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M-STRONGLY SOLID MONOIDS OF GENERALIZED HYPERSUBSTITUTIONS OF TYPE $\tau = (2)$

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Abstract. The purpose of this paper is to characterize M-strongly solid monoids of generalized hypersubstitutions of type $\tau = (2)$ which is the extension of M-solid monoids of hypersubstitutions of the same type.

1 Introduction

The concept of a generalized hypersubstitution is a generalization of the concept of a hypersubstitution. It is used to study strong hyperidentities and strongly solid varieties. Firstly, we give briefly the concept of the monoid of all generalized hypersubstitutions.

Let $X := \{x_1, x_2, x_3, ...\}$ be a countably infinite set of symbols called *variables*. Let $(f_i)_{i \in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i*-ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity, written as $(n_i)_{i \in I}$ and is called a *type*.

An n*-ary term of type* τ is defined inductively as follows :

(i) The variables $x_1, x_2, ..., x_n$ are n-ary terms of type τ.

(ii) If $t_1, t_2, ..., t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, ..., t_{n_i})$ is an n-ary term of type τ.

By $W_\tau(X_n)$, we denote the smallest set which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii). Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ and is called *the set of all terms of type* τ.

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A *generalized hypersubstitution* of type $\tau = (n_i)_{i\in I}$ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define first the concept of a *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, ..., t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, .., t_m) := x_j$.

(iii) If $t = f_i(s_1, .., s_{n_i})$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), .., S^m(s_{n_i}, t_1, ..., t_m))$.

Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_i, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mapping and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, ..., x_{n_i})$. In [3], S. Leeratanavalee and K. Denecke proved that :

Proposition 1. ([3]) *For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\sigma[t], \sigma[t_1], ..., \sigma[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 2. ([3]) $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ *is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of* $\underline{Hyp_G(\tau)}$.

As usual, instead of $f(x, y)$ we write also xy.

Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. Let $t \in W_\tau(X_n)$ for $n \in \mathbb{N}$ and $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ be an algebra of type τ. The *n-ary term operation* $t^{\mathcal{A}} : A^n \to A$ of type τ is inductively defined by

(i) $t^{\mathcal{A}}(a_1, a_2, ..., a_n) := a_i$ if $t = x_i \in X_n$.

(ii) $t^{\mathcal{A}}(a_1, a_2, ..., a_n) := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, a_2, ..., a_n), .., t_{n_i}^{\mathcal{A}}(a_1, a_2, ..., a_n))$ if t is a compound term $f_i(t_1, t_2, ..., t_{n_i})$.

Let s, t be n-ary terms of type τ and $\mathcal{A}$ be an algebra of type $\tau = (n_i)_{i \in I}$. An *equation* of type τ is a pair (s,t) ; such pair are commonly written as $s \approx t$. The set of all equations of type τ is denoted by $E_\tau(X)$.

An equation $s \approx t$ is an *identity* of $\mathcal{A}$, denoted by $\mathcal{A} \models s \approx t$ if $s^{\mathcal{A}} = t^{\mathcal{A}}$.

Let K be a class of algebras of type τ. The class K satisfies an equation $s \approx t$, denoted by $K \models s \approx t$, if for every $\mathcal{A} \in K, \mathcal{A} \models s \approx t$.

Let Σ be a set of equations of type τ. The class K is said to satisfy Σ, denoted by $K \models \Sigma$, if $K \models s \approx t$ for every $s \approx t \in \Sigma$. Let

$$IdK := \{s \approx t \in E_\tau(X) | K \models s \approx t\},$$

$$Mod\Sigma := \{\mathcal{A} \in Alg(\tau) | \mathcal{A} \models \Sigma\}.$$

We denote the class of all algebras of type τ by $Alg(\tau)$. Let V be a nonempty subset of $Alg(\tau)$. V is called a *variety* if $V = ModIdV$.

Theorem 3. *A non-empty subset V of $Alg(\tau)$ is a variety if and only if $V = Mod\Sigma$ for some $\Sigma \subseteq E_\tau(X)$.*

Let M be a submonoid of $Hyp_G(\tau)$. An identity $s \approx t$ of a variety V of type τ is called an *M-strong hyperidentity* if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V for any $\sigma \in M$. If every identity satisfied in the variety V is an M-strong hyperidentity, we call the variety V be an *M-strongly solid*. A single semigroup S is called M-strongly solid if the variety $V(S)$ generated by S is M-strongly solid.

Definition 4. *Let M be a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$. M is said to be M-strongly solid if the reduct $(M; \circ_G)$ is M-strongly solid.*

2 M-strongly solid submonoids of $Hyp_G(2)$ which M is implied to $\{\sigma_{id}\}$

Throughout this paper, we restrict ourselves to study on the type $\tau = (2)$. Let f be a binary operation symbol. By σ_t we denote the generalized hypersubstitution which maps f to the term t in $W_{(2)}(X)$. Let $\mathbb{O}^+$ and $\mathbb{E}^+$ be the set of all positive odd integers and the set of all positive even integers, respectively. For $s \in W_{(2)}(X)$ and $2 < m \in \mathbb{N}$ we denote :

$$
\begin{array}{rcl}
s^d & := & \text{the dual term of } s \text{ obtained by rearranging all variables} \\
 & & \text{occurring in } s \text{ from right to left}, \\
s' & := & \text{the term obtained by interchanging of } x_1 \text{ and } x_2 \text{ occurring} \\
 & & \text{in } s, \\
s^* & := & \text{the term obtained from } s \text{ by replacing of letter } x_1 \text{ by } x_m, \\
s^{**} & := & \text{the term obtained from } s \text{ by replacing of letter } x_2 \text{ by } x_m, \\
var(s) & := & \text{the set of all variables occurring in } s, \\
\ell(s) & := & \text{the length of } s, \\
leftmost(s) & := & \text{the first variable (from the left) occurring in } s, \\
rightmost(s) & := & \text{the last variable occurring in } s, \\
W_{x_1} & := & \{s \in W_{(2)}(X) | var(s) = \{x_1\}\}, \\
W_{x_2} & := & \{s \in W_{(2)}(X) | var(s) = \{x_2\}\}, \\
W & := & \{s \in W_{(2)}(X) | x_1, x_2 \notin var(s)\}, \\
W^G_{(2)}(\{x_1\}) & := & \{s \in W_{(2)}(X) | x_1 \in var(s), x_2 \notin var(s)\}, \\
W^G_{(2)}(\{x_2\}) & := & \{s \in W_{(2)}(X) | x_2 \in var(s), x_1 \notin var(s)\}, \\
\overline{W^G_{(2)}(\{x_1\})} & := & W^G_{(2)}(\{x_1\}) \backslash W_{x_1}, \\
\overline{W^G_{(2)}(\{x_2\})} & := & W^G_{(2)}(\{x_2\}) \backslash W_{x_2}, \\
\overline{W^G_{(2)}(\{x_i\})_{x_j}} & := & \{s \in \overline{W^G_{(2)}(\{x_i\})} | leftmost(s) = x_j \text{ where } i \in \{1,2\}, j \in \mathbb{N}\}, \\
\overline{W^G_{(2)}(\{x_i\})^{x_k}} & := & \{s \in \overline{W^G_{(2)}(\{x_i\})} | rightmost(s) = x_k \text{ where } i \in \{1,2\}, k \in \mathbb{N}\}, \\
\overline{W^G_{(2)}(\{x_i\})^{x_k}_{x_j}} & := & \{s \in \overline{W^G_{(2)}(\{x_i\})} | leftmost(s) = x_j \text{ and } rightmost(s) = x_k \\
 & & \text{where } i \in \{1,2\}, j, k \in \mathbb{N}\}, \\
P_G(2) & := & \{\sigma_{x_i} \in Hyp_G(2) | i \in \mathbb{N}, x_i \in X\}, \\
D^G_i & := & \{\sigma_{t_i}, \sigma_{t_i^d} | \sigma_{t_i} \in G\}, \\
P^{ab}_i & := & \{\sigma_{x^i_a}, \sigma_{x^i_b} | a, b, i \in \mathbb{N}\}, \\
G & := & \{\sigma_s \in Hyp_G(2) | s \in W_{(2)}(X) \backslash X, x_1, x_2 \notin var(s)\}, \\
G_{x_m} & := & \{\sigma_s \in G | leftmost(s) = x_m, 2 < m \in \mathbb{N}\}, \\
G^{x_m} & := & \{\sigma_s \in G | rightmost(s) = x_m, 2 < m \in \mathbb{N}\}, \\
G^{x_n}_{x_m} & := & \{\sigma_s \in G | leftmost(s) = x_m \text{ and } rightmost(s) = x_n, \\
 & & 2 < m, n \in \mathbb{N}\}, \\
T_i & := & \{\sigma_{t_i}, \sigma_{t_i'}\} \text{ where } t_i \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}} \text{ or } \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}, \\
B_i & := & \{\sigma_{t_i}, \sigma_{t_i'}, \sigma_{t_i^d}, \sigma_{(t_i')^d}\} \text{ where } t_i \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}} \text{ or } \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}, \\
C_i & := & \{\sigma_{t_i}, \sigma_{t_i^*}\} \text{ where } t_i \in \overline{W^G_{(2)}(\{x_1\})_{x_m}}.
\end{array}
$$

Let M be an M-strongly solid submonoid of $Hyp_G(2)$. Clearly, $(xy)z \approx x(yz)$ is an identity in $V(M)$ and for all $\sigma \in Hyp_G(2), \ell(\hat{\sigma}[xy]) \leq \ell(\hat{\sigma}[\hat{\sigma}[xy]])$. Since $\hat{\sigma}[(xy)z] = S^2(\sigma(f), S^2(\sigma(f), x, y), z)$, $\hat{\sigma}[x(yz)] = S^2(\sigma(f), x, S^2(\sigma(f), y, z))$ and if there exist x_1, x_2 occurring in $\sigma(f)$ k times and l times respectively, then after substitution there will be x occurs k^2 times and z occurs l times in $\hat{\sigma}[(xy)z]$ and x occurs k times and z occurs l^2 times in $\hat{\sigma}[x(yz)]$. Since M is M-strongly solid, so $\hat{\sigma}[(xy)z] \approx \hat{\sigma}[x(yz)]$ is an identity in $V(M)$. Thus there exist $a, b \in \mathbb{N}, a \neq b$ such that $x^a \approx x^b$ is an identity in $V(M)$. Hence $\sigma^a = \sigma^b$. If $\ell(\hat{\sigma}[xy]) < \ell(\hat{\sigma}[\hat{\sigma}[xy]])$, then $\sigma^a \neq \sigma^b$ which is a contradiction. Therefore $\ell(\hat{\sigma}[xy]) = \ell(\hat{\sigma}[\hat{\sigma}[xy]])$.

$$\begin{aligned} \text{Let } U &= \{\sigma | \sigma \in Hyp_G(2) \text{ and } \ell(\hat{\sigma}[xy]) = \ell(\hat{\sigma}[\hat{\sigma}[xy]])\} \\ &= \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \{\sigma_t | t \in W_{x_1} \cup W_{x_2} \cup W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}. \end{aligned}$$

Thus $M \subseteq U$.

Proposition 5. *Let M be an M-strongly solid submonoid of $Hyp_G(2)$. If M is one of all subcases from Case 1-5, then M is implied to $\{\sigma_{id}\}$.*

Case 1: For $i, m, n, k \in \mathbb{N}$ $(m, n, k > 2)$.

1.1. $M = \{\sigma_{id}, \sigma_{x_1}\} \cup A$, *where A is one of these sets :* $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_2\})_{x_m}}, \sigma_{x_m}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$.

1.2. $M = \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup A$, *where A is one of these sets :* $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}\}$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}\}$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, $\left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{x^i_1}\}\right)$, $\left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{x^i_2}\}\right)$.

1.3. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \subseteq M \subseteq A$, *where A is either* $\{\sigma_{id}, \sigma_{x_1}\} \cup \{\sigma_t | t \in W_{x_2}\}$ *or* $\{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_t | t \in W_{x_1}\}$.

1.4. $\{\sigma_{id}, \sigma_{x_1}, \sigma_t, \sigma_{x_m}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_t, \sigma_{x_m}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $\sigma_t \in G_{x_m}$.

1.5. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}, \sigma_{x_n}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}, \sigma_{x_n}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $\sigma_t \in G^{x_n}_{x_m}$.

1.6. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $t \in \overline{W^G_{(2)}(\{x_1\})^{x_m}_{x_1}}$.

1.7. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_{x_m}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $t \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_m}}$.

1.8. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists i} T_i\right) \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\forall i} T_i\right) \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$.

1.9. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_{m_1}x_{m_2}...x_{m_r}}, \sigma_{x_1x_{m_2}...x_{m_r}}, \sigma_{x_{m_1}x_{m_2}...x_{m_{r-1}}x_2}, \sigma_{x_{m_1}}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_{m_1}x_{m_2}...x_{m_r}}, \sigma_{x_1x_{m_2}...x_{m_r}}, \sigma_{x_{m_1}x_{m_2}...x_{m_{r-1}}x_2}, \sigma_{x_{m_1}}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $m_l > 2 \;\; \forall l \in \mathbb{N}, r \in \mathbb{N}$.

1.10. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_{m_1}x_{m_2}...x_{m_r}}, \sigma_{x_1x_{m_2}...x_{m_r}}, \sigma_{x_{m_1}x_{m_2}...x_{m_{r-1}}x_2}, \sigma_{x_{m_1}}, \sigma_{x_{m_n}}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_{m_1}x_{m_2}...x_{m_r}}, \sigma_{x_1x_{m_2}...x_{m_r}}, \sigma_{x_{m_1}x_{m_2}...x_{m_{r-1}}x_2}, \sigma_{x_{m_1}}, \sigma_{x_{m_n}}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $m_l > 2 \;\; \forall l \in \mathbb{N}, r \in \mathbb{N}$.

Case 2: For $i, m, k \in \mathbb{N}$ $(m, k > 2)$.

2.1. $\{\sigma_{id}\} \subset M \subseteq \{\sigma_{id}\} \cup A$, *where A is one of these sets :* $\{\sigma_t | t \in W_{x_1}\}$, $\left(\bigcup_{\forall i} P_i^{12}\right)$, $\left(\bigcup_{\forall i} P_i^{1m}\right)$.

2.2. $M = \{\sigma_{id}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right) \cup A$, *where A is either* $\left(\bigcup_{\exists i}\{\sigma_{x_1^i}\}\right)$ *or* $\left(\bigcup_{\exists i} P_i^{12}\right)$.

Case 3: For $i, a, m, k \in \mathbb{N}, a > 1$ $(m, k > 2)$.

3.1. $\{\sigma_{id}\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_t | t \in W_{x_2}\}$.

3.2. $\{\sigma_{id}, \sigma_t, \sigma_s\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W_{x_2} \cup \overline{W_{(2)}^G(\{x_2\})^{x_2}}\}$, *where* $t \in W_{x_2}$ *and* $s \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}$.

3.3. $\{\sigma_{id}, \sigma_{x_m}\} \cup \left(\bigcup_{\exists a}\{\sigma_{x_2^a}\}\right) \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_m}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_t | t \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}\}$.

3.4. $\{\sigma_{id}\} \cup \left(\bigcup_{\exists a}\{\sigma_{x_2^a}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | \sigma_{t_i} \in G^{x_m}\}\right) \cup \{\sigma_{x_m^a}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_k}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_t | \sigma_t \in G^{x_m}\} \cup \{\sigma_{x_m^a} | a > 1\}$.

3.5. $\{\sigma_{id}\} \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | \sigma_{t_i} \in G^{x_m}\}\right) \cup \left(\bigcup_{\exists a}\{\sigma_{x_2^a}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{s_i} | s_i \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}\}\right) \cup \left(\bigcup_{\exists a}\{\sigma_{x_m^a}\}\right) \cup \{\sigma_{t_i^{**}}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_k}\} \cup \{\sigma_t | \sigma_t \in G^{x_m}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_s | s \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}\} \cup \{\sigma_{x_m^a}\} \cup \{\sigma_{t^{**}}\}$.

Case 4: For $i, k \in \mathbb{N}$ $(k > 2)$.

4.1. $M = \{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists i} T_i\right)$.

4.2. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \subseteq M \subseteq A$, *where* A *is either* $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_2}\} \cup \{\sigma_{x_1^i} | i \in \mathbb{O}^+\}$ *or* $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_{x_1^i} | i \in \mathbb{E}^+\}$.

4.3. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \subseteq M \subseteq A$, *where* A *is either* $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}\} \cup \{\sigma_{x_2^i} | i \in \mathbb{O}^+\}$ *or* $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_{x_2^i} | i \in \mathbb{E}^+\}$.

4.4. $M = \{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists k} \{\sigma_{x_k}\}\right) \cup A$, *where* A *is one of these sets :*

$$\left(\bigcup_{\exists i \in \mathbb{O}^+} \{\sigma_{x_1^i}\}\right), \left(\bigcup_{\exists i \in \mathbb{E}^+} \{\sigma_{x_1^i}\}\right), \left(\bigcup_{\exists i \in \mathbb{O}^+} \{\sigma_{x_2^i}\}\right), \left(\bigcup_{\exists i \in \mathbb{E}^+} \{\sigma_{x_2^i}\}\right), \left(\bigcup_{\exists i \in \mathbb{E}^+} P_i^{12}\right).$$

Case 5: $\{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\exists i \in \mathbb{O}^+} P_i^{12}\right) \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\forall i \in \mathbb{O}^+} P_i^{12}\right) \cup \left(\bigcup_{\exists k} \{\sigma_{x_k}\}\right)$, *where* $i, k \in \mathbb{N}$ *and* $k > 2$.

Proof. For Case 1, we have $V(M) \models xyx \approx yx$. Since M is M-strongly solid, so $\hat{\sigma}[xyx] \approx \hat{\sigma}[yx] \in IdV(M)$ for all $\sigma \in M$. Since $\sigma_{x_1} \in M$ for all M, we get $\hat{\sigma}_{x_1}[\hat{\sigma}_t[xyx]] \approx \hat{\sigma}_{x_1}[\hat{\sigma}_t[yx] \in IdV(M)$ for all $\sigma_t \in M$. Thus $x \approx y \in IdV(M)$. Therefore M is implied to $\{\sigma_{id}\}$.

The proof of Case 2-5 are similar to the proof of Case 1 in which for Case 2 $V(M) \models x^2 \approx x, xyx \approx yx$ and $\sigma_{x_1^a} \in M$ where $a \in \mathbb{N}$, therefore M is implied to $\{\sigma_{id}\}$. Case 3 $V(M) \models x^2 \approx x, xyx \approx xy$ and $\sigma_{x_2^a} \in M$ where $a \in \mathbb{N}$, therefore M is implied to $\{\sigma_{id}\}$. Case 4 $V(M) \models x^2y^2x^2 \approx y^2x^2$ and $\sigma_{x_2} \in M$ where $a \in \mathbb{N}$, therefore M is implied to $\{\sigma_{id}\}$. Case 5 $V(M) \models x^3 \approx x, x^2y^2x^2 \approx y^2x^2$ and $\sigma_{x_2^a} \in M$ where $a \in \mathbb{O}^+$, therefore M is implied to $\{\sigma_{id}\}$.

□

3 M-strongly solid submonoids of $Hyp_G(2)$ which M is not implied to $\{\sigma_{id}\}$

In this section, we consider $M \subseteq U$ where M is M-strongly solid submonoid and M is not implied to $\{\sigma_{id}\}$. Since M has a lot of elements. It is difficult to write all submonoids M of U in the exactly form. But there are some cases where it is clear that M is not implied to $\{\sigma_{id}\}$.

Proposition 6. *Let* M *be an* M*-strongly solid submonoid of* $Hyp_G(2)$*. If* M *is one of all subcases from Case 1-2, then* M *is not implied to* $\{\sigma_{id}\}$.

Case 1:

1.1. $M = \{\sigma_{id}, \sigma_{x_2x_1}\}$.

1.2. $\{\sigma_{id}\} \subset M \subseteq \{\sigma_{id}\} \cup A$, *where* A *is one of these sets :* $\{\sigma_t | t \in W\}$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}$, $\{\sigma_t | t \in \overline{W^G_{(2)}(\{x_2\})^{x_2}}\}$, $\left(\bigcup_{\forall i \in \mathbb{N}} T_i\right)$ *and* $|M| \geq 2$.

1.3. $\{\sigma_{id}, \sigma_{x_2x_1}\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup A$, *where* A *is either* $\left(\bigcup_{\forall i \in \mathbb{N}} D^G_i\right)$ *or* $\left(\bigcup_{\forall i \in \mathbb{N}} B_i\right)$ *and* $|M| \geq 4$.

1.4. $\{\sigma_{id}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W \cup A\}$, *where* $t \in W$, $s \in A$ *and* A *is either* $\overline{W^G_{(2)}(\{x_1\})}$ *or* $\overline{W^G_{(2)}(\{x_2\})}$.

1.5. $A \cup \{\sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq A \cup \{\sigma_u | u \in W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})}\}$, $v \in \overline{W^G_{(2)}(\{x_2\})}$ *and* A *is either* $\{\sigma_{id}\}$ *or* $\{\sigma_{id}, \sigma_{x_2x_1}\}$.

Case 2: For $i, a, b, m, k \in \mathbb{N}, a, b > 1$ $(m, k > 2)$.

2.1. $\{\sigma_{id}, \sigma_{x_2x_1}\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\forall i \in \mathbb{E}^+} P^{12}_i\right)$ *and* $|M| \geq 4$.

2.2. $\{\sigma_{id}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}\} \cup A \cup \{\sigma_u | u \in W \cup \overline{W^G_{(2)}(\{x_1\})_{x_m}}\}$, *where* $s \in W$, $v \in \overline{W^G_{(2)}(\{x_1\})_{x_m}}$, $t \in A$ *and* A *is either* $\{\sigma^a_{x_1} | a > 1\}$ *or* $\{\sigma^b_{x_2} | b > 1\}$.

2.3. $M = \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\exists a} P^{12}_a\right) \cup \left(\bigcup_{\exists i} B_i\right)$.

2.4. $\{\sigma_{id}, \sigma^a_{x_1}, \sigma_t\} \cup A \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma^a_{x_1} | a > 1\} \cup \{\sigma_s | s \in W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})_{x_m}}\}$, *where* $t \in W$ *and* A *is either* $\{\sigma_v, \sigma_u\}$ *with* $v \in \overline{W^G_{(2)}(\{x_1\})}$, $u \in \overline{W^G_{(2)}(\{x_2\})^{x_k}_{x_m}}$ *or* $\{\sigma_e, \sigma_f\}$ *with* $e \in \overline{W^G_{(2)}(\{x_1\})_{x_k}}$, $f \in \overline{W^G_{(2)}(\{x_2\})_{x_m}}$.

2.5. $\{\sigma_{id}, \sigma^a_{x_2}, \sigma_t\} \cup A \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma^a_{x_2} | a > 1\} \cup \{\sigma_s | s \in W \cup \overline{W^G_{(2)}(\{x_1\})^{x_m}} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$ *and* A *is either* $\{\sigma_v, \sigma_u\}$ *with* $v \in \overline{W^G_{(2)}(\{x_1\})^{x_m}_{x_k}}$, $u \in \overline{W^G_{(2)}(\{x_2\})}$ *or* $\{\sigma_e, \sigma_f\}$ *with* $e \in \overline{W^G_{(2)}(\{x_1\})^{x^m}}$, $f \in \overline{W^G_{(2)}(\{x_2\})^{x_k}}$.

2.6. $\{\sigma_{id}, \sigma^a_{x_1}, \sigma^b_{x_2}, \sigma_t\} \cup A \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma^a_{x_1} | a > 1\} \cup \{\sigma^b_{x_2} | b > 1\} \cup \{\sigma_s | s \in W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$ *and* A *is either* $\{\sigma_v, \sigma_u\}$ *with* $v \in \overline{W^G_{(2)}(\{x_1\})_{x_m}}$, $u \in \overline{W^G_{(2)}(\{x_2\})}$ *or* $\{\sigma_e, \sigma_f\}$ *with* $e \in \overline{W^G_{(2)}(\{x_1\})}$, $f \in \overline{W^G_{(2)}(\{x_2\})^{x_m}}$.

2.7. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma^a_{x_1}, \sigma^b_{x_2}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \{\sigma_{x_1^a} | a > 1\} \cup \{\sigma_{x_2^b} | b > 1\} \cup \{\sigma_u | u \in W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})}$, $v \in \overline{W^G_{(2)}(\{x_2\})}$.

Proof. Let M be an M-strongly solid submonoid of $Hyp_G(2)$ and let M is one of all subcases in Case 1, we have $\sigma_t \notin M$ where $t \in W_{x_1}$ or W_{x_2}. And for M in the Case 2, we have σ_{x_1} or $\sigma_{x_2} \notin M$ and M is not idempotent. So we get $x \approx y \notin IdV(M)$. Therefore M is not implied to $\{\sigma_{id}\}$. □

Next, we consider the remaining cases which M can be classified into three groups by using $V(M)$ as a tool.

Proposition 7. *Let M be an M-strongly solid submonoid of $Hyp_G(2)$ and i, j, a, m, $k \in \mathbb{N}, a > 1$ with $m, k > 2$. If M is one of the following cases, then $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $leftmost(g) = leftmost(h)$.*

1. $\{\sigma_{id}, \sigma_{x_1^a}, \sigma_t, \sigma_{x_m^a}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_m}\} \cup \{\sigma_{x_1^a}\} \cup \{\sigma_s | \sigma_s \in G\}$, *where* $\sigma_t \in G_{x_m}$.

2. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_1^a}, \sigma_t, \sigma_{x_m}, \sigma_{x_m^a}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_k}\} \cup \{\sigma_{x_1^a}\} \cup \{\sigma_s | \sigma_s \in G\}$, *where* $\sigma_t \in G_{x_m}$.

3. $\{\sigma_{id}, \sigma_{x_m}, \sigma_{x_k}, \sigma_{x_1^a}, \sigma_t, \sigma_{x_m^a}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_m}, \sigma_{x_k}\} \cup \{\sigma_{x_1^a}\} \cup \{\sigma_s | s \in W\}$, *where* $\sigma_t \in G_{x_m}$.

4. $\{\sigma_{id}, \sigma_t, \sigma_s\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W_{x_1} \cup \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}$, *where* $t \in W_{x_1}$, $s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}$ *and in case of* $|M| = 3$, *then* $\sigma_{x_1} \notin M$.

5. $M = \{\sigma_{id}, \sigma_{x_m}\} \cup \left(\bigcup_{\exists a} \{\sigma_{x_1^a}\}\right) \cup \left(\bigcup_{\exists i} \{\sigma_{t_i} | t_i \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}\right)$.

6. $\{\sigma_{id}\} \cup \left(\bigcup_{\exists a} \{\sigma_{x_1^a}\}\right) \cup \left(\bigcup_{\exists i} \{\sigma_{t_i} | \sigma_{t_i} \in G_{x_m}\}\right) \cup \left(\bigcup_{\exists j} \{\sigma_{s_j} | s_j \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}\right) \cup \{\sigma_{x_m^a}, \sigma_{s_j^*}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_k}\} \cup \{\sigma_{x_1^a} | a > 1\} \cup \{\sigma_t | \sigma_t \in G_{x_m}\} \cup \{\sigma_s | s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\} \cup \{\sigma_{x_m^a}, \sigma_{s_j^*}\}$.

7. $\left(\{\sigma_{id}, \sigma_{x_1}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W \cup W_{x_1} \cup \overline{W^G_{(2)}(\{x_1\})}\}\right) \setminus M_1$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})}$ *and* $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_m}, \sigma_u\} \subseteq M_1 \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_u\} \cup \left(\bigcup_{\exists k} \{\sigma_{x_k}\}\right)$, *where* $u \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}$.

Proof. Let M be an M-strongly solid submonoid of $Hyp_G(2)$. Clearly, $(xy)z \approx x(yz)$ is an identity in $V(M)$. Let $g \approx h$ be an arbitrary identity in $V(M)$ and $c, d \in \mathbb{N}$, where $c \neq d$. From $\sigma_{x_1^c} \in M$ and $\sigma_{x_1^c} \circ \sigma_{x_1^c} = \sigma_{x_1^c} \circ \sigma_{x_1^d} = \sigma_{x_1^c}$ and for all $\sigma \in M, \hat{\sigma}[g] \approx \hat{\sigma}[h] \in IdV(M)$ it follows that the first variables in g and h are the same. Thus $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $leftmost(g) = leftmost(h)$. □

The examples of M and $V(M)$ corresponding to Proposition 7 as follows.

1. $\{\sigma_{id}, \sigma_{x_1^a}, \sigma_t, \sigma_{x_m^a}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_m}\} \cup \{\sigma_{x_1^a}\} \cup \{\sigma_s | \sigma_s \in G\}$, where $\sigma_t \in G_{x_m}$.
2. $\{\sigma_{id}, \sigma_t, \sigma_s\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W_{x_1} \cup \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}$, where $t \in W_{x_1}$, $s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}$ and in case of $|M| = 3$, then $\sigma_{x_1} \notin M$.
3. $M = \{\sigma_{id}, \sigma_{x_m}\} \cup \left(\bigcup_{\exists a}\{\sigma_{x_1^a}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | t_i \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}\right)$.
4. $\{\sigma_{id}\} \cup \left(\bigcup_{\exists a}\{\sigma_{x_1^a}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | \sigma_{t_i} \in G_{x_m}\}\right) \cup \left(\bigcup_{\exists j}\{\sigma_{s_j} | s_j \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\}\right) \cup \{\sigma_{x_m^a}, \sigma_{s_j^*}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_k}\} \cup \{\sigma_{x_1^a} | a > 1\} \cup \{\sigma_t | \sigma_t \in G_{x_m}\} \cup \{\sigma_s | s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}\} \cup \{\sigma_{x_m^a}, \sigma_{s_j^*}\}$.

For M in each case, we have $V(M) \subseteq Mod\{(xy)z \approx x(yz), x^2 \approx x, xyx \approx xy\}$.

Proposition 8. *Let M be an M-strongly solid submonoid of $Hyp_G(2)$ and $a, m, k \in \mathbb{N}, , a > 1$ with $m, k > 2$. If M is one of the following cases, then $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $rightmost(g) = rightmost(h)$.*

1. $\{\sigma_{id}, \sigma_{x_2}, \sigma_{x_m}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_t\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $\sigma_t \in G$.
2. $\{\sigma_{id}, \sigma_{x_2^a}, \sigma_{x_m}, \sigma_{x_k}\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_t | t \in W\}$.
3. $\{\sigma_{id}, \sigma_{x_2}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_v | v \in W\}$, *where* $\sigma_t, \sigma_s \in G$.
4. $\{\sigma_{id}, \sigma_{x_2}, \sigma_{x_2^a}, \sigma_t\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_s | s \in W\}$, *where* $\sigma_t \in G$.
5. $M = \{\sigma_{id}, \sigma_{x_2}, \sigma_t\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $t \in \overline{W^G_{(2)}(\{x_2\})^{x_2}}$.
6. $\left(\{\sigma_{id}, \sigma_{x_2}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W \cup W_{x_2} \cup \overline{W^G_{(2)}(\{x_2\})}\}\right) \setminus M_1$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_2\})}$ *and* $\{\sigma_{id}, \sigma_{x_2}, \sigma_{x_m}, \sigma_v\} \subseteq M_1 \subseteq \{\sigma_{id}, \sigma_{x_2}, \sigma_v\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right)$, *where* $v \in \overline{W^G_{(2)}(\{x_2\})^{x_2}}$.

7. $\{\sigma_{id}, \sigma_{x_2^a}, \sigma_{x_m}, \sigma_{x_k}, \sigma_t\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \{\sigma_s | s \in W \cup \overline{W_{(2)}^G(\{x_2\})^{x_2}}\}$, *where* $t \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}$.

Proof. Let M be an M-strongly solid submonoid of $Hyp_G(2)$. Clearly, $(xy)z \approx x(yz)$ is an identity in $V(M)$. Let $g \approx h$ be an arbitrary identity in $V(M)$ and $c, d \in \mathbb{N}$, where $c \neq d$. From $\sigma_{x_2^c} \in M$ and $\sigma_{x_2^c} \circ \sigma_{x_2^c} = \sigma_{x_2^c} \circ \sigma_{x_2^d} = \sigma_{x_2^c}$ and for all $\sigma \in M, \hat{\sigma}[g] \approx \hat{\sigma}[h] \in IdV(M)$ it follows that the last variables in g and h are the same. Thus $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $rightmost(g) = rightmost(h)$. ☐

The examples of M and $V(M)$ corresponding to Proposition 5 (4.3) are as follows.

1. $\{\sigma_{id}, \sigma_{x_2}, \sigma_{x_m}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_t\} \cup \{\sigma_{x_2^a} | a > 1\} \cup \left(\bigcup_{\exists k} \{\sigma_{x_k}\}\right)$, where $\sigma_t \in G$.

2. $M = \{\sigma_{id}, \sigma_{x_2}, \sigma_t\} \cup \left(\bigcup_{\exists k} \{\sigma_{x_k}\}\right)$, where $t \in \overline{W_{(2)}^G(\{x_2\})^{x_2}}$.

For M in each case, we have $V(M) \subseteq Mod\{(xy)z \approx x(yz), x^2 \approx x, xyx \approx yx\}$.

Proposition 9. *Let M be an M-strongly solid submonoid of $Hyp_G(2)$ and $i, j, a, m, n, k \in \mathbb{N}, a > 1$ and $m, n, k > 2$. If M is one of the following cases, then $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $leftmost(g) = leftmost(h)$ and $rightmost(g) = rightmost(h)$.*

1. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_v | v \in W_{x_1} \cup W_{x_2} \cup W \cup \overline{W_{(2)}^G(\{x_2\})}\}$, *where* $t \in W$, $s \in \overline{W_{(2)}^G(\{x_2\})}$ *and* M *is not implied to* σ_{id}.

2. $\{\sigma_{id}, \sigma_{x_1^a}, \sigma_t, \sigma_s, \sigma_v\} \subseteq M \subseteq \{\sigma_{id}\} \cup \{\sigma_{x_1^a} | a > 1\} \cup \{\sigma_u | u \in W \cup \overline{W_{(2)}^G(\{x_1\})_{x_1}} \cup \overline{W_{(2)}^G(\{x_2\})_{x_m}^{x_2}}\}$, *where* $t \in W$, $s \in \overline{W_{(2)}^G(\{x_1\})_{x_1}}$, $v \in \overline{W_{(2)}^G(\{x_2\})_{x_m}^{x_2}}$.

3. $\{\sigma_{id}, \sigma_{x_1}, \sigma_t, \sigma_s, \sigma_v\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}\} \cup \{\sigma_u | u \in W \cup \overline{W_{(2)}^G(\{x_1\})} \cup \overline{W_{(2)}^G(\{x_2\})_{x_m}}\}$, *where* $t \in W$, $s \in \overline{W_{(2)}^G(\{x_1\})}$, $v \in \overline{W_{(2)}^G(\{x_2\})_{x_m}}$ *and* M *do not implies to* σ_{id}.

4. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists i} \{\sigma_{t_i} | \sigma_{t_i} \in G_{x_m}^{x_k}\}\right) \cup \{\sigma_{x_m}, \sigma_{x_k}\} \cup A \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}\} \cup \{\sigma_t | t \in W \cup W_{x_2}\}$, *where* A *is either* $\left(\bigcup_{\exists a} \{\sigma_{x_1^a}\}\right) \cup \{\sigma_{x_k^a}\}$ *or* $\left(\bigcup_{\exists a} \{\sigma_{x_2^a}\}\right) \cup \{\sigma_{x_k^a}\}$.

5. $\{\sigma_{id}\} \cup \left(\bigcup_{\exists i} P_i^{12}\right) \cup \left(\bigcup_{\exists j} \{\sigma_{t_j} | \sigma_{t_i} \in G_{x_m}^{x_k}\}\right) \cup \{\sigma_{x_m^i}, \sigma_{x_k^i}\} \subseteq M \subseteq \{\sigma_{id}\} \cup \left(\bigcup_{\forall i} P_i^{12}\right) \cup \{\sigma_{t_j} | \sigma_{t_i} \in G_{x_m}^{x_k}\} \cup \{\sigma_{x_m^i}, \sigma_{x_k^i}\} \cup \left(\bigcup_{\exists n} \{\sigma_{x_n}\}\right)$.

6. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | \sigma_{t_i} \in G^{x_k}_{x_m}\}\right) \cup \{\sigma_{x_m}, \sigma_{x_k}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_t | t \in W\}$.

7. $M = \{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right) \cup \left(\bigcup_{\exists i \in \mathbb{O}^+} P^{12}_i\right) \cup \left(\bigcup_{\exists j \in \mathbb{E}^+} P^{12}_j\right)$.

8. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | \sigma_{t_i} \in G^{x_k}_{x_m}\}\right) \cup \{\sigma_{x_m}, \sigma_{x_k}\} \cup \{\sigma_{t^d_i}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_{t_i} | \sigma_{t_i} \in G^{x_k}_{x_m}\} \cup \{\sigma_{x_m}, \sigma_{x_k}\} \cup \{\sigma_{t^d_i}\} \cup \left(\bigcup_{\exists n}\{\sigma_{x_n}\}\right)$.

9. $\{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\exists i}\{\sigma_{t_i}, \sigma_{t^d_i} | \sigma_{t_i} \in G^{x_k}_{x_m}\}\right) \cup \left(\bigcup_{\exists j} P^{12}_j\right) \cup \{\sigma_{x^j_m}, \sigma_{x^j_k}\} \subseteq M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \left(\bigcup_{\forall i}\{\sigma_{t_i}, \sigma_{t^d_i} | \sigma_{t_i} \in G^{x_k}_{x_m}\}\right) \cup \left(\bigcup_{\forall j} P^{12}_j\right) \cup \{\sigma_{x^j_m}, \sigma_{x^j_k}\} \cup \left(\bigcup_{\exists n}\{\sigma_{x_n}\}\right)$.

10. $M = \{\sigma_{id}\} \cup \left(\bigcup_{\exists i} P^{12}_i\right) \cup \left(\bigcup_{\exists j}\{\sigma_{t_j} | t_j \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}\}\right)$.

11. $M = \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup A$, *where A is either* $\left(\bigcup_{\exists a}\{\sigma_{x^a_1}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | t_i \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}\}\right)$ *or* $\left(\bigcup_{\exists a}\{\sigma_{x^a_2}\}\right) \cup \left(\bigcup_{\exists i}\{\sigma_{t_i} | t_i \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}\}\right)$.

12. $\{\sigma_{id}, \sigma_{x_2}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_v | v \in W \cup \in \overline{W^G_{(2)}(\{x_1\})^{x_m}}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})^{x_m}}$.

13. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x^a_1}, \sigma_{x_m}, \sigma_t\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_s | s \in W \cup W_{x_1} \cup \overline{W^G_{(2)}(\{x_1\})}\}$, *where* $t \in \overline{W^G_{(2)}(\{x_1\})}$.

14. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup A \subset M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \{\sigma_t | t \in W \cup \overline{W^G_{(2)}(\{x_1\})}\}$, *where A is either* $\{\sigma_s, \sigma_u\}$ *with* $\sigma_s \in G, u \in \overline{W^G_{(2)}(\{x_1\})}$ *or* $\{\sigma_{x_k}, \sigma_v\}$ *with* $v \in \overline{W^G_{(2)}(\{x_1\})_{x_k}}$.

15. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_m}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}\} \cup \left(\bigcup_{\exists k}\{\sigma_{x_k}\}\right) \cup \{\sigma_u | u \in \overline{W^G_{(2)}(\{x_1\})}\}$, *where* $t, s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}$.

16. $\{\sigma_{id}, \sigma_t, \sigma_s, \sigma_v, \sigma_u\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_w | w \in W_{x_1} \cup W_{x_2} \cup \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}} \cup \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}\}$, *where* $t \in W_{x_1}$, $s \in W_{x_2}$, $v \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}$, $u \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}$ *and* M *is not implied to* σ_{id}.

17. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \{\sigma_v | v \in W_{x_1} \cup W_{x_2} \cup \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}} \cup \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}\}$, *where* $t \in \overline{W^G_{(2)}(\{x_1\})^{x_1}_{x_1}}$, $s \in \overline{W^G_{(2)}(\{x_2\})^{x_2}_{x_2}}$.

18. $\{\sigma_{id}, \sigma_{x_2}, \sigma_{x_m}, \sigma_t, \sigma_s\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2}\} \cup \{\sigma_v | v \in \overline{W^G_{(2)}(\{x_1\})^{x_k}} \cup \overline{W^G_{(2)}(\{x_2\})} \cup W\}$, *where* $t \in \overline{W^G_{(2)}(\{x_1\})^{x_k}}$, $s \in \overline{W^G_{(2)}(\{x_2\})}$.

19. $\{\sigma_{id}, \sigma_{x_2^a}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma^a_{x_1} | a > 1\} \cup \{\sigma_u | u \in W \cup \overline{W^G_{(2)}(\{x_1\})^{x_m}} \cup \overline{W^G_{(2)}(\{x_2\})^{x_2}}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})^{x_m}_{x_1}}$, $v \in \overline{W^G_{(2)}(\{x_2\})^{x_2}}$.

20. $\{\sigma_{id}, \sigma_{x_1^a}, \sigma_{x_2^a}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma^a_{x_1} | a > 1\} \cup \{\sigma^b_{x_2} | b > 1\} \cup \{\sigma_u | u \in W \cup \overline{W^G_{(2)}(\{x_1\})_{x_1}} \cup \overline{W^G_{(2)}(\{x_2\})^{x_2}}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})_{x_1}}$, $v \in \overline{W^G_{(2)}(\{x_2\})^{x_2}}$.

21. $\{\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}\} \cup \{\sigma_u | u \in W_{x_1} \cup W_{x_2} \cup W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})}$, $v \in \overline{W^G_{(2)}(\{x_2\})}$ *and* M *is not implied to* σ_{id}.

22. $\{\sigma_{id}, \sigma_{x_2x_1}, \sigma_{x_1}, \sigma_{x_2}, \sigma_t, \sigma_s, \sigma_v\} \subset M \subseteq \{\sigma_{id}, \sigma_{x_2x_1}\} \cup \{\sigma_u | u \in W_{x_1} \cup W_{x_2} \cup W \cup \overline{W^G_{(2)}(\{x_1\})} \cup \overline{W^G_{(2)}(\{x_2\})}\}$, *where* $t \in W$, $s \in \overline{W^G_{(2)}(\{x_1\})}$, $v \in \overline{W^G_{(2)}(\{x_2\})}$.

Proof. Let M be an M-strongly solid submonoid of $Hyp_G(2)$. Clearly, $(xy)z \approx x(yz)$ is an identity in $V(M)$. Let $g \approx h$ be an arbitrary identity in $V(M)$and $c, d \in \mathbb{N}$ where $c \neq d$. From $\sigma_{x_1^c}, \sigma_{x_2^d} \in M$ and $\sigma_{x_1^c} \circ \sigma_{x_2^d} = \sigma_{x_2^c}$, $\sigma_{x_2^c} \circ \sigma_{x_1^d} = \sigma_{x_1^c}$ and for all $\sigma \in M, \hat{\sigma}[g] \approx \hat{\sigma}[h] \in IdV(M)$ it follows that the first variables in g and h are the same and the last variables in g and h are the same. Thus $V(M) \subseteq Mod\{(xy)z \approx x(yz), g \approx h\}$ where $leftmost(g) = leftmost(h)$and $rightmost(g) = rightmost(h)$. □

4 M-strongly solid monoids of generalized hypersubstitutions of type $\tau = (2)$

From the previous section, we can characterize M-strongly solid monoids of generalized hypersubstitutions of type $\tau = (2)$ which are implied to $\{\sigma_{id}\}$ and M-strongly solid submonoids which are not implied to $\{\sigma_{id}\}$. So in this section, we collect M-strongly solid monoids of generalized hypersubstitutions of type $\tau = (2)$.

Theorem 10. *Let M be a submonoid of $Hyp_G(2)$. Then the following are equivalent:*

(i) M is M-strongly solid.

(ii) M is one of all cases in Proposition 5 and Proposition 6-9.

Proof. Let M be an M-strongly solid submonoid of $Hyp_G(2)$. Then (ii) follows from Proposition 5 and Proposition 6-9.

On the other hand, if M is one of the cases in Proposition 5. We get M is implied to $\{\sigma_{id}\}$. So $V(M) = I$ is the trivial variety. Clearly, I is M-strongly solid.

For M is one of Case 1 in Proposition 6, we consider $M = \{\sigma_{id}, \sigma_{x_2x_1}\}$, we have the commutative law is an identity in the variety $V(M)$. And $\hat{\sigma}_{xy}[u] = \hat{\sigma}_{yx}[v]$ is also an identity in $V(M)$. Thus $V(M)$ is M-strongly solid. And if $M = \{\sigma_{id}\} \cup \{\sigma_t | t \in W\}$,then we have $t = \hat{\sigma}_t[u] = \hat{\sigma}_t[v] = t$. Thus $V(M)$ is M-strongly solid. For other cases. Let $g \approx h$ be an arbitrary identity in $V(M)$. Then we can derive new identities $\hat{\sigma}[g] \approx \hat{\sigma}[h] \in IdV(M)\ \forall \sigma \in M$. Consequently, $V(M)$ is M-strongly solid.

Next, if M is one of the cases in Proposition 7. We get $V(M) \subseteq Mod\{(xy)z \approx x(yz), u \approx v\}$ where $leftmost(u) = leftmost(v)$. For all $\sigma \in M$, we have $\hat{\sigma}[(xy)z] = \hat{\sigma}[x(yz)]$ and $\hat{\sigma}[u] = \hat{\sigma}[v]$. Consequently, $V(M)$ is M-strongly solid.

The proof for Proposition 8 and Proposition 9 are similar to Proposition 7. □

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