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MULTIPLE PERIODIC SOLUTIONS FOR A FOURTH-ORDER DISCRETE HAMILTONIAN SYSTEM

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Abstract. By means of a three critical points theorem proposed by Brezis and Nirenberg and a general version of Mountain Pass Theorem, we obtain some multiplicity results for periodic solutions of a fourth-order discrete Hamiltonian system

$$\Delta^4 u(t-2) + \nabla F(t, u(t)) = 0, \text{ for all } t \in \mathbb{Z}.$$

1 Introduction

Consider the nonlinear fourth-order discrete Hamiltonian system

$$\Delta^4 u(t-2) + \nabla F(t, u(t)) = 0, \quad \forall t \in \mathbb{Z}, \tag{1.1}$$

where $\Delta u(t) = u(t+1) - u(t), \Delta^2 u(t) = \Delta(\Delta u(t)), F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}, F(t,x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and T-periodic in t for all $x \in \mathbb{R}^N$. T is a positive integer and $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

The theory of nonlinear difference equations (including discrete Hamiltonian systems) has been widely used to study discrete models in many fields such as computer science, economics, neural networks, ecology and so on. Many scholars studied the qualitative properties of difference equations such as stability, oscillation and boundary value problems (see e.g. [1, 2, 3] and references cited therein). But results on periodic solutions of difference equations are relatively rare and the results usually obtained by analytic techniques or various fixed point theorems (see e.g. [4]).

We may think of (1.1) as being a discrete analogue of the following fourth-order Hamiltonian system

$$\frac{d^4 x(t)}{dt^4} + \nabla F(t, x(t)) = 0, \quad \forall t \in \mathbb{R},$$

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where $F : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}, F(t,x)$ is continuously differentiable in x for every $t \in \mathbb{R}$ and $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x. As is known to us, the development of the study of periodic solutions of differential equations is relatively rapid. There have been many approaches to study periodic solutions of differential equations, such as fixed-point theory, coincidence degree theory, critical point theory and so on. However, there are few known techniques for studying the existence of periodic solutions of discrete systems. The authors in [1] studied the existence of periodic solutions of a second order nonlinear difference equation by using the critical point theory for the first time. The paper [5] shows that critical point theory is an effective approach to the study of periodic solutions of second order difference equations. In [6], Xue and Tang studied the existence of periodic solutions of superquadratic second-order discrete Hamiltonian systems. Compared to second-order discrete Hamiltonian systems, the study of higher-order discrete Hamiltonian systems, and in particular, fourth-order discrete Hamiltonian systems, has received considerably less attention (see e.g. [7, 8] and the references cited therein).

In [9], Thandapanı and Arockıasamy studied the following fourth-order difference equation of the form

$$\Delta^2(r_n\Delta^2 y_n) + f(n, y_n) = 0, \quad n \in \mathbb{N},$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \cdots\}$, n_0 is a nonnegative integer, and the real sequence $\{r_n\}$ and the function f satisfy the following conditions:

(a) $r_n > 0$ for all $n \in \mathbb{N}(n_0)$ and $\sum\limits_{n=n_0}^{\infty} \frac{n}{r_n} < \infty$;

(b) $f : \mathbb{N}(n_0)\times\mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$ and all $n \in \mathbb{N}(n_0)$, and $f(n,\cdot) \not\equiv 0$ eventually.

In [10], Cai, Yu and Guo studied the existence of the periodic solutions of the equation

$$\Delta^2(r_{n-2}\Delta^2 x_{n-2}) + f(n, x_n) = 0, \quad n \in \mathbb{N}, \tag{1.2}$$

where $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function in the second variable and $f(n+T, z) = f(n, z)$ for all $(n,z) \in \mathbb{Z}\times\mathbb{R}, r_n > 0, r_{n+T} = r_n$ for a given positive integer T and for all $n \in \mathbb{Z}$. They obtained the following result.

Theorem 1. *([10], Theorem 1.1)Assume that the following conditions are satisfied.*

(A_1) *For all $z \in R$ and all $t \in Z$, one has $\int_0^z f(t,s)ds \le 0$ and $\lim\limits_{z\to 0} \frac{f(t,z)}{z} = 0$.*

(A_2) *There exist $R_2 > 0$ and $\beta > 2$ such that, for every $t \in Z$ and every $z \in R$ with $|z| \ge R_2$, one has $zf(t,z) \le \beta\int_0^z f(t,s)ds < 0$.*

Then, equation (1.2) *has at least two nontrivial T-periodic solutions.*

But when $N > 1$, the existence and multiplicity of periodic solutions for problem (1.1) have not been studied by critical point theory.

In this paper, we study the existence and multiplicity of periodic solutions for problem (1.1) when $N \geq 1$. Our results are superior to those obtained in references [9] and [10].

2 Preliminaries and statements

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure. From this framework structure, we can reduce the problem of finding T-periodic solutions of problem (1.1) to the one of seeking the critical points of a corresponding functional.

For $a, b \in \mathbb{Z}$, we define $\mathbb{Z}(a) = \{a, a+1, a+2, \cdots\}, \mathbb{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$.

For a given positive integer T, we define H_T as follows:

$$H_T = \{u : \mathbb{Z} \to \mathbb{R}^N | u(t+T) = u(t), t \in \mathbb{Z}\}.$$

H_T can be equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ as follows:

$$\langle u, v \rangle = \sum_{t=1}^{T} \big(u(t), v(t)\big), \quad \forall\, u, v \in H_T,$$

$$\|u\| = \sqrt{\sum_{t=1}^{T} |u(t)|^2}, \quad \forall\, u \in H_T,$$

where $(\cdot, \cdot)$ and $| \cdot |$ denote the usual inner product and the usual norm in $\mathbb{R}^N$, respectively. It is easy to see that $(H_T, \langle \cdot, \cdot \rangle)$ is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{NT}$. For any $u \in H_T$, set

$$\|u\|_\theta = \big(\sum_{t=1}^{T} |u(t)|^\theta\big)^{\frac{1}{\theta}}, \quad \forall\, \theta > 1.$$

Then $\| \cdot \|$ and $\| \cdot \|_\theta$ are equivalent. That is, there exist positive constants C_1, C_2 such that

$$C_1 \|u\|_\theta \leq \|u\| \leq C_2 \|u\|_\theta, \ \forall u \in H_T.$$

In order to make a variational structure of problem (1.1), we need the following lemma.

Lemma 2. *For any $u,v\in H_T$,*

$$\sum_{t=1}^{T}\big(\Delta^4u(t-2),v(t)\big)=\sum_{t=1}^{T}\big(\Delta^2u(t),\Delta^2v(t)\big)=\sum_{t=1}^{T}\big(\Delta^4v(t-2),u(t)\big).$$

Proof. In fact, for any $u,v\in H_T$, since $u(t+T)=u(t), v(t+T)=v(t)$, thus

$$\begin{aligned}
\sum_{t=1}^{T}\big(\Delta^4u(t-2),v(t)\big) &= \sum_{t=1}^{T}\big(\Delta^2(\Delta^2u(t-2)),v(t)\big)\\
&= \sum_{t=1}^{T}\big(\Delta^2\big(u(t)-2u(t-1)+u(t-2)\big),v(t)\big)\\
&= \sum_{t=1}^{T}\big(\Delta^2\big(u(t)-u(t-1)\big),v(t)\big)\\
&\quad -\sum_{t=1}^{T}\big(\Delta^2\big(u(t-1)-u(t-2)\big),v(t)\big)\\
&= \sum_{t=1}^{T}\big(\Delta^2\big(\Delta u(t-1)\big),v(t)\big)-\sum_{t=0}^{T-1}\big(\Delta^2\big(\Delta u(t-1)\big),v(t+1)\big)\\
&= -\sum_{t=1}^{T}\big(\Delta^2\big(\Delta u(t-1)\big),\Delta v(t)\big)\\
&= \sum_{t=1}^{T}\big(\Delta^2u(t),\Delta^2v(t)\big).
\end{aligned}$$

Similarly, one can show that $\sum_{t=1}^{T}\big(\Delta^4v(t-2),u(t)\big)=\sum_{t=1}^{T}\big(\Delta^2u(t),\Delta^2v(t)\big)$. The proof is complete. □

Consider the functional

$$I(u)=\frac{1}{2}\sum_{t=1}^{T}|\Delta^2u(t)|^2+\sum_{t=1}^{T}F\big(t,u(t)\big), \tag{2.1}$$

where $F(t,x)$ is the same as that in (1.1). Clearly, $I\in C^1(H_T,\mathbb{R})$. For any $v\in H_T$, one has

$$\langle I'(u),v\rangle=\sum_{t=1}^{T}\big(\Delta^2u(t),\Delta^2v(t)\big)+\sum_{t=1}^{T}\big(\nabla F\big(t,u(t)\big),v(t)\big).$$

Hence $u\in H_T$ is a critical point of I if and only if

$$\sum_{t=1}^{T}\big(\Delta^2u(t),\Delta^2v(t)\big)=-\sum_{t=1}^{T}\big(\nabla F\big(t,u(t)\big),v(t)\big).$$

It follows from Lemma 2 that

$$\sum_{t=1}^{T}\big(\Delta^4 u(t-2), v(t)\big) = -\sum_{t=1}^{T}\big(\nabla F\big(t, u(t)\big), v(t)\big).$$

By the arbitrariness of v, we conclude that

$$\Delta^4 u(t-2) + \nabla F(t, u(t)) = 0, \quad \forall t \in \mathbb{Z}.$$

Since $u \in H_T$ is T-periodic, and $F(t,x)$ is T-periodic in t, hence $u \in H_T$ is a critical point of I if and only if for any $t \in \mathbb{Z}$, $\Delta^4 u(t-2) + \nabla F(t, u(t)) = 0$. That is the functional I is the variational framework of problem (1.1). So, we can reduce the existence of periodic solutions of problem (1.1) to the existence of critical points of I on H_T.

In this paper, we need the following definition and theorems.

Definition 3. *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. I is said to be satisfying P.S. condition on X if any sequence $\{x_n\} \in X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.*

Theorem 4. *([11], Theorem 4) Let X be a Banach space with a direct sum decomposition $X_1 \bigoplus X_2$ with $dim X_2 < \infty$. Let I be a C^1 function on X with $I(0) = 0$, satisfying P.S. condition and assume that, for some $R > 0$,*

$$I(u) \geq 0, \qquad \forall \;\; u \in X_1, \quad \|u\| \leq R \tag{2.2}$$

and

$$I(u) \leq 0, \qquad \forall \;\; u \in X_2, \quad \|u\| \leq R. \tag{2.3}$$

Assume also that I is bounded blow and $\inf_X I < 0$. Then I has at least two nonzero critical points.

Theorem 5. *([12], Theorem 9.12) Let E be a Banach space. Let $I \in C^1(E,\mathbb{R})$ be an even functional which satisfies the P.S. condition and $I(0) = 0$. If $E = V \bigoplus W$, where V is finite dimensional, and I satisfies*

- (I_1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \bigcap W} \geq \alpha$, where $B_\rho = \{x \in E : \|x\| < \rho\}$,*
- (I_2) *for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \backslash B_{R(\widetilde{E})}$,*

then I possesses an unbounded sequence of critical values.

3 Main results

Theorem 6. *Assume that $F(t,x)$ satisfies*

(F_1) *There exists a positive integer $T \geq 2$ such that $F(t+T,x) = F(t,x)$ for all $(t,x) \in \mathbb{Z} \times \mathbb{R}^N$.*

(F_2) *There exist constants $R_1 > 0,\ \gamma > 0$ and $\beta > 0$ such that for any $|x| \geq R_1$,*
$$F(t,x) \geq \beta |x|^2 - \gamma.$$

(F_3) *There exist some constants $\delta > 0,\ k \in Z[0, [\frac{T}{2}] - 1]$ such that*
$$-\frac{1}{2}\lambda_{k+1}|x|^2 \leq F(t,x) \leq -\frac{1}{2}\lambda_k |x|^2$$
for all $|x| \leq \delta$ and $t \in \mathbb{Z}[1,T]$, where $\lambda_k = 2\cos 2k\omega - 8\cos k\omega + 6, \omega = \frac{2\pi}{T}, [\cdot]$ denotes the Gauss Function.

Then the problem (1.1) has at least three T-periodic solutions.

Remark 7. *Take $F(t,x) = -\frac{1}{2}\lambda_1 |x|^2 + \frac{1}{2}|x|^3$. Then $F(t,x)$ satisfies all the conditions of Theorem 6.*

Denote
$$N_k = \{u \in H_T |\ \ \Delta^4 u(t-2) = \lambda_k u(t)\},$$
where $\lambda_k = 2\cos 2k\omega - 8\cos k\omega + 6,\ \ k \in \mathbb{Z}[0, [\frac{T}{2}]],\ \ \omega = \frac{2\pi}{T}$, then N_k is the subspace of H_T and $\lambda_k \geq 0$ for all $k \in \mathbb{Z}[0, [\frac{T}{2}]]$.

In order to prove Theorem 6, we need to prove the following lemmas.

Lemma 8. *It follows from the definition of N_k that*

(1) $N_k \perp N_j,\ \ k \neq j,\ k,j \in \mathbb{Z}[0, [\frac{T}{2}]]$,

(2) $H_T = \bigoplus\limits_{k=0}^{[\frac{T}{2}]} N_k$.

Proof. (1) By the definition of N_k and Lemma 2, for any $u \in N_k,\ v \in N_j,\ k \neq j,\ k,j \in \mathbb{Z}[0, [\frac{T}{2}]]$, we obtain

$$\begin{aligned}
\lambda_k \langle u, v \rangle &= \lambda_k \sum_{t=1}^{T} \big(u(t), v(t)\big) \\
&= \sum_{t=1}^{T} \big(\Delta^4 u(t-2), v(t)\big) \\
&= \sum_{t=1}^{T} \big(\Delta^2 u(t), \Delta^2 v(t)\big) \\
&= \sum_{t=1}^{T} \big(\Delta^4 v(t-2), u(t)\big) \\
&= \lambda_j \langle u, v \rangle.
\end{aligned}$$

Sine $\lambda_k \neq \lambda_j$, thus $\langle u, v\rangle = 0$, then (1) is verified.

(2) For $k \in \mathbb{Z}[0, [\frac{T}{2}]]$, we define

$$W_k = \left\{ a\cos k\omega t + b \sin k\omega t \mid a, b \in \mathbb{R}^N,\ t \in \mathbb{Z},\ k \in Z[0, [\frac{T}{2}]] \right\}.$$

Then $W_k \subset N_k$. In fact, for any $w = a\cos k\omega t + b\sin k\omega t \in W_k$, we get

$$\begin{aligned}
\Delta^4 u(t-2) &= u(t+2) - 4u(t+1) + 6u(t) - 4u(t-1) + u(t-2)\\
&= a\big(\cos k\omega(t+2) + \cos k\omega(t-2)\big) + b\big(\sin k\omega(t+2) + \sin k\omega(t-2)\big)\\
&\quad -4\big[a\big(\cos k\omega(t+1) + \cos k\omega(t-1)\big) + b\big(\sin k\omega(t+1) + \sin k\omega(t-1)\big)\big]\\
&\quad +6\big(a\cos k\omega t + b\sin k\omega t\big)\\
&= 2\cos 2k\omega\big(a\cos k\omega t + b\sin k\omega t\big) - 8\cos k\omega\big(a\cos k\omega t + b\sin k\omega t\big)\\
&\quad +6\big(a\cos k\omega t + b\sin k\omega t\big)\\
&= \big(2\cos 2k\omega - 8\cos k\omega + 6\big)u(t)\\
&= \lambda_k u(t),
\end{aligned}$$

which implies that $u \in N_k$, thus $W_k \subset N_k$, furthermore, we have $\bigoplus_{k=0}^{[\frac{T}{2}]} W_k \subset \bigoplus_{k=0}^{[\frac{T}{2}]} N_k \subset H_T$. It is easy to get that

$$\dim W_0 = N;$$

$$\dim W_k = 2N, \text{ when } k \in \mathbb{Z}, 0 < k < \tfrac{T}{2};$$

$$\dim W_{[\frac{T}{2}]} = N, \text{ when } T \text{ is even.}$$

Thus, we have $\dim \bigoplus_{k=0}^{[\frac{T}{2}]} W_k = NT = \dim H_T$. Then the result (2) holds. □

Lemma 9. *Let* $H_k = \bigoplus_{j=0}^{k} N_j,\ H_k^{\perp} = \bigoplus_{j=k+1}^{[\frac{T}{2}]} N_j,\ k \in \mathbb{Z}[0, [\frac{T}{2}]-1]$, *then*

$$\sum_{t=1}^{T} |\Delta^2 u(t)|^2 \leq \lambda_k \|u\|^2 \qquad \forall u \in H_k, \tag{3.1}$$

$$\sum_{t=1}^{T} |\Delta^2 u(t)|^2 \geq \lambda_{k+1} \|u\|^2 \qquad \forall u \in H_k^{\perp}. \tag{3.2}$$

Proof. For any $u_k \in N_k$, it follows from Lemma 2 that

$$\sum_{t=1}^{T}\big(\Delta^2 u_k(t), \Delta^2 u_k(t)\big) = \sum_{t=1}^{T}\big(\Delta^4 u_k(t-2), u_k(t)\big) = \lambda_k \sum_{t=1}^{T}\big(u_k(t), u_k(t)\big).$$

Since $\lambda_k = 2\cos 2k\omega - 8\cos k\omega + 6, \omega = \frac{2\pi}{T}, k \in \mathbb{Z}[0, [\frac{T}{2}]]$, we have $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{[\frac{T}{2}]} \leq 16$ and $\lambda_{[\frac{T}{2}]} = 16$ when T is even; $\lambda_{[\frac{T}{2}]} = 2\cos(\frac{2}{T}\pi) + 8\cos(\frac{1}{T}\pi) + 6$ when T is odd. For any $u \in H_k$, there exist some constants $a_j, j \in \mathbb{Z}[0,k]$, such that $u = \sum_{j=0}^{k} a_j u_j$, where $u_j \in N_j$. It follows from Lemma 2 and Lemma 8 that

$$\begin{aligned}
\sum_{t=1}^{T} |\Delta^2 u(t)|^2 &= \sum_{t=1}^{T}\Big(\sum_{j=0}^{k} a_j \Delta^2 u_j(t), \sum_{j=0}^{k} a_j \Delta^2 u_j(t)\Big) \\
&= \sum_{t=1}^{T}\sum_{j=0}^{k} a_j^2 \big(\Delta^2 u_j(t), \Delta^2 u_j(t)\big) \\
&= \sum_{t=1}^{T}\sum_{j=0}^{k} a_j^2 \lambda_j \big(u_j(t), u_j(t)\big) \\
&\leq \lambda_k \sum_{t=1}^{T}\sum_{j=0}^{k} \big(a_j u_j(t), a_j u_j(t)\big) \\
&= \lambda_k \sum_{t=1}^{T}\Big(\sum_{j=0}^{k} a_j u_j(t), \sum_{j=0}^{k} a_j u_j(t)\Big) \\
&= \lambda_k \|u\|^2.
\end{aligned}$$

Then (3.1) is verified. By using the same method, we can get (3.2). The proof is complete. □

Now we prove Theorem 6.

Proof. We shall apply Theorem 4 to the functional I. Clearly $I \in C^1(H_T, \mathbb{R})$. By (F_3), we can get $F(t,0) = 0$, so we can say that $I(0) = 0$. Now, we will verify that I satisfies the rest conditions of Theorem 5.

Firstly, we show that I satisfies the P.S. condition.

Let $\{u_k\}_{k\in\mathbb{Z}(1)} \subset H_T$, $I'(u_k) \to 0$ as $k \to \infty$ and $\{I(u_k)\}$ is bounded.

Set

$$\Gamma_1 = \big\{t | t \in \mathbb{Z}[1,T], |u_k(t)| \geq R_1\big\}, \quad \Gamma_2 = \big\{t | t \in \mathbb{Z}[1,T], |u_k(t)| < R_1\big\}.$$

Then, by (F_2), we obtain

$$\begin{aligned}
I(u_k) &= \frac{1}{2}\sum_{t=1}^{T} |\Delta^2 u_k(t)|^2 + \sum_{t=1}^{T} F(t, u_k(t)) \\
&\geq \sum_{t\in\Gamma_1}\big(\beta |u_k(t)|^2 - \gamma\big) + \sum_{t\in\Gamma_2} F(t, u_k(t)) \\
&\geq \beta\|u_k\|^2 - T\gamma + \sum_{t\in\Gamma_2}\big(F(t, u_k(t) - \beta|u_k(t)|^2\big).
\end{aligned}$$

The continuity of $F(t,x)-\beta|x|^2$ with x implies that there exists a positive constant M such that for any $t\in\mathbb{Z}[1,T],\ |x|<R_1,\ F(t,x)-\beta|x|^2\geq -M$. Then we have

$$I(u_k)\geq \beta\|u_k\|^2-T(\gamma+M).$$

Since $\{I(u_k)\}$ is bounded, then $\{u_k\}_{k\in\mathbb{Z}(1)}$ is bounded. As a consequence in finite dimensional space H_T, $\{u_k\}_{k\in\mathbb{Z}(1)}$ has a convergent subsequence, thus the P.S. condition holds.

Secondly, we claim that I has a local linking at 0, that is, I satisfies (2.2) and (2.3). Also we have $\inf_{u\in H_T} I(u)\leq 0$.

It follows from (F_3) and Lemma 9 that

$$\begin{aligned} I(u) &= \tfrac{1}{2}\sum_{t=1}^{T}|\Delta^2 u(t)|^2+\sum_{t=1}^{T}F(t,u(t)) \\ &\leq \tfrac{1}{2}\lambda_k\|u\|^2-\tfrac{1}{2}\lambda_k\sum_{t=1}^{T}|u(t)|^2 \\ &= 0 \end{aligned}$$

for all $u\in H_k$ with $\|u\|\leq\delta$.

Similar to the above, we conclude that

$$\begin{aligned} I(u) &= \tfrac{1}{2}\sum_{t=1}^{T}|\Delta^2 u(t)|^2+\sum_{t=1}^{T}F(t,u(t)) \\ &\geq \tfrac{1}{2}\lambda_{k+1}\|u\|^2-\tfrac{1}{2}\lambda_{k+1}\sum_{t=1}^{T}|u(t)|^2 \\ &= 0 \end{aligned}$$

for all $u\in H_k^{\perp}$ with $\|u\|\leq\delta$.

Then I has a local linking at 0, which implies that 0 is a critical point of I. At the same time, we get that $\inf\limits_{u\in H_T} I(u)\leq 0$.

In the case that $\inf\limits_{u\in H_T} I(u)<0$, our result follows from Theorem 4.

In the case that $\inf\limits_{u\in H_T} I(u)=0$, from the above, we have $I(u)=\inf\limits_{u\in H_T} I(u)=0$ for all $u\in H_k$ with $\|u\|\leq\delta$, which implies that all $u\in H_k$ with $\|u\|\leq\delta$ are minimum points of I. Hence all $u\in H_k$ with $\|u\|\leq\delta$ are solutions of problem (1.1), and (1.1) has infinite solutions in H_T. Therefore, Theorem 6 is verified. The proof of Theorem 6 is complete. □

Theorem 10. *Assume that the following conditions are satisfied*

(F_4) *$F(t,x)$ is even in x and there exists a positive integer $T\geq 2$ such that $F(t+T,x)=F(t,x)$ for all $(t,x)\in\mathbb{Z}\times\mathbb{R}^N$;*

(F_5) *there exist constants* $a_1 > 0, a_2 > 0$ *and* $\beta > 2$ *such that for all* $x \in \mathbb{R}^N$,

$$F(t,x) \leq -a_1|x|^\beta + a_2;$$

(F_6) $F(t,x) = o(|x|^2)$ *as* $|x| \to 0$ *uniformly in* t.

Then the problem (1.1) *has an infinite number of* T*-periodic solutions.*

Proof. $I \in C^1(H_T, \mathbb{R})$, by (2.1), ($F_4$) and ($F_6$), I is an even functional and $I(0) = 0$.

We divide our proof into three parts in order to show Theorem 10.

Firstly, we show that I satisfies the P.S. condition.

Let $\{u_k\}_{k\in\mathbb{Z}(1)} \subset H_T$ and $\{I(u_k)\}$ is bounded. Then, there exists $M_1 > 0$ such that for all $k \in \mathbb{Z}(1)$,

$$|I(u_k)| \leq M_1. \tag{3.3}$$

On the other hand, by (F_5), for every $u \in H_T$, we have

$$\begin{aligned} I(u) &\leq \frac{1}{2}\sum_{t=1}^{T}\big(\Delta u(t) - \Delta u(t-1)\big)^2 - a_1\sum_{t=1}^{T}|u(t)|^\beta + a_2T \\ &\leq \frac{1}{2}\sum_{t=1}^{T}2\big(|\Delta u(t)|^2 + \Delta|u(t-1)|^2\big) - a_1\|u\|_\beta^\beta + a_2T \\ &= 2\sum_{t=1}^{T}(\Delta u(t))^2 - a_1\|u\|_\beta^\beta + a_2T \\ &\leq 2\sum_{t=1}^{T}2\big(|u(t+1)|^2 + |u(t)|^2\big) - a_1\|u\|_\beta^\beta + a_2T \\ &= 8\sum_{t=1}^{T}|u(t)|^2 - a_1\|u\|_\beta^\beta + a_2T \\ &= 8\|u\|^2 - a_1\|u\|_\beta^\beta + a_2T \\ &\leq 8\|u\|^2 - a_1(\frac{1}{C_2})^\beta\|u\|^\beta + a_2T. \end{aligned} \tag{3.4}$$

Hence, by (3.3) and (3.4), we have for all $k \in \mathbb{Z}(1)$,

$$-M_1 \leq I(u_k) \leq 8\|u_k\|^2 - a_1\left(\frac{1}{C_2}\right)^\beta\|u_k\|^\beta + a_2T.$$

That is,

$$a_1\left(\frac{1}{C_2}\right)^\beta\|u_k\|^\beta - 8M_2\|u_k\|^2 \leq M_1 + a_2T, \qquad \forall k \in \mathbb{Z}(1).$$

By $\beta > 2$, $\{u_k\}$ is bounded on H_T. Since H_T is finite dimensional, $\{u_k\}$ has a convergent subsequence, and the P.S. condition holds.

Secondly, we verify the condition (I_2) of Theorem 5.

For arbitrary finite dimensional subspace $\widetilde{E} \subset H_T$, any given $\varphi \in \widetilde{E}$, $\|\varphi\| = 1$ and $\lambda > 0$, by the proof of (3.4), we have

$$\begin{aligned} I(\lambda\varphi) &\leq 8\lambda^2\|\varphi\|^2 - a_1\lambda^\beta(\tfrac{1}{C_2})^\beta\|\varphi\|^\beta + a_2T \\ &= 8\lambda^2 - a_1\lambda^\beta(\tfrac{1}{C_2})^\beta + a_2T \\ &\to -\infty \quad (\lambda \to +\infty). \end{aligned}$$

So there exists $R(\widetilde{E}) > 0$ such that $I \leq 0$ on $\widetilde{E}\backslash B_{R(\widetilde{E})}$.

Finally, we verify the condition (I_1) of Theorem 5.

Take $k_0 \in [0, [\frac{T}{2}] - 1]$, by Lemma 8 and Lemma 9, we have $H_T = H_{k_0} \bigoplus H_{k_0}^\perp$.

By the condition (F_6), we obtain

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0.$$

Hence, $\forall\epsilon > 0$, there exists $\delta > 0$ such that for every x with $|x| \leq \delta$,

$$|F(t,x)| \leq \epsilon|x|^2.$$

For any $u \in H_{k_0}^\perp$ with $\|u\| \leq \delta$, then $|u(t)| \leq \delta$, $t \in \mathbb{Z}(1,T)$. Then, we have

$$\begin{aligned} I(u) &= \tfrac{1}{2}\sum_{t=1}^{T}(\Delta^2u(t))^2 + \sum_{t=1}^{T} F(t,u(t)) \\ &\geq \tfrac{\lambda_{k_0+1}}{2}\|u\|^2 - \epsilon\sum_{t=1}^{T}|u(t)|^2 \\ &= \tfrac{\lambda_{k_0+1}}{2}\|u\|^2 - \epsilon\|u\|^2. \end{aligned}$$

Take $\epsilon = \frac{1}{4}\lambda_{k_0+1}$ and $\alpha = \frac{1}{4}\lambda_{k_0+1}\delta^2$, then

$$I(u) \geq \alpha, \quad \forall u \in H_{k_0}^\perp \bigcap \partial B_\delta.$$

By Theorem 5, I possesses infinite critical points, that is, problem (1.1) has infinite nontrivial T-periodic solutions. $\square$

Remark 11. *Take $F(t,x) = -|x|^4$. Then $F(t,x)$ satisfy all conditions of Theorem 10.*

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