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# A COMMON FIXED POINT THEOREM IN MENGER SPACE USING IMPLICIT RELATION

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**Abstract**. The main purpose of this paper is to prove a common fixed point theorem for two pairs of weakly compatible mappings in Menger space using implicit relation.

## 1 Introduction

The concept of probabilistic metric space was first introduced and studied by Menger [5], which is a generalization of the metric space and also the study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [11, 12]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [1].

In 1972, V. M. Sehgal and A. T. Bharucha-Reid [13] initiated the study of contraction mappings on probabilistic metric spaces. Several interesting and elegant results have been obtained by various authors in this direction. In 1986, Jungck [3] introduced the notion of compatible mappings in metric spaces. Mishra [7] extended the notion of compatibility to probabilistic metric spaces. And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [4]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true. Recently, Singh and Jain [15] established a common fixed point theorem in Menger space through weak compatibility.

In [6], Mihet established a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. This implicit relation is similar to that in [10]. In [10], Popa used the family $F_4$ of implicit real functions to find the fixed points of two pairs of semicompatible maps in a d-complete topological space. Here, $F_4$ denotes the family of real continuous functions $F : (\mathbb{R}^+)^4 \to \mathbb{R}$ satisfying the following properties:

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$(F_h)$ There exists $h \geq 1$ such that for every $u \geq 0$, $v \geq 0$ with $F(u,v,u,v) \geq 0$ or $F(u,v,v,u) \geq 0$, we have $u \geq hv$.

$(F_u)$ $F(u,u,0,0) < 0$, for all $u > 0$.

Many authors [2, 6, 9, 10, 16] proved common fixed point theorems using implicit relation on various spaces. In this paper we establish a common fixed point theorem for two pairs of weakly compatible mappings in Menger space using implicit relation.

First we recall some definitions and known results in Menger spaces.

## 2 Preliminaries

**Definition 1.** *[12] A triangular norm* $\mathrm{T}$ *(shortly t-norm) is a binary operation on the unit interval* $[0,1]$ *and the following conditions are satisfied: for all* $a,b,c,d \in [0,1]$*,*

*(i)* $\mathrm{T}(a,1) = a$ *for all* $a \in [0,1]$*;*

*(ii)* $\mathrm{T}(a,b) = \mathrm{T}(b,a)$*;*

*(iii)* $\mathrm{T}(a,b) \leq \mathrm{T}(c,d)$ *for* $a \leq c$, $b \leq d$*;*

*(iv)* $\mathrm{T}(\mathrm{T}(a,b),c) = \mathrm{T}(a,\mathrm{T}(b,c))$*;*

*Examples of t-norms are* $\mathrm{T}(a,b) = \min\{a,b\}$, $\mathrm{T}(a,b) = ab$ *and* $\mathrm{T}(a,b) = \max\{a+b-1,0\}$.

**Definition 2.** *[12] A mapping* $\mathcal{F} : \mathbb{R} \to \mathbb{R}^+$ *is called a distribution function if it is non-decreasing and left continuous with* $\inf_{t\in\mathbb{R}} F(t) = 0$ *and* $\sup_{t\in\mathbb{R}} F(t) = 1$.

*We shall denote by* $\Im$ *the set of all distribution functions while* $H$ *will always denote the specific distribution function defined by*

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

**Definition 3.** *[12] The ordered pair* $(X,\mathcal{F})$ *is called a probabilistic metric space (shortly PM-space) if* $X$ *is a nonempty set of elements and* $\mathcal{F}$ *is a mapping from* $X \times X$ *to* $\Im$*, the collection of all distribution functions. The value of* $\mathcal{F}$ *at* $(x,y) \in X \times X$ *is represented by* $F_{x,y}$*. The functions* $F_{x,y}$ *are assumed to satisfy the following conditions: for all* $x,y,z \in X$ *and* $t,s > 0$*,*

*(i)* $F_{x,y}(t) = 1$ *for all* $t > 0$ *if and only* $x = y$*;*

*(ii)* $F_{x,y}(0) = 0$*;*

*(iii)* $F_{x,y}(t) = F_{y,x}(t)$*;*

*(iv) if* $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$ *then* $F_{x,z}(t+s) = 1$,

*The ordered triple* $(X,\mathcal{F},\mathrm{T})$ *is called a Menger space if* $(X,\mathcal{F})$ *is a PM-space,* $\triangle$ *is a t-norm and the following inequality holds:*

*(v)* $F_{x,y}(t+s) \geq \mathrm{T}(F_{x,z}(t), F_{z,y}(s))$*, for all* $x,y,z \in X$ *and* $t,s > 0$.

*Every metric space* $(X,d)$ *can always be realized as a PM space by considering* $\mathcal{F} : X \times X \to \Im$ *defined by* $F_{x,y}(t) = H(t - d(x,y))$ *for all* $x,y \in X$.

**Definition 4.** *[12] Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger space with continuous t-norm.*

*(i) A sequence $\{x_n\}$ in $X$ is said to be converge to a point $x$ in $X$ if and only if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $N$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.*

*(ii) A sequence $\{x_n\}$ in $X$ is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $N$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$.*

*(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.*

**Definition 5.** *[7] Self maps $A$ and $B$ of a Menger space $(X, \mathcal{F}, \mathrm{T})$ are said to be compatible if and only if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Ax_n, Bx_n \to x$ for some $x$ in $X$.*

**Definition 6.** *[15] Self maps $A$ and $B$ of a Menger space $(X, \mathcal{F}, \mathrm{T})$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.*

**Remark 7.** *[15] Two compatible self-maps are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.*

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

**Example 8.** *Let $(X, d)$ be a metric space defined by $d(x, y) = |x - y|$, where $X = [0, 6]$ and $(X, \mathcal{F}, \mathrm{T})$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $t > 0$. We define self maps $A$ and $B$ as follows:*

$$A(x) = \begin{cases} 6 - x, & \text{if } 0 \leq x < 3; \\ 6, & \text{if } 3 \leq x \leq 6. \end{cases} \quad B(x) = \begin{cases} x, & \text{if } 0 \leq x < 3; \\ 6, & \text{if } 3 \leq x \leq 6. \end{cases}$$

*Taking $x_n = 3 - \frac{1}{n}$ . We get $Ax_n = 3 + \frac{1}{n}$, $Bx_n = 3 - \frac{1}{n}$. Thus, $Ax_n \to 3$, $Bx_n \to 3$. Hence $x = 3$. Further $ABx_n = 3 + \frac{1}{n}$, $BAx_n = 6$. Now; $\lim_{n\to\infty} F_{ABx_n,BAx_n}(t) = \lim_{n\to\infty} F_{3+\frac{1}{n},6}(t) = \frac{t}{t+3} < 1$, for all $t > 0$. Hence $(A, B)$ is not compatible.*

*Coincidence points of $A$ and $B$ are in $[3, 6]$. Now for any $x \in [3, 6]$. $Ax = Bx = 6$ and $AB(x) = A(6) = 6 = B(6) = BA(x)$. Thus $(A, B)$ is weakly compatible.*

**Lemma 9.** *[8, 14] Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger probabilistic metric space and define $E_{\lambda,F} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

*for each $\lambda \in (0, 1)$ and $x, y \in X$. Then we have*

*(i) For any $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,F}(x_1, x_n) \leq E_{\lambda,F}(x_1, x_2) + \ldots + E_{\lambda,F}(x_{n-1}, x_n),$$

*for any* $x_1, \dots, x_n \in X$.

*(ii) The sequence* $\{x_n\}_{n\in N}$ *is convergent with respect to Menger probabilistic metric* $F$ *if and only if* $E_{\lambda,F}(x_n, x) \to 0$. *Also the sequence* $\{x_n\}$ *is a Cauchy sequence with respect to Menger probabilistic metric* $F$ *if and only if it is a Cauchy sequence with* $E_{\lambda,F}$.

## 3 Implicit Relation

Let $\Phi$ be the class of all real continuous functions $\varphi : (\mathbb{R}^+)^4 \to \mathbb{R}$, non-decreasing in the first argument and satisfying the following conditions:

(a) $u, v \geq 0, \varphi(u, v, u, v) \geq 0$ or $\varphi(u, v, v, u) \geq 0$ implies that $u \geq v$.

(b) $\varphi(u, u, 1, 1) \geq 0$ for all $u \geq 1$.

**Example 10.** *[9] Define* $\varphi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. *Then* $\varphi \in \Phi$.

**A characterization of $\Phi$ in linear form [2, 9, 16].** If $a, b, c, d \in \mathbb{R}$ with $a + b + c + d = 0, a > 0, a + b > 0, a + c > 0$ and $a + d > 0$ then $\varphi(t_1, t_2, t_3, t_4) = at_1 + bt_2 + ct_3 + dt_4 \in \Phi$.

*Proof.* For $u, v \geq 0$ and $\varphi(u, v, v, u) \geq 0$, we have

$(a + d) = -(b + c)$, where $a + d > 0$ and $b + c < 0$.

The above expression can also be written as, $(a + d)u \geq -(b + c)v$,

$(a + d)u + (b + c)v \geq 0$ for $u, v \geq 0$,

$(a + d)u \geq -(b + c)v$

$(a + d)u \geq (a + d)v$.

Hence, $u \geq v$, since $a + d > 0$.

Similarly, we can also prove

$(a + c)u + (b + d)v \geq 0$.

That is, $(a + c)u - (a + c)v \geq 0$. Hence, $u \geq v$ as $(a + c) > 0$.

Also, $\varphi(u, u, 1, 1) \geq 0$ gives $(a + b)u + (c + d)1 \geq 0$.

That is, $(a + b)u \geq -(c + d)$,

$(a + b)u \geq (a + b)$, as $a + b + c + d = 0$. Hence $u \geq 1$, as $a + b > 0$, $\varphi$ is non-decreasing in the first argument. □

## 4 Result

**Theorem 11.** *Let* $A, L, M$ *and* $S$ *be self maps on a complete Menger space* $(X, \mathcal{F}, \mathrm{T})$ *and satisfy the following conditions:*

*(i)* $L(X) \subseteq S(X), M(X) \subseteq A(X)$;

*(ii) One of* $S(X)$ *or* $A(X)$ *is a complete subspace of* $X$;

*(iii) The pairs* $(L, A)$ *and* $(M, S)$ *are weakly compatible;*

*(iv)* $\varphi(F_{Lx,My}(kt), F_{Ax,Sy}(t), F_{Lx,Ax}(t), F_{My,Sy}(kt)) \geq 0$;

*(v)* $\varphi(F_{Lx,My}(kt), F_{Ax,Sy}(t), F_{Lx,Ax}(kt), F_{My,Sy}(t)) \geq 0$;
*for some* $\varphi \in \Phi$, *there exists* $k \in (0,1)$ *such that for all* $x, y \in X$ *and* $t > 0$.
*In addition assume that*

$$E_{\lambda,F}(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1-\lambda\},$$

*for each* $\lambda \in (0,1)$ *and* $x, y \in X$.

*Then* $A, L, M$ *and* $S$ *have a unique common fixed point in* $X$.

*Proof.* Let $x_0$ be an arbitrary element in $X$. From condition (i) there exist $x_1, x_2 \in X$ such that $Lx_0 = Sx_1 = y_0$ and $Mx_1 = Ax_2 = y_2$. Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, 3, 4, \ldots$.

Put $x = x_{2n}$ and $y = x_{2n+1}$ in inequality (iv), then we get
$\varphi\left(F_{Lx_{2n},Mx_{2n+1}}(kt), F_{Ax_{2n},Sx_{2n+1}}(t), F_{Lx_{2n},Ax_{2n}}(t), F_{Mx_{2n+1},Sx_{2n+1}}(kt)\right) \geq 0$,
$\varphi\left(F_{y_{2n+1},y_{2n+2}}(kt), F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n+2},y_{2n+1}}(kt)\right) \geq 0$,
using (a), we get
$F_{y_{2n+2},y_{2n+1}}(kt) \geq F_{y_{2n+1},y_{2n}}(t)$.
Similarly, by putting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (v), then we get
$\varphi\left(F_{y_{2n+3},y_{2n+2}}(kt), F_{y_{2n+1},y_{2n+2}}(t), F_{y_{2n+3},y_{2n+2}}(kt), F_{y_{2n+1},y_{2n+2}}(t)\right) \geq 0$,
using (a), we get
$F_{y_{2n+3},y_{2n+2}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t)$.
Thus, for any $n$, we have
$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t)$.
Consequently,
$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(\frac{t}{k})$.
By repeated application of above inequality, we get

$$\begin{aligned} F_{y_n,y_{n+1}}(t) &\geq F_{y_{n-1},y_n}(\tfrac{t}{k}) \\ &\geq \ldots \geq F_{y_0,y_1}(\tfrac{t}{k^n}), \end{aligned}$$

for $n = 1, 2, 3, \ldots.$, which implies that

$$\begin{aligned} E_{\lambda,F}(y_n, y_{n+1}) &= \inf\{t > 0 : F_{y_n,y_{n+1}}(t) > 1-\lambda\} \\ &\leq \inf\{t > 0 : F_{y_0,y_1}(\tfrac{t}{k^n}) > 1-\lambda\} \\ &= k^n \inf\{t > 0 : F_{y_0,y_1}(t) > 1-\lambda\} \\ &= k^n E_{\lambda,F}(y_0, y_1), \text{ for every } \lambda \in (0,1). \end{aligned}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. For every $\mu \in (0,1)$, there exists $\gamma \in (0,1)$ such that, for $m \geq n$,

$$\begin{aligned} E_{\mu,F}(y_n, y_m) &= E_{\gamma,F}(y_{m-1}, y_m) + E_{\gamma,F}(y_{m-2}, y_{m-1}) + \ldots + E_{\gamma,F}(y_n, y_{n+1}) \\ &= k^n E_{\gamma,F}(y_0, y_1) \textstyle\sum_{i=n}^{m-1} k^i \to 0, \end{aligned}$$

as $m, n \to \infty$. Thus by Lemma (9), $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete then $\{y_n\}$ converges to $z \in X$. That is, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Mx_{2n+1} = \lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Ax_{2n} = z$.

Suppose $S(X)$ is a complete subspace of $X$ then for some $v \in X$ we have $S(v) = z$. Put $x = x_{2n}$ and $y = v$ in (iv), then we get

$\varphi\left(F_{Lx_{2n},Mv}(kt), F_{Ax_{2n},Sv}(t), F_{Lx_{2n},Ax_{2n}}(t), F_{Mv,Sv}(kt)\right) \geq 0,$
as $n \to \infty$, we have
$\varphi\left(F_{z,Mv}(kt), F_{z,z}(t), F_{z,z}(t), F_{Mv,z}(kt)\right) \geq 0,$
$\varphi\left(F_{z,Mv}(kt), 1, 1, F_{Mv,z}(kt)\right) \geq 0.$

Using (a), we have $F_{z,Mv}(kt) \geq 1$, for all $t > 0$. Hence $F_{z,Mv}(t) = 1$. Thus $z = Mv$. Therefore, $z = Mv = Sv$. From weak compatibility of $(M, S)$, we have $M(S(v)) = S(M(v))$, hence $Mz = Sz$. Now put $x = x_{2n}$ and $y = z$ in (iv), then we get

$\varphi\left(F_{Lx_{2n},Mz}(kt), F_{Ax_{2n},Sz}(t), F_{Lx_{2n},Ax_{2n}}(t), F_{Mz,Sz}(kt)\right) \geq 0,$
as $n \to \infty$, we have
$\varphi\left(F_{z,Sz}(kt), F_{z,Sz}(t), F_{z,z}(t), F_{Sz,Sz}(kt)\right) \geq 0,$
As $\varphi$ is non-decreasing in the first argument, we have
$\varphi\left(F_{z,Sz}(t), F_{Sz,z}(kt), 1, 1\right) \geq 0.$

Using (b), we have $F_{z,Sz}(t) \geq 1$, for all $t > 0$. Thus $F_{z,Sz}(t) = 1$, we have, $z = Sz$. Therefore, $z = Mz = Sz$. Since $M(X) \subseteq A(X)$ then there exists $w \in X$ such that $A(w) = Mz = Sz = z$. Now put $x = w$ and $y = z$ in (iv), then we get

$\varphi\left(F_{Lw,Mz}(kt), F_{Aw,Sz}(t), F_{Lw,Aw}(t), F_{Mz,Sz}(kt)\right) \geq 0,$
$\varphi\left(F_{Lw,z}(kt), F_{z,z}(t), F_{Lw,z}(t), F_{z,z}(kt)\right) \geq 0,$
$\varphi\left(F_{Lw,z}(kt), 1, F_{Lw,z}(t), 1\right) \geq 0.$
As $\varphi$ is non-decreasing in the first argument, we have
$\varphi\left(F_{Lw,z}(t), 1, F_{Lw,z}(t), 1\right) \geq 0.$

Using (a), we have $F_{Lw,z}(t) \geq 1$, for all $t > 0$. Hence $F_{Lw,z}(t) = 1$. Therefore $z = Lw = Aw$. Also it is given that the pair $(L, A)$ is weakly compatible, then $L(A(w)) = A(L(w))$, that is $Lz = Az$. Now put $x = z$ and $y = x_{2n+1}$ in (iv), then we get

$\varphi\left(F_{Lz,Mx_{2n+1}}(kt), F_{Az,Sx_{2n+1}}(t), F_{Lz,Az}(t), F_{Mx_{2n+1},Sx_{2n+1}}(kt)\right) \geq 0,$
$\varphi\left(F_{Lz,z}(kt), F_{Lz,z}(t), F_{Lz,Lz}(t), F_{z,z}(kt)\right) \geq 0,$
$\varphi\left(F_{Lz,z}(kt), F_{Lz,z}(t), 1, 1\right) \geq 0.$
As $\varphi$ is non-decreasing in the first argument, we have
$\varphi\left(F_{Lz,z}(t), F_{Lz,z}(t), 1, 1\right) \geq 0.$

Using (b), we have $F_{Lz,z}(t) \geq 1$, for all $t > 0$. Thus $F_{Lz,z}(t) = 1$, we have, $z = Lz$. Therefore, $z = Lz = Az$. Now, combine all the results it is clear that $z = Az = Lz = Mz = Sz$. That is $z$ is the common fixed point.

The proof is similar when $A(X)$ is assumed to be a complete subspace of $X$.

**Uniqueness.** Let $u(u \neq z)$ be another common fixed point of $A, L, M$ and $S$. Now, taking $x = z$ and $y = u$ in (iv), then we get

$\varphi(F_{Lz,Mu}(kt), F_{Az,Su}(t), F_{Lz,Az}(t), F_{Mu,Su}(kt)) \geq 0,$
$\varphi(F_{z,u}(kt), F_{z,u}(t), F_{z,z}(t), F_{u,u}(kt)) \geq 0,$
$\varphi(F_{z,u}(kt), F_{z,u}(t), 1, 1) \geq 0.$
As $\varphi$ is non-decreasing in the first argument, we have
$\varphi(F_{z,u}(t), F_{z,u}(t), 1, 1) \geq 0.$

Using (b), we have $F_{z,u}(t) \geq 1$, for all $t > 0$. Thus $F_{z,u}(t) = 1$, we have, $z = u$ and so the uniqueness of the common fixed point. $\square$

Now, we give an example which illustrates Theorem 11.

**Example 12.** *Let $X = [0, 30]$ with the metric $d$ defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

*for all $x, y \in X$. Clearly $(X, \mathcal{F}, \mathrm{T})$ is a complete Menger space. Define $A, L, M$ and $S : X \to X$ by*

$$A(x) = \begin{cases} 0, & \text{if } x = 0; \\ 12, & \text{if } 0 < x \leq 15; \\ x - 9, & \text{if } 15 < x \leq 30. \end{cases} \quad S(x) = \begin{cases} 0, & \text{if } x = 0; \\ 6, & \text{if } 0 < x \leq 15; \\ x - 6, & \text{if } 15 < x \leq 30. \end{cases}$$

$$L(x) = \begin{cases} 0, & \text{if } x = 0; \\ 6, & \text{if } 0 < x \leq 30. \end{cases} \quad M(x) = \begin{cases} 0, & \text{if } x = 0; \\ 9, & \text{if } 0 < x \leq 30. \end{cases}$$

*Then $A, L, M$ and $S$ satisfy all the conditions of Theorem 11 with $k \in (0, 1)$ and have a unique common fixed point $0 \in X$. It may be noted in this example that the mappings $L$ and $A$ commute at coincidence point $0 \in X$. So $L$ and $A$ are weakly compatible maps. Similarly, $M$ and $S$ are weakly compatible maps. To see the pairs $(L, A)$ and $(M, S)$ are not compatible, let us consider a sequence $\{x_n\}$ defined as $x_n = 15 + \frac{1}{n}, n \geq 1$, then $x_n \to 15$ as $n \to \infty$. Then $\lim_{n\to\infty} Lx_n = 6, \lim_{n\to\infty} Ax_n = 6$ but $\lim_{n\to\infty} F_{LAx_n,ALx_n}(t) = \frac{t}{t+|6-12|} \neq 1$. Thus the pair $(L, A)$ is not compatible. Also $\lim_{n\to\infty} Mx_n = 9, \lim_{n\to\infty} Sx_n = 9$ but $\lim_{n\to\infty} F_{MSx_n,SMx_n}(t) = \frac{t}{t+|9-6|} \neq 1$. So the pair $(M, S)$ is not compatible. All the mappings involved in this example are discontinuous even at the common fixed point $x = 0$.*

On taking $A = S$ and $L = M$ in Theorem 11 then we get the interesting result.

**Corollary 13.** *Let $A$ and $L$ be self maps on a complete Menger space $(X, \mathcal{F}, \mathrm{T})$ and satisfy the following conditions:*
*(i) $L(X) \subseteq A(X)$;*
*(ii) $A(X)$ is a complete subspace of $X$;*
*(iii) The pair $(L, A)$ is weakly compatible;*
*(iv) $\varphi(F_{Lx,Ly}(kt), F_{Ax,Ay}(t), F_{Lx,Ax}(t), F_{Ly,Ay}(kt)) \geq 0$;*
*(v) $\varphi(F_{Lx,Ly}(kt), F_{Ax,Ay}(t), F_{Lx,Ax}(kt), F_{Ly,Ay}(t)) \geq 0$;*
*for some $\varphi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$.*
*In addition assume that*

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

*for each* $\lambda \in (0,1)$ *and* $x, y \in X$.

*Then* $A$ *and* $L$ *have a unique common fixed point in* $X$.

**Corollary 14.** *Let* $A$ *and* $L$ *be self maps on a complete Menger space* $(X, \mathcal{F}, \mathrm{T})$ *and satisfy the following conditions:*

*(i)* $L(X) \subseteq A(X)$;

*(ii)* $A(X)$ *is a complete subspace of* $X$;

*(iii) The pair* $(L, A)$ *is weakly compatible;*

*(iv)* $aF_{Lx,Ly}(kt) + bF_{Ax,Ay}(t) + cF_{Lx,Ax}(t) + dF_{Ly,Ay}(kt)) \geq 0$;

*(v)* $aF_{Lx,Ly}(kt) + bF_{Ax,Ay}(t) + cF_{Lx,Ax}(kt) + dF_{Ly,Ay}(t)) \geq 0$;

*for all* $x, y \in X, t > 0$, *and for some* $k \in (0,1)$, *some fixed* $a, b, c, d \in R$ *such that* $a > 0, a+b > 0, a+c > 0, a+d > 0$ *and* $a+b+c+d = 0$.

*In addition assume that*

$$E_{\lambda,F}(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

*for each* $\lambda \in (0,1)$ *and* $x, y \in X$.

*Then* $A$ *and* $L$ *have a unique common fixed point in* $X$.

*Proof.* Using the characterization of $\Phi$ in Corollary 13, the result follows. □

**Corollary 15.** *Let* $A$ *and* $L$ *be self maps on a complete Menger space* $(X, \mathcal{F}, \mathrm{T})$ *and satisfy the following conditions:*

*(i)* $L(X) \subseteq A(X)$;

*(ii)* $A(X)$ *is a complete subspace of* $X$;

*(iii) The pair* $(L, A)$ *is weakly compatible;*

*(iv)* $F_{Lx,Ly}(kt) \geq b_0 F_{Ax,Ay}(t) + c_0 F_{Lx,Ax}(t)$;

*for all* $x, y \in X, t > 0$, *and for some* $k \in (0,1)$, *where* $b_0, c_0 \in [0,1]$ *with* $b_0 + c_0 = 1$.

*In addition assume that*

$$E_{\lambda,F}(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

*for each* $\lambda \in (0,1)$ *and* $x, y \in X$.

*Then* $A$ *and* $L$ *have a unique common fixed point in* $X$.

*Proof.* Choosing $a = 1, d = 0, b = -b_0$ and $c = -c_0$, in Corollary 14, and using the fact that $F$ is a non-decreasing function, the condition (v) of Corollary 14 is trivially satisfied and the result follows. □

If we take $A = I$(identity map) and relax some conditions in Corollary 15 then we get an important result as follows.

**Corollary 16.** *Let* $L$ *be a self map on a complete Menger space* $(X, \mathcal{F}, \mathrm{T})$ *and satisfy the following condition:*

$$F_{Lx,Ly}(kt) \geq b_0 F_{x,y}(t) + c_0 F_{Lx,x}(t);$$

*for all* $x, y \in X, t > 0$, *and for some* $k \in (0,1)$, *where* $b_0, c_0 \in [0,1]$ *with* $b_0 + c_0 = 1$.

*In addition assume that*

$$E_{\lambda,F}(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\},$$

*for each* $\lambda \in (0,1)$ *and* $x, y \in X$.

*Then* $L$ *has a unique common fixed point in* $X$.

**Remark 17.** *If we take* $b_0 = 1$ *and* $c_0 = 0$ *in Corollary 16 then we get the Banach contraction principle in setting of probabilistic metric space.*

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