Cu+D}{1-u^2} + \frac{Gu+J}{(1-u^2)^2} + \frac{K}{(b-a)u-(b+a)} + \frac{L}{[(b-a)u-(b+a)]^2} \tag{4.3}$$

with [36]

$$C = \frac{(b-a)(b+a)}{2ab}, \tag{4.4}$$

$$D = n^2 + (b+a-1)n + \frac{1}{4}[(b+a)^2 - 2(b+a) - 4] + \frac{b^2+a^2}{2ab}, \tag{4.5}$$

$$G = \frac{(b-a)(b+a)}{2}, \quad J = -\frac{1}{2}(b^2+a^2-2), \tag{4.6}$$

$$K = \frac{(b-a)^2(b+a)}{2ab}, \quad L = -2(b-a)^2. \tag{4.7}$$

Below we explore certain possibilities stemming from specific choices of the function $u(x)$.

4.1 The Scarf-I potential

In the first case we derive the previous result of the Scarf-I potential given in [36]. For this purpose we set

$$u'e^{-F(x)} = \lambda\sqrt{1-u^2}, \quad \lambda = \text{const}. \tag{4.8}$$

then it immediately follows that

$$u(x) = \sin\theta, \quad \theta := (\lambda\int^x e^{F(s)}ds). \tag{4.9}$$

As a result (3.11) becomes

$$2(E-V(x)) = \frac{\lambda^2}{4} - \frac{3\lambda^2}{4}\sec^2\theta + e^{-2F(x)}\left[(\beta+\frac{1}{2})f' - (2\beta+\frac{5}{4}+4\alpha(\alpha+\beta+1))f^2\right]$$

$$+\lambda^2\left[Cu+D+\frac{Gu+J}{1-u^2}\right] + \lambda^2(1-u^2)\left[\frac{K}{(b-a)u-(b+a)} + \frac{L}{[(b-a)u-(b+a)]^2}\right]. \quad (4.10)$$

In order to simplify this expression we may set the coefficients of f' and f^2 to be zero which once again yield the values given in (3.16) for the ambiguity parameters. Next equating the coefficients of the constant terms on both sides we find after defining the change of parameters:

$$a = A - B - \frac{1}{2}, \;\; b = A + B - \frac{1}{2} \;\; \Rightarrow \;\; A = \frac{1}{2}(b+a+1), \;\; B = \frac{1}{2}(b-a),$$

that the eigenvalue may be expressed as

$$2E = \lambda^2(n-1+A)^2, \quad n = 1, 2, 3, ... \quad (4.11)$$

which upon scaling, ($\lambda^2 = 2$), can be simply written as

$$E_\nu = (\nu + A)^2, \;\; \nu = 0, 1, 2, ...$$

On the other hand the potential $V(x)$ can be expressed as

$$V(x) = V_1(x) + V_2(x)$$

where

$$V_1(x) = [A(A-1)+B^2]\sec^2\theta - B(2A-1)\sec\theta\tan\theta, \quad (4.12)$$

$$V_2(x) = \frac{2(2A-1)}{[2B\sin\theta-(2A-1)]} + \frac{2[(2A-1)^2 - 4B(2A-1)\sin\theta + 4B^2]}{[2B\sin\theta-(2A-1)]^2}. \quad (4.13)$$

The potential $V_1(x)$ represents the Scarf-I potential with the value of θ usually restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. However, as $\theta = 2\int^x e^{F(s)}ds$ we have

$$0 < \int^x e^{F(s)}ds < \frac{\pi}{4}.$$

Notice that as, $\theta \to \pm\frac{\pi}{2}$, the second potential i.e., $V_2(x)$ approaches a constant value so that overall the potential $V(x)$ behaves as a Scarf-I potential. The explicit form of the wave function follows from (3.6) and (3.10) and is given by

$$\psi_\nu(x) = N_\nu \frac{e^{F(x)/2}}{\sqrt{2}} \frac{(1-\sin\theta)^{\frac{A-B}{2}}(1+\sin\theta)^{\frac{A+B}{2}}}{[2B\sin\theta-(2A-1)]} P_{\nu+1}^{(A-B-\frac{1}{2}, A+B-\frac{1}{2})}(\sin\theta). \quad (4.14)$$

Up to this point we have basically reproduced the essential results of [36] regarding the exceptional Jacobi polyomials and it follows that the normalization factor is given, in this case, by

$$\frac{N_\nu}{2} = \frac{B}{2^{A-2}} \left(\frac{\nu!(2\nu+2A)\Gamma(\nu+2A)}{(\nu+A-B+\frac{1}{2})((\nu+A+B+\frac{1}{2})\Gamma((\nu+A-B+\frac{1}{2})\Gamma((\nu+A+B+\frac{1}{2})} \right)^{1/2}. \tag{4.15}$$

4.2 Case II: A new potential

From (4.10) it will be observed that a second possibility exists which arises from the choice

$$u'e^{-F(x)} = \lambda(1-u^2), \tag{4.16}$$

and gives

$$u(x) = \tanh\theta(x), \tag{4.17}$$

where, as before, $\theta(x) = \lambda \int^x e^{F(s)ds}$. It now follows from (3.11) that

$$2(E - V(x)) = e^{-2F(x)} \left[-\lambda^2 e^{2F(x)} + (\beta + \frac{1}{2})f' - (2\beta + \frac{5}{4} + 4\alpha(\alpha+\beta+1))f^2 \right]$$

$$+ \lambda^2 \left[(Cu+D)(1-u^2) + (Gu+J) + \frac{K(1-u^2)^2}{(b-a)u-(b+a)} + \frac{L(1-u^2)^2}{[(b-a)u-(b+a)]^2} \right]. \tag{4.18}$$

The choice $\beta = -1/2, \alpha = \gamma = -1/4$ causes the coefficients of f' and f^2 to vanish and upon equating the coefficient of the constant term (after setting $\lambda^2 = 2$) we obtain the energy eigenvalue as

$$E_\nu(A,\delta) = \left[(\nu+A)^2 - \frac{1}{4} \right] - \left(1 + \frac{1}{\delta^2} \right) \left(A - \frac{1}{2} \right)^2 + 2(1-2\delta^2), \quad \nu = 0,1,2... \tag{4.19}$$

In arriving at this expression we have made use of the definitions (4.4)-(4.7) and have redefined the parameters a and b by the following

$$A = \frac{1}{2}(b+a+1), \quad \delta = \frac{b+a}{b-a}, \quad \nu = n-1.$$

As $a \neq b$ therefore the presence of the factor $(b-a)$ in the denominators is not a cause of undue concern. Unlike the previous case we note here the presence of both the parameters a and b (or alternatively A and δ) in the expression for the energy eigenvalue. The explicit form of the corresponding potential is

$$V(x) = U_1(x) + U_2(x)$$

where

$$U_1(x) = \left[(\nu + A)^2 - \frac{1}{4}\right] \tanh^2\theta + \left[\frac{1}{2\delta}(2A-1)^2 - 2\delta\right] \tanh\theta \tag{4.20}$$

$$U_2(x) = \frac{6\delta(1-\delta^2)}{(\tanh\theta - \delta)} - \frac{2(1-\delta^2)^2}{(\tanh\theta-\delta)^2}. \tag{4.21}$$

The corresponding wave function is

$$\psi_\nu(x) = N_\nu \frac{e^{F(x)/2}}{2^{1/4}} \frac{(1+\tanh\theta)^{\frac{1}{2}(A+B-\frac{1}{2})}(1-\tanh\theta)^{\frac{1}{2}(A-B-\frac{1}{2})}}{[2B\tanh\theta - (2A-1)]} P_{\nu+1}^{(A-B-\frac{1}{2}, A+B-\frac{1}{2})}(\tanh\theta), \tag{4.22}$$

whence the normalization factor follows from the requirement

$$\frac{|N_\nu|^2}{2} \int dy \frac{(1-\sin y)^{A-B-1}(1+\sin y)^{A+B-1}}{[2B\sin y - (2A-1)]^2} \left[P_{\nu+1}^{(A-B-\frac{1}{2}, A+B-\frac{1}{2})}(\sin y)\right]^2 = 1, \tag{4.23}$$

where we have made the change of variables, $\tanh\theta = \sin y$, keeping in mind that $\theta(x) = \sqrt{2}\int^x e^{F(s)}ds$.

4.3 Case III: Another new potential

A third possibility consists in setting

$$u'(x) = \lambda e^{F(x)}[(b-a)u - (b+a)], \tag{4.24}$$

which implies $u(x) = \delta + e^\theta$ where $\delta = (b+a)/(b-a)$ and $\theta = \lambda(b-a)\int^x e^{F(s)}ds$. Upon substitution in (3.11) and using (4.3) the energy eigenvalue after simplification has the following form

$$\frac{2E}{\lambda^2} = -(D + \frac{1}{4})(b-a)^2 + 2C(b^2 - a^2) + L - K(b+a).$$

Using the values of the constants C, D, L and K as stated in (4.4)-(4.7) to simplify this expression we obtain finally

$$E_\nu = -\frac{\lambda^2}{2}(b-a)^2(\nu + A)^2, \quad \nu = 0, 1, 2, ... \tag{4.25}$$

where as before $A = (b + a + 1)/2$. On the other hand from the non constant terms it follows that the potential is a rational function of $u(x)$ and is given by

$$V(x) = -\frac{\lambda^2(b-a)^2}{2(1-u^2)^2}\left[B_0 + B_1 u + B - 2u^2 + B_3 u^3\right], \tag{4.26}$$

with

$$B_0 = J\delta^2 + w_2, \quad B_1 = G\delta^2 - 2\delta J + w_1, \tag{4.27}$$

$$B_2 = J - w_2 - 2\delta G, \quad B_3 = G - w_1, \tag{4.28}$$

$$w_1 = C(\delta^2 + 1) - 2\delta D, \quad w_2 = D(\delta^2 + 1) - 2\delta C. \tag{4.29}$$

Once again the values of the constants appearing in the above equations are explicitly given in terms of the parameters a and b by eqns.(4.4)-(4.7).

5 Conclusion

The purpose of this survey is twofold; firstly, to unveil the contribution of the Jacobi last multiplier to the study of exactly solvable position-dependent mass models, and secondly, to describe a procedure for quantization of the Liénard type II equation. With regard to the first we have shown here how the JLM can be used to express the Hamiltonian of the Liénard II as an exactly solvable position-dependent mass systems. This leads us naturally to our second goal in which we have proposed an adaption of the techniques for exactly solvable systems to quantize the Liénard type II equation. The eigenfunctions are obtained in terms of associated Laguerre, exceptional Laguerre functions and the exceptional Jacobi polynomials.
Acknowledgement. We wish to thank Barnana Roy, Bijan Bagchi and Prasanta Panigrahi for enlightening discussions and constant encouragement. We are grateful to Andreas Fring for his interest and various exchange of scientific ideas over emails. A part of this work was done while one of us (PG) was visiting IHES. PG wishes to express his gratitude to the members of IHES for their warm hospitality.

References

[1] F. Arias de Saavedra, J. Boronat, A. Polls and A. Fabrocini *Effective mass of one 4He atom in liquid 3He* , Phys. Rev B **50** (1994) 4248.

[2] B. Bagchi, S. Das, S. Ghosh and S. Poria, *Nonlinear dynamics of a position-dependent mass driven Duffing-type oscillator*, (2013) J. Phys. A Math: Theor. **46** 032001. MR3007509. Zbl 1273.81102.

[3] B. Bagchi, *Position-dependent mass models and their nonlinear characterization* J. Phys. A: Math. Theor. **40** (2007) F1041–F1045 (arXiv:0706.0607 [quant-ph]). MR2441860(2009k:81070). Zbl 1127.81007.

[4] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury, *New approach to (quasi)-exactly solvable Schrödinger equations with a position-dependent effective mass* Mod. Phys. Lett. A **19** (2004) 2765. MR2217356(2006k:81399).

[5] D. J. Ben Daniel and C. B. Duke, *Space-charge effects on electron tunneling,* Phys. Rev B **152**(2) (1966) 683-692.

[6] A. Bhattacharjie and E. C. G. Sudarshan, *A class of solvable potentials*, Nuovo Cimento **25** (1962) 864. MR0152289(27 #2269). Zbl 0361.46061.

[7] S. Bochner, *Über Sturm-Liouvillsche Polynomsysteme*, Math. Z. **29** (1929) 730-736. MR1545034. JFM 55.0260.01.

[8] J. F. Cariñena, M. F. Rañada and M. Santander, *A quantum exactly solvable non-linear oscillator with quasi-harmonic behaviour*, Ann. Phys. **322** (2007) 434-59. (arXiv:math-ph/0604008) MR2289586(2008a:81244).

[9] J. F. Cariñena, A. M. Perelomov, M. F. Rañada and M. Santander, *A quantum exactly solvable nonlinear oscillator related to the isotonic oscillator*, J. Phys. A: Math. Theor. **41** (2008) 085301. MR2451306(2009f:81060). Zbl 1138.81380.

[10] S. Cruz, Y. Cruz and O. Rosas-Ortiz, *Position-dependent mass oscillators and coherent states*, J. Phys. A: Math. Theor. **42**, 185205 (2009). MR2591199(2011a:81127). Zbl 1162.81388.

[11] R. De, R. Dutt and U. Sukhatme *Mapping of shape invariant potentials under point canonical transformations*, J. Phys. A: Math. Gen. **25** (1992) L843-L850. MR1172067(93k:81038).

[12] A de Souza Dutra and C A S Almeida, *Exact solvability of potentials with spatially dependent effective masses*, Phys Lett. A **275** (2000) 25. MR1810988(2001k:81354). .

[13] M. R. Geller and W. Kohn, *Quantum mechanics of electrons in crystals with graded composition*, Phys. Rev. Lett. **70** (1993) 3103.

[14] A. Ghose Choudhury and Partha Guha, *Quantization of the Liénard II equation and Jacobi's last multiplier*, J. Phys. A: Math. Theor. **46** 165202 (2013). Zbl 1276.81078.

[15] A. Ghose Choudhury, P. Guha and B. Khanra, *On the Jacobi last multiplier, integrating factors and the Lagrangian formulation of differential equations of the Painlevé-Gambier classification*, J. Math. Anal. Appl. **360** (2009), no. 2, 651–664. MR2561261(2011c:34224). Zbl 1183.34138.

[16] I. I. Goldman and V. D. Krivchenkov, *Problems in quantum mechanics* Pergamon, London 1961. MR0129305(23 #B2341). Zbl 0094.23407.

[17] D. Gómez-Ullate N. Kamran and R. Milson, *Quasi-exact solvability in a general polynomial setting* Inverse Problems **23** (2007) 1915–1942, (arXiv:nlin/0610065 [nlin.SI]) MR2353321(2008g:81107). Zbl 1129.34056.

[18] D. Gómez-Ullate, N. Kamran and R. Milson, *An extension of Bochner's problem: exceptional invariant subspaces* J. Approx. Theory **162** (2010) 987-1006, (arXiv:0805.3376 [math-ph]). MR2610341(2011d:33016). Zbl 1214.34079.

[19] D. Gómez-Ullate, N. Kamran and R. Milson, *An extended class of orthogonal polynomials defined by a Sturm-Liouville problem*, J. Math. Anal. Appl. **359** (2009) 352-67, (arXiv:0807.3939 [math-ph]). MR2542180(2010j:34064). Zbl 1183.34033.

[20] B. Gonul, B. Gonul, D. Tutcu and O. Ozer, *Supersymmetric approach to exactly solvable systems with position-dependent masses*, Mod.Phys.Lett. A **17** (2002) 2057. MR1940547(2003i:81083). Zbl 1083.81525.

[21] T. Gora and F. Williams, *Theory of electronic states and transport in graded mixed semiconductors*, Phys. Rev, **177**(3)(1969) 1179-1182.

[22] C. G. J. Jacobi, *Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica,* Giornale Arcadico di Scienze, Lettere ed Arti **99** (1844), 129-146.

[23] C. G. J. Jacobi, *Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi,*J. Reine Angew. Math **27** (1844), 199-268, *Ibid* **29** (1845), 213-279 and 333-376. MR1578395. Zbl 027.0793cj.

[24] N. Kamran and P. J. Olver, *Lie algebras of differential operators and Lie-algebraic potentials*, J. Math. Anal. Appl. **145** (1990) 342-356. MR1038161(91b:34145). Zbl 0693.34021.

[25] G. Lévai and O. Özer, *An exactly solvable Schrödinger equation with finite positive position-dependent effective mass*, J. Math. Phys. **51** (2010) 092103. MR2742794(2011j:81087). Zbl 1309.81079.

[26] G. Lévai, *A search for shape-invariant solvable potentials* J. Phys. A: Math. Gen. **22** (1989) 689. MR0986845(90k:81260). Zbl 0687.35081.

[27] T. L. Li and K. J. Kuhn, *Band-offset ratio dependence on the effective-mass Hamiltonian based on a modified profile of the GaAs-AlxGa1-xAs quantum well*, Phys. Rev B **47** (1993) 12760.

[28] B. Midya and B. Roy, *A generalized quantum nonlinear oscillator*, J. Phys. A: Math. Theor. **42** (2009) 285301. MR2519734(2011d:81111). Zbl 1167.81372.

[29] B. Midya and B. Roy, *Exceptional orthogonal polynomials and exactly solvable potentials in position-dependent-mass Schrödinger Hamiltonians*, Phys. Lett. A **373** (2009) 4117-22. MR2552404(2010j:81301). Zbl 1234.81072.

[30] A. Mostafazadeh, *$\mathcal{PT}$-symmetric cubic anharmonic oscillator as a physical model*, J. Phys. A: Math. Gen. **38** (2005) 6557-70 (arXiv:quant-ph/0411137); MR2167044(2006e:81060a). Zbl 1072.81020. J. Phys. A: Math. Gen. 38 8185 (erratum). MR2169337(2006e:81060b).

[31] R. L. Norton *Cam design and Manufacturing Handbook*, Second Edition, Industrial Press, Inc. 2009.

[32] M. C. Nucci and P. G. L. Leach, *Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story,* J. Phys. A: Math. Gen. **37**, (2004), 7743-7753. MR2090011(2005g:70012). Zbl 1065.70007.

[33] S. Odake and R. Sasaki, *Infinitely many shape invariant potentials and new orthogonal polynomials*, Phys. Letts. B **679**, (2009) 414. MR2569488(2011b:81102).

[34] S. Odake and R. Sasaki, *Infinitely many shape-invariant potentials and cubic identities of the Laguerre and Jacobi polynomials* J. Math.Phys. **51**, (2010) 053513. MR2666991(2011k:81105). Zbl 1310.81059.

[35] A. Puente, L. Serra and M. Casas, *Dipole excitation of Na clusters with a non-local energy density functional*, Z Phys. D **31** (1994) 283.

[36] C. Quesne, J. Phys. A: Math. Theor. **41**, 392001 (2008). MR0470692. Zbl 0361.46061.

[37] C. Quesne, *Solvable Rational Potentials and Exceptional Orthogonal Polynomials in Supersymmetric Quantum Mechanics*, SIGMA **5** (2009), 084, 24pp. MR2559677 (2011c:81119). Zbl 1188.81071.

[38] C. Quesne, *Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry*, J. Phys. A: Math. Theor. **41** (2008) 392001. MR2439200(2010f:42054). Zbl 1192.81174.

[39] V. Chithiika Ruby, M. Senthilvelan and M. Lakshmanan, *Exact quantization of a PT-symmetric (reversible) Liénard-type nonlinear oscillator*, J. Phys. A: Math. Theor. **45** (2012) 382002. MR2971248. Zbl 1252.81055.

[40] M. Sabatini, *On the period function of $x'' + f(x)x'^2 + g(x) = 0$*. J. Differential Equations **196** (2004), no. 1, 151-168. MR2025190(2004i:34069). Zbl 1048.34068.

[41] A. G. M. Schmidt, *Time evolution for harmonic oscillators with position-dependent mass* Phys. Scr. **75** (2007) 480-483. MR2299232(2007m:81074).

[42] L. Serra and E. Lipparini, *Spin response of unpolarized quantum dots*, Europhys. Lett. 40 (1997) 667.

[43] O. Von Roos, *Position-dependent effective masses in semiconductor theory*, Phys. Rev. B **27** (1983) 7547-7552.

[44] Y. Weissman and J. Jortner, *The isotonic oscillator*, Phys. Letts. **A70** (1979) 177-179. MR0587593(81i:81021).

[45] E. Witten, *Dynamical breaking of supersymmetry* Nucl. Phys. B **188** (1981) 513-54. Zbl 1258.81046.

[46] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Cambridge University Press, Cambridge, 1988. MR0992404(90a:01112).

[47] Q. G. Zhu and H. Kroemer, *Interface connection rules for effective-mass wave-function at an abrupt heterojunction between two different semiconductors*, Phys. Rev B **27** (1983) 3519.

A. Ghose Choudhury
E-mail: aghosechoudhury@gmail.com
Department of Physics,
Surendranath College,
Mahatma Gandhi Road,
Calcutta-700009, India.

Partha Guha
E-mail: partha@bose.res.in
S. N. Bose National Centre for Basic Sciences,
JD Block, Sector III, Salt Lake,
Kolkata - 700098, India.

License