FUNCTION VALUED METRIC SPACES

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Abstract. In this paper we introduce the notion of an $\mathcal{F}$-metric, as a function valued distance mapping, on a set X and we investigate the theory of $\mathcal{F}$-metric spaces. We show that every metric space may be viewed as an $\mathcal{F}$-metric space and every $\mathcal{F}$-metric space (X, δ) can be regarded as a topological space (X, τ_δ). In addition, we prove that the category of the so-called extended $\mathcal{F}$-metric spaces properly contains the category of metric spaces. We also introduce the concept of an $\bar{\mathcal{F}}$-metric space as a completion of an $\mathcal{F}$-metric space and, as an application to topology, we prove that each normal topological space is $\bar{\mathcal{F}}$-metrizable.

1 Introduction

The celebrated paper of Zadeh [15], motivated some authors to define and discuss some notions of a fuzzy metric on a set and fuzzy norm on a linear space. A probabilistic metric space is a fuzzy generalization of metric spaces where the distance is no longer defined on positive real numbers, but on distribution functions. For an account on probabilistic metric spaces the reader is referred to the book [13]. Katsaras [6] in 1984 introduced a concept of a fuzzy norm. Later, Felbin [4] introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real to each element of the fuzzy linear space so that the corresponding metric associated to this fuzzy norm is of the Kaleva type [5] fuzzy metric. In 2003, following [3], Bag and Samanta in [1] and [2], introduced and studied an idea of a fuzzy norm on a linear space in such a manner that its corresponding fuzzy metric is of Kramosil and Michalek type [8]. All of these can be assumed as norms whose values are mappings.

On the other hand, if $\mathcal{A}$ is a C^*-algebra then a Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module $\mathcal{E}$ (which is at the same time a complex vector space) equipped with an $\mathcal{A}$-valued inner product. The reader is referred to [9, 10, 14] and [12] for further information on Hilbert C^*-modules. If $\mathcal{A}$ is a commutative C^*-algebra then an $\mathcal{A}$-valued inner product is a function valued inner product.

This point of view to the metric, norm and inner product motivates us to work on function valued metrics on sets. Whence, in light of the definition of a usual metric and the definition of a Hilbert C^*-module, one can find an interesting idea to define the notion of an $\mathcal{F}$-metric δ on a set X as a mapping from $X \times X$ into the positive cone $\mathcal{A}^+$ of a C^*-algebra $\mathcal{A}$. Here, we do this in the case of commutative unital C^*-algebras, i.e. $\mathcal{A} = \mathcal{C}(\Omega)$, where Ω is a compact Hausdorff topological space. The partial order $\leq$ defined on the self-adjoint part $\mathcal{A}_s$ of $\mathcal{A}$ helps us to consider the triangle inequality for $\mathcal{F}$-metrics. Once an $\mathcal{F}$-metric is defined on a set X, we may construct the open neighborhood at a point $x \in X$ with radius $r \in \mathcal{A}^+$. Moreover, we are interested in viewing an $\mathcal{F}$-metric space (X, δ) as a topological space (X, τ_δ) and this is done by using some open neighborhoods whose radiuses form a downwards directed set. The downwards directed assumption helps us to prove that the intersection of a finitely many number of open neighborhoods is again an open neighborhood. Obviously, we expect that each metric space is an $\mathcal{F}$-metric space, for every commutative C^*-algebra $\mathcal{A}$. However, if each $\mathcal{A}$-metric space (X, δ) is metrizable then the notion of an $\mathcal{F}$-metric space is in vain, since we can consequently infer that these two theories coincide. So it will be nice, and indeed necessary, to find an example of an $\mathcal{F}$-metric space (X, δ) which is not metrizable. We do these in the following and as an application to topology, we prove that each normal topological space is $\bar{\mathcal{F}}$-metrizable.

In this paper we assume that Ω is a compact Hausdorff topological space and $\mathcal{A} = \mathcal{C}(\Omega)$, the commutative unital algebra of all continuous mappings $f : \Omega \to \mathbb{C}$ with the norm $\|.\|_\infty$ defined by $\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$. The unit of $\mathcal{A}$ is denoted by ι, i.e. $\iota : \Omega \to \mathbb{C}$ is defined by $\iota(\omega) = 1$. A simple modification gives similar results concerning the non-unital case.

2 Preliminaries

An element f in $\mathcal{A}$ is called positive (denoted by $0 \leq f$ or $f \geq 0$) if its range, $f(\Omega)$, is a subset of the non-negative real numbers $\mathbb{R}^+$. It is strictly positive (denoted by $0 \prec f$ or $f \succ 0$) if $f(\Omega)$ is a subset of the positive real numbers $\mathbb{R}^{++}$. Note that $f \succ 0$ is *not* equivalent to $f > 0$ (i.e. $f \geq 0$ and $f \neq 0$). The set of all positive elements and the set of all strictly positive elements of $\mathcal{A}$ are denoted by $\mathcal{A}^+$ and $\mathcal{A}^{++}$, respectively. Obviously, $f \in \mathcal{A}^+$ is strictly positive if and only if it is invertible in $\mathcal{A}$. There is a nice criterion for invertabilty of a positive element of $\mathcal{A}$ as follows (see Proposition 3.2.12 of [11]). It can be viewed as the C^*-Archimedean property.

Proposition 1. *A positive element f of $\mathcal{A}$ is invertible if and only if $f \geq \lambda\iota$ for some $\lambda \in \mathbb{R}^{++}$.*

A simple argument shows that $\mathcal{A}^{++}$ is downwards directed, i.e. if $f, g \in \mathcal{A}^{++}$ then there is an $h \in \mathcal{A}^{++}$ such that $h \preceq f$ and $h \preceq g$. Moreover, if $f \in \mathcal{A}^{++}$ then

there is an $f_0 \in \mathcal{A}^{++}$ such that $f_0 \prec f$. Furthermore, $\mathcal{A}^{++}$ is a cone, in the sense that the sum of two strictly positive elements of $\mathcal{A}$ is again strictly positive and if $f \in \mathcal{A}^{++}$ then so is λf for all $\lambda \in \mathbb{R}^{++}$.

Let $f, g \in \mathcal{A}^+$. We use the notation $f \lhd g$ to show that $f(\omega) < g(\omega)$ for all $\omega \in \Omega$ with $g(\omega) \neq 0$. The relation $\lhd$ is transitive on $\mathcal{A}^+$ and $f \lhd g$ implies $f + h \lhd g + h$ for all real valued mapping h with $f + h \in \mathcal{A}^+$. Note that if $g \in \mathcal{A}^{++}$ then $f \lhd g$ is equivalent to $f \prec g$.

We also need a version of the axiom of completeness for non-empty bounded subsets of $\mathcal{A}$. For some reasons we have to consider a topology, other than the Euclidean one, to reach into this axiom. Consider the topological space $(\mathbb{R}, \tau_u)$, where τ_u is the upper limit topology on $\mathbb{R}$. Then a function $f : \Omega \to \mathbb{R}$ is upper semi-continuous if and only if $f : \Omega \to (\mathbb{R}, \tau_u)$ is continuous (see, for example [7]). This, together the fact that Ω is compact, imply that $\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\} < \infty$ for each upper semi-continuous mapping $f : \Omega \to \mathbb{R}$. Let $\mathcal{C}^{1/2}(\Omega)$ be the set of all upper semi-continuous mappings $f : \Omega \to \mathbb{R}$. Proposition 1.5.12 of [11] states that

Proposition 2. *A pointwise infimum of any number of elements in $\mathcal{C}^{1/2}(\Omega)$ and a supremum of finitely many elements will again define an element in $\mathcal{C}^{1/2}(\Omega)$. Furthermore, $\mathcal{C}^{1/2}(\Omega)$ is stable under addition and under multiplication with positive real numbers. Finally, $\mathcal{C}^{1/2}(\Omega)$ is closed under uniform convergence.*

Let $\mathcal{A}_u^+ = \mathcal{C}^{1/2}(\Omega)^+$, the space of all positive valued upper semi-continuous mappings on Ω, and $\mathcal{A}_u^{++} = \mathcal{C}^{1/2}(\Omega)^{++}$, the space of all strictly positive valued upper semi-continuous mappings on Ω. Clearly $\mathcal{A}^+ \subseteq \mathcal{A}_u^+, \mathcal{A}^{++} \subseteq \mathcal{A}_u^{++}$. We then can prove the following axiom of completeness.

Theorem 3. *Let $\mathcal{F}$ be a non-empty subset of $\mathcal{A}_u^+$. Then $\inf \mathcal{F}$ exists in $\mathcal{A}_u^+$. In other words, there is an $f_0 \in \mathcal{A}_u^+$ such that $f_0 \leq f$ for each $f \in \mathcal{F}$ and if g is any lower bound for $\mathcal{F}$ then $g \leq f_0$.*

Proof. For each $f \in \mathcal{F}$ and $\omega \in \Omega$ we have $f(\omega) \geq 0$. Thus the set $\{f(\omega) : f \in \mathcal{F}\}$ is a non-empty bounded below subset of $\mathbb{R}^+$ and so its infimum exists. Let

$$f_0(\omega) = \inf\{f(\omega) : f \in \mathcal{F}\}.$$

Then $f_0 \in \mathcal{A}_u^+$, by Proposition 2. Now let g be a lower bound for $\mathcal{F}$. Hence $g(\omega) \leq f(\omega)$ and so $g(\omega)$ is a lower bound for the set $\{f(\omega) : f \in \mathcal{F}\}$. Thus $f_0(\omega) \leq g(\omega)$ or equivalently $f_0 \leq g$. $\qquad\square$

3 $\mathcal{F}$-metrics

The notion of positive elements and similarity between the cone of positive elements of $\mathcal{A}_u$ and $\mathbb{R}^+$ tempt us to introduce the following notion.

Definition 4. *Let X be a set. A mapping $\delta : X \times X \to \mathcal{A}_u^+$ is called an $\mathcal{F}$-metric, or an $\mathcal{A}$-metric, if for all $x, y, z \in X$ the following conditions hold:*
(i) $\delta(x, y) = 0$ if and only if $x = y$;
(ii) $\delta(x, y) = \delta(y, x)$;
(iii) $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$. (triangle inequality)
In this case (X, δ) is called an $\mathcal{F}$-metric space, or an $\mathcal{A}$-metric space.

To illustrate the notion, let us give some examples.

Example 5. Let f be a non-zero positive element of $\mathcal{A}_u$. Then

$$\delta(x, y) = \begin{cases} f & \text{if x} \neq \text{y} \\ 0 & \text{otherwise} \end{cases}$$

gives an $\mathcal{A}$-metric δ on X which is called the discrete $\mathcal{F}$-metric on X constructed via f.

Example 6. Let $X = \mathcal{A}$. Then $\delta(f, g) = |f - g|$ defines an $\mathcal{F}$-metric on X. For the triangle inequality we have

$$\begin{aligned} \delta(f, g)(\omega) &= |f - g|(\omega) \\ &\leq |f - h|(\omega) + |h - g|(\omega) \\ &= \delta(f, h)(\omega) + \delta(h, g)(\omega), \end{aligned}$$

for all $\omega \in \Omega$ and $f, g, h \in X$. Thus $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$.

Example 7. Let $\Omega = \mathbb{C}$ and $X = \mathbb{C}$. For each $x \in X$, let f_x be the constant function $f_x(\omega) = x, (\omega \in \Omega)$. Then $\delta(x, y) = |f_x - f_y|$ is an $\mathcal{F}$-metric on X. Note that this is nothing but the Euclidean metric on $\mathbb{C}$.

Let (X, δ) be an $\mathcal{A}$-metric space. Similar to the case of metric spaces, we can define the ball $N_r(x)$ centered at $x \in X$ with radius $r \in \mathcal{A}_u^{++}$ by $N_r(x) = \{y \in X : \delta(x, y) \lhd r\}$. Interior points of a subset of X and open sets are defined in a usual manner. Note that $N_r(x)$ is an open set. In fact, if $y \in N_r(x)$ then $\delta(x, y) \lhd r$ and for the strictly positive element $r_0 = r - \delta(x, y)$ we have $N_{r_0}(y) \subseteq N_r(x)$. This shows that y is an interior point of $N_r(x)$.

The following theorem guarantees that the open subsets of an $\mathcal{F}$-metric space form a topology.

Theorem 8. *Let (X, δ) be an $\mathcal{A}$-metric space. Then the family of all open subsets of X with respect to δ forms a topology on X.*

Proof. We need to show that for an arbitrary family $\{N_{r_\gamma}(x_\gamma) : \gamma \in \Gamma\}$ of open balls, the set $U = \bigcup_{\gamma \in \Gamma} N_{r_\gamma}(x_\gamma)$ is open. Let $x \in U$. Then there is a $\gamma \in \Gamma$ such that $x \in N_{r_\gamma}(x_\gamma)$. Thus there is an r_0 such that $N_{r_0}(x) \subseteq N_{r_\gamma}(x_\gamma) \subseteq U$. So x is an interior point of U.

In addition, we have to show that $V = \bigcap_{j=1}^{n} N_{r_j}(x_j)$ is open. Let $x \in V$. Then $x \in N_{r_j}(x_j)$ and so there are $s_j \in \mathcal{A}_u^{++}$ such that $N_{s_j}(x) \subseteq N_{r_j}(x_j)$. Pick an $r_0 \in \mathcal{A}_u^{++}$ such that $r_0 \lhd s_j$ for all $1 \leq j \leq n$. (Note that $\mathcal{A}_u^{++}$ is downwards directed and so r_0 exists.) We then have $N_{r_0}(x) \subseteq V$. $\qquad\square$

The topology mentioned in the above theorem is called the topology on X induced by the $\mathcal{F}$-metric δ and is denoted by τ_δ.

We can consider a subset of $\mathcal{A}_u^+ \setminus \{0\}$ as the set of radiuses of our open balls as follows.

Definition 9. *Let (X, δ) be an $\mathcal{A}$-metric space. A non-empty subset $\mathcal{R}$ of $\mathcal{A}_u^+ \setminus \{0\}$ is called allowance with respect to δ if it is downwards directed, $\lambda\mathcal{R} \subseteq \mathcal{R}$ for each $\lambda \in \mathbb{R}^{++}$, and $\delta(x, y) \lhd r$ for some $r \in \mathcal{R}$ implies the existence of an element $r_0 \in \mathcal{R}$ and a $\lambda \in \mathbb{R}^{++}$ such that $r_0 + \delta(x, y) \lhd r \lhd \lambda r_0$. The $\mathcal{R}$-extended topology on X induced by δ, denoted by $\tau_\delta^{\mathcal{R}k}$, is defined to be the topology on X generated by the topological base $\{N_r^\delta(x)\}_{r \in \mathcal{R}k, x \in X}$.*

Note that the third condition helps us to prove that open balls are indeed open and so we have the following theorem.

Theorem 10. *Let (X, δ) be an $\mathcal{A}$-metric space and $\mathcal{R}$ be an allowance set with respect to δ. Then $\tau_\delta^{\mathcal{R}k}$ is indeed a topology on X.*

Definition 11. *Let $\mathcal{R}_w$ be the set of all $\lambda r \in \mathcal{A}_u^+$ such that $\lambda \in \mathbb{R}^{++}$ and $r \in \mathcal{A}_u^+$ has a finite support. Then $\mathcal{R}_w$ is an allowance set with respect to any $\mathcal{A}$-metric and is called the weak allowance subset of $\mathcal{A}_u^+$.*

The following theorem shows that we can view a metric space as an $\mathcal{F}$-metric space. To avoid any confusion, we will show the balls in (X, d) by N^d and the balls in (X, δ) by N^δ.

Theorem 12. *Suppose that (X, d) is a metric space and $0 \neq f \in \mathcal{A}_u^+$. If $\delta_f : X \times X \to \mathcal{A}_u^+$ is defined by $\delta_f(x, y) = d(x, y)f$, then (X, δ_f) is an $\mathcal{F}$-metric space. Let $\mathcal{R}$ be an allowance set with respect to δ_f.*
(i) If $\mathcal{R}$ has the Archimedean property, then $\tau_{\delta_f}^{\mathcal{R}k} = \tau_d$;
(ii) If there is an r in $\mathcal{R}$ such that $\lambda f \not\lhd r$ for any $\lambda \in \mathbb{R}^{++}$, then $\tau_{\delta_f}^{\mathcal{R}k}$ is the discrete topology.

Proof. Put $\delta = \delta_f$. To prove the triangle inequality for δ one should note that a multiple of a positive element of $\mathcal{A}_u$ by a positive real number is again a positive element of $\mathcal{A}_u$.

(i) We shall show that the family $\{N_{\lambda f}^\delta(x)\}_{\lambda \in \mathbb{R}^{++}, x \in X}$ forms a topological base for $\tau_\delta^{\mathcal{R}k}$. To see this, let $N_r^\delta(x)$ be an arbitrary open ball in $\tau_\delta^{\mathcal{R}k}$ and $y \in N_r^\delta(x)$. Since $\mathcal{R}$ is allowance, there is an $r_0 \in \mathcal{R}$ such that $r_0 \lhd r - \delta(x, y)$. By the Archimedean

property of $r_0 \in \mathcal{R}$, there is a $\lambda_0 \in \mathbb{R}^{++}$ such that $\lambda_0 f \lhd r_0$. Now $N_{\lambda_0 f}^\delta(y) \subseteq N_r^\delta(x)$, since for $z \in N_{\lambda_0 f}^\delta(y)$ we have $\delta(z,x) \leq \delta(z,y) + \delta(y,x) \lhd \lambda_0 f + \delta(x,y) \lhd r_0 + \delta(x,y) \lhd r$.

Let $N_{\lambda f}^\delta(x)$ be an arbitrary open ball in $\tau_\delta^{\mathcal{R}k}$ and $y \in N_{\lambda f}^\delta(x)$. Let $\lambda_0 = \lambda - d(x,y)$. Then $\lambda_0 \in \mathbb{R}^{++}$, since $\delta(x,y) \lhd \lambda f$ implies that $d(x,y)f \lhd \lambda f$ and so $\lambda - d(x,y) > 0$. We assert that $N_{\lambda_0}^d(y) \subseteq N_{\lambda f}^\delta(x)$. To see this, let $z \in N_{\lambda_0}^d(y)$. We have $d(z,y) < \lambda_0$ and so $\delta(z,x) \leq \delta(z,y) + \delta(y,x) = d(z,y)f + d(y,x)f \lhd \lambda_0 f + d(y,x)f = (\lambda - d(x,y) + d(x,y))f = \lambda f$. This shows that y is an interior point of $N_{\lambda f}^\delta(x)$ with respect to d. Thus $N_{\lambda f}^\delta(x) \in \tau_d$.

On the other hand, if $N_\lambda^d(x)$ is an arbitrary basis element of the topology τ_d and $y \in N_\lambda^d(x)$ then for $\lambda_0 = \lambda - d(x,y)$ we have $N_{\lambda_0 f}^\delta(y) \subseteq N_\lambda^d(x)$ and so $N_\lambda^d(x)$ is open with respect to $\tau_\delta^{\mathcal{R}k}$. Thus the topologies coincide.

(ii) If $\delta(x,y) \lhd r$ then $d(x,y)f \lhd r$, and so $d(x,y) = 0$ which implies $x = y$. Hence $N_r^\delta(x) = \{x\}$ for each $x \in X$ and the topology $\tau_\delta^{\mathcal{R}k}$ is then discrete. $\qquad \square$

We illustrate the above result by the following example.

Example 13. Let $X = \mathbb{C}$ and $\Omega = [0,1]$. If $u : \Omega \to \mathbb{C}$ is defined by $u(\omega) = \omega$ then $\delta(x,y) = |x-y|u$ is an $\mathcal{F}$-metric on X. The positive cone $\mathcal{A}^+$ is an allowance set with respect to δ. For the element $r \in \mathcal{A}^+$ defined by $r(\omega) = \omega^2$ we have $\lambda u \not\lhd r$ for any $\lambda \in \mathbb{R}^{++}$. It follows from Theorem 12 (ii) that $\tau_\delta^{\mathcal{A}k^+}$ is the discrete topology.

The following question arises from Theorem 12. For an $\mathcal{A}$-metric space (X, δ) can we find an element $f \in \mathcal{A}_u^+$ such that $\delta(x,y) = d(x,y)f$ for some metric d on X? Our next example provides a negative answer to the question.

Example 14. Consider $X = \mathcal{C}[0,1]$ equipped with $\delta(f,g) = |f-g|$ as a $\mathcal{C}[0,1]$-metric space. If there are a usual metric d on X and a (positive) element h in $\mathcal{C}[0,1]_u$ such that $\delta(f,g) = d(f,g)h$ for all $f,g \in \mathcal{C}[0,1]$, then $|f| = d(f,2f)h$ for all $f \in \mathcal{C}[0,1]$ which is clearly impossible.

We now want to find a sufficient condition on an $\mathcal{F}$-metric space to be metrizable. The following theorem states that if $\mathcal{R}$ has the Archimedean property then an $\mathcal{R}$-extended $\mathcal{F}$-metric space is nothing but a metric space.

Theorem 15. *Let (X, δ) be an $\mathcal{R}$-extended $\mathcal{A}$-metric space. If we define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \|\delta(x,y)\|$ then (X, d) is a metric space. Furthermore, if $\mathcal{R}$ has the Archimedean property then $\tau_d = \tau_\delta^{\mathcal{R}k}$.*

Proof. For the triangle inequality, we have

$$
\begin{aligned}
d(x,y) &= \|\delta(x,y)\| \\
&\leq \|\delta(x,z) + \delta(z,y)\| \\
&\leq \|\delta(x,z)\| + \|\delta(z,y)\| \\
&= d(x,z) + d(z,y),
\end{aligned}
$$

since $0 \le f \le g$ implies $\|f\| \le \|g\|$ for $f, g \in \mathcal{A}_u$.

Now suppose that $\mathcal{R}$ has the Archimedean property. Let $N_r^\delta(x)$ be an arbitrary basis open set in $\tau_\delta^{\mathcal{R}k}$ and $y \in N_r^\delta(x)$. Since $\mathcal{R}$ is allowance, there is an $f \lhd r - \delta(x, y)$ in $\mathcal{R}$ and so there is a $\lambda_0 \in \mathbb{R}^{++}$ such that $\lambda_0 \iota \lhd f$. We have $N_{\lambda_0}^d(y) \subseteq N_r^\delta(x)$ since for $z \in N_{\lambda_0}^d(y)$,

$$
\begin{aligned}
\delta(z, x) \quad &\le \quad \delta(z, y) + \delta(y, x) \\
&\le \quad \|\delta(z, y)\| \iota + \delta(x, y) \\
&\le \quad d(z, y)\iota + \delta(x, y) \\
&\lhd \quad \lambda_0 \iota + \delta(x, y) \\
&\lhd \quad f + \delta(x, y) \\
&= \quad r.
\end{aligned}
$$

Thus $N_r^\delta(x) \in \tau_d$.

Now let $N_\lambda^d(x)$ be an arbitrary basis open set in τ_d and $y \in N_\lambda^d(x)$. Let r be a fixed element of $\mathcal{R}$ and $r_0 = \frac{\lambda_0 r}{\|r\|}$, where $\lambda_0 = \frac{1}{2}(\lambda - d(x, y))$. Then $r_0 \in \mathcal{R} \cap \mathcal{A}^{++}$ and $N_{r_0}^\delta(y) \subseteq N_\lambda^d(x)$. To see this, let $z \in N_{r_0}^\delta(y)$. We have $\delta(z, y) \lhd r_0$ and since $r_0(\omega) \ne 0, (\omega \in \Omega)$ then $\delta(z, y)(\omega) < r_0(\omega)$ for all $\omega \in \Omega$. Thus $\|\delta(z, y)\| \le \|r_0\|$. Hence

$$
\begin{aligned}
d(z, x) \quad &\le \quad d(z, y) + d(y, x) \\
&= \quad \|\delta(z, y)\| + d(x, y) \\
&\le \quad \|r_0\| + d(x, y) \\
&= \quad \lambda_0 + d(x, y) \\
&< \quad \lambda.
\end{aligned}
$$

This shows that y is an interior point of $N_\lambda^d(x)$ with respect to $\tau_\delta^{\mathcal{R}k}$ and so $N_\lambda^d(x) \in \tau_\delta^{\mathcal{R}k}$. $\qquad \square$

Let us denote the category of all metric spaces and the category of all extended $\mathcal{F}$-metric spaces by Met and $\mathcal{F}$-ExtMet, respectively. We now aim to show that Met $\subsetneq$ $\mathcal{F}$-ExtMet, i.e. the inclusion is proper. Our proof is constructive.

Example 16. Let (X, τ_p) be the space of all continuous mappings $f : \Omega \to [0, 1]$ as a subspace of $\prod_{\omega \in \Omega}[0, 1] = [0, 1]^\Omega$ with the product topology, or equivalently the topology of pointwise convergence, induced by the Euclidean metric on $[0, 1]$. Then X is a Hausdorff non-metrizable topological space. We will show that X is $\mathcal{A}$-metrizable. Define $\delta : X \times X \to \mathcal{A}^+$ by $\delta(f, g) = |f - g|$. Then δ is an $\mathcal{A}$-metric (see Example 6). Let $\mathcal{R} = \mathcal{R}_w$, the weak allowance subset of $\mathcal{A}_u^+$. Then $\tau_\delta^{\mathcal{R}k} = \tau_p$. Suppose that $N_{\lambda r}(f)$ be a basis $\tau_\delta^{\mathcal{R}k}$-open set, where $\lambda \in \mathbb{R}^{++}$ and $r \in \mathcal{A}_u^+$ has a finite support. So there are $\omega_1, \ldots, \omega_n \in \Omega$ such that $r(\omega_j) \ne 0 (1 \le j \le n)$ and

$r(\omega) = 0$ for all $\omega \in \Omega \setminus \{\omega_1, \ldots, \omega_n\}$. Let $g \in N_{\lambda r}(f)$. Thus $|g - f|(\omega_j) < \lambda r(\omega_j)$. Put

$$U_{\omega_j} = (g(\omega_j) - \lambda r(\omega_j) + |f - g|(\omega_j), g(\omega_j) + \lambda r(\omega_j) - |f - g|(\omega_j)).$$

Then U_{ω_j} is open in $[0, 1]$ with respect to the Euclidean topology. Let $U_\omega = [0, 1]$ for $\omega \in \Omega \setminus \{\omega_1, \ldots \omega_n\}$. Then $U = \prod_{\omega \in \Omega} U_\omega$ is a τ_p-open neighborhood of g, since $g(\omega) \in U_\omega$ for all $\omega \in \Omega$. We have $U \subseteq N_{\lambda r}(f)$. To see this, let $h \in U$. Then $|h - f|(\omega_j) < \lambda r(\omega_j)$ and so $\delta(h, f) = |h - f| \lhd \lambda r$. Hence $\tau_\delta^{\mathcal{R}k} \subseteq \tau_p$.

On the other hand, if $U = \prod_{\omega \in \Omega}$ is a τ_p-open subset of X then there is a finite set $F = \{\omega_1, \ldots, \omega_n\}$ such that U_ω is an open subset of $[0, 1]$ for $\omega \in F$ and is equal to $[0, 1]$ for all $\omega \in \Omega \setminus F$. Let $f \in U$. Then $f(\omega_j) \in U_{\omega_j}$ and so there is an $r_{\omega_j} \in \mathbb{R}^{++}$ such that $(f(\omega_j) - r_{\omega_j}, f(\omega_j) + r_{\omega_j}) \subseteq U_{\omega_j}$. Let $r : \Omega \to \mathbb{R}$ be defined by $r(\omega_j) = r_{\omega_j}$ and 0 elsewhere. Hence $N_r(f) \subseteq U$. Thus U is $\tau_\delta^{\mathcal{R}k}$-open.

By the same argument, we can show that the space of all upper semi-continuous mappings $f : \Omega \to [0, 1]$ as a subspace of $[0, 1]^\Omega$ equipped with the topology of pointwise convergence is $\mathcal{F}$-metrizable.

Let (X, δ) be an $\mathcal{A}$-metric and $\mathcal{R}$ be an allowance set with respect to δ. We want to give some simple facts concerning the convergence of nets in X and the continuity of mappings on $(X, \tau_\delta^{\mathcal{R}k})$.

Lemma 17. *Let (X, δ) be an $\mathcal{A}$-metric space and $\{x_\gamma\}_{\gamma \in \Gamma}$ be a net in X. Then $\{x_\gamma\}$ converges to $x \in X$ with respect to $\tau_\delta^{\mathcal{R}k}$ if and only if for every $\varepsilon \in \mathcal{R}$ there is a $\gamma_\varepsilon \in \Gamma$ such that $\delta(x_\gamma, x) \lhd \varepsilon$, for all $\gamma \geq \gamma_\varepsilon$.*

Definition 18. *Let (X, δ) be an $\mathcal{R}$-extended $\mathcal{A}$-metric space and $\{x_\gamma\}$ be a net in X. $\{x_\gamma\}$ is called $\mathcal{R}$-Cauchy if for each $\varepsilon \in \mathcal{R}$ there is a $\gamma_\varepsilon \in \Gamma$ such that $\delta(x_\gamma, x_{\gamma'}) \lhd \varepsilon$, for all $\gamma, \gamma' \geq \gamma_\varepsilon$.*

Lemma 19. *Let $\mathcal{A}_1 = \mathcal{C}(\Omega_1)$ and $\mathcal{A}_2 = \mathcal{C}(\Omega_2)$, where Ω_1 and Ω_2 are compact Hausdorff topological spaces. Suppose that (X, δ_1) is an $\mathcal{A}_1$-metric space and (Y, δ_2) is an $\mathcal{A}_2$-metric space. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be allowance sets with respect to δ_1 and δ_2, respectively. If $\varphi : (X, \tau_{\delta_1}^{\mathcal{R}k_1}) \to (Y, \tau_{\delta_2}^{\mathcal{R}k_2})$ is a mapping and $x_0 \in X$, then φ is continuous at x_0 if and only if for each $\varepsilon \in \mathcal{R}_2$ there is an $\eta \in \mathcal{R}_1$ such that $\delta_1(x, x_0) \lhd \eta$ implies $\delta_2(\varphi(x), \varphi(x_0)) \lhd \varepsilon$, for each $x \in X$.*

Let $\delta : X \times X \to \mathcal{A}_u^+$ be an $\mathcal{A}$-metric and $\mathcal{R}$ be an allowance set with respect to δ. For fixed $x \in X$, the mapping $\delta_x : (X, \tau_\delta^{\mathcal{R}k}) \to (\mathcal{A}_u^+, \|.\|_\infty)$ defined by $\delta_x(y) = \delta(x, y)$ is obviously continuous. If we consider $\mathcal{A}_u^+$ as a set then we can put the final topology $t_{\delta,x}^{\mathcal{R}k}$ on $\mathcal{A}_u^+$ as the finest topology for which δ_x is continuous. Thus $t_{\delta,x}^{\mathcal{R}k} \subseteq t_{\|.\|_\infty}$, where $t_{\|.\|_\infty}$ is the topology induced by $\|.\|_\infty$. Moreover, a subset $\mathcal{U}$ of $\mathcal{A}_u^+$ is $t_{\delta,x}^{\mathcal{R}k}$-open if and only if $\delta_x^{-1}(\mathcal{U})$ is $\tau_\delta^{\mathcal{R}k}$-open in X. This shows that for a net $\{y_\gamma\}_{\gamma \in \Gamma}$ in $(X, \tau_\delta^{\mathcal{R}k})$, $y_\gamma \to x$ if and only if $\delta(x, y_\gamma) \to 0$ in $(\mathcal{A}_u^+, t_{\delta,x}^{\mathcal{R}k})$. From now on, as we now have several topologies on X and $\mathcal{A}_u$, we denote the topologies on X by the Greek letters

$(\tau, \tau_\delta$, etc.) and those on $\mathcal{A}_u$ by the Gothic letters ($\mathfrak{t}, \mathfrak{t}_\delta$, etc.). Also, we denote the τ-closure of a subset Y of X by τ-cl(Y) and the $\mathfrak{t}$-closure of a subset $\mathcal{F}$ of $\mathcal{A}_u$ by $\mathfrak{t}$-cl($\mathcal{F}$).

Let (X, δ) be an $\mathcal{A}$-metric space, $\mathcal{R}$ be an allowance set with respect to δ and let Y be a subset of X. For fixed $x \in X$, the set $\mathcal{F} = \{\delta(x, y) : y \in Y\}$ is a subset of $\mathcal{A}_u^+$. Thus, regarding to discussions mentioned above, we can consider $\mathfrak{t}_{\delta,x}^{\mathcal{R}}$-cl($\mathcal{F}$).

Definition 20. *Let (X, δ) be an $\mathcal{A}$-metric space, $\mathcal{R}$ be an allowance set with respect to δ, Y be a subset of X, x be a fixed element of X and $\mathcal{F} = \{\delta(x, y) : y \in Y\}$. Then $\mathfrak{t}_{\delta,x}^{\mathcal{R}}$-cl($\mathcal{F}$) is called the distance of the point $x \in X$ to the set Y and is denoted by $\delta(x, Y)$.*

Theorem 21. *Let (X, δ) be an $\mathcal{A}$-metric space, $\mathcal{R}$ be an allowance set with respect to δ and Y be a subset of X. Then $0 \in \delta(x, Y)$ if and only if $x \in \tau_\delta^{\mathcal{R}}$-cl($Y$).*

Proof. Let $0 \in \delta(x, Y)$. Since $0 \in \mathfrak{t}_{\delta,x}^{\mathcal{R}}$-cl($\mathcal{F}$) there is a net $\{y_\gamma\}_{\gamma \in \Gamma}$ such that $\delta(x, y_\gamma) \to 0$ with respect to $\mathfrak{t}_{\delta,x}^{\mathcal{R}k}$, or equivalently $y_\gamma \to x$ with respect to $\tau_\delta^{\mathcal{R}k}$. This implies $x \in \tau_\delta^{\mathcal{R}}$-cl($Y$).

Conversely, if $x \in \tau_\delta^{\mathcal{R}}$-cl($Y$) then there is a net $\{y_\gamma\}_{\gamma \in \Gamma}$ in $(X, \tau_\delta^{\mathcal{R}k})$ such that $y_\gamma \to x$ or equivalently $\delta(x, y_\gamma) \to 0$ with respect to $\mathfrak{t}_{\delta,x}^{\mathcal{R}k}$. Thus $0 \in \mathfrak{t}_{\delta,x}^{\mathcal{R}}$-cl($\mathcal{F}$) $= \delta(x, Y)$. $\square$

4 Completion of an $\mathcal{F}$-metric space ($\bar{\mathcal{F}}$-metric spaces)

Naturally we can say that an $\mathcal{R}$-extended $\mathcal{F}$-metric (X, δ) is $\mathcal{F}$-complete if each $\mathcal{R}$-Cauchy net of X converges in X. In the case of ordinary metrics, when we want to find the completion of a metric space, sometimes we have to extend the range of the metric into a large subspace of $\mathbb{R}$. For example, if we consider the Euclidean metric $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ then its completion $\tilde{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has $\mathbb{R}$ as its range. Thus if we want to construct a complete $\mathcal{F}$-metric space (X, δ) it is natural to have its range in $(\mathbb{R}^+)^\Omega$ instead of $\mathcal{A}_u^+$. So we have to generalize the notion of an $\mathcal{F}$-metric as follows.

Definition 22. *Let X be a set. A mapping $\delta : X \times X \to (\mathbb{R}^+)^\Omega$ is called an $\bar{\mathcal{F}}$-metric, or a $\bar{\mathcal{A}}$-metric, if for all $x, y, z \in X$ the following conditions hold:*
(i) $\delta(x, y) = 0$ if and only if $x = y$;
(ii) $\delta(x, y) = \delta(y, x)$;
(iii) $\delta(x, y)(\omega) \le \delta(x, z)(\omega) + \delta(z, y)(\omega)$ for all $\omega \in \Omega$.
In this case (X, δ) is called an $\bar{\mathcal{F}}$-metric space, or an $\bar{\mathcal{A}}$-metric space.

The terminology refers to the fact that $\mathcal{A}^+$ is dense in $((\mathbb{R}^+)^\Omega, \mathfrak{t}_p)$, where $\mathfrak{t}_p$ is the product topology on $(\mathbb{R}^+)^\Omega$. To see this note that by Proposition 1.5.13 of [11], since Ω is compact Hausdorff and hence normal, each upper semi-continuous mapping $f : \Omega \to \mathbb{R}^+$ is the pointwise infimum of continuous mappings. Thus finite

support positive valued mappings are in the $\mathfrak{t}_p$-closure of $\mathcal{A}^+$. Now for $f \in (\mathbb{R}^+)^\Omega$ consider the net $\{r_F\}_F$, where $r : \Omega \to \mathbb{R}^+$ has finite support F and $r_F(\omega) \leq f(\omega)$ for all $\omega \in \Omega$, which is directed by $r_F \leq r'_{F'}$ if and only if $r'_{F'}|_F = r_F$. Then f is the pointwise limit of the net $\{r_F\}_F$.

All notions, notations and facts concerning $\mathcal{F}$-metrics and $\mathcal{R}$-extended $\mathcal{F}$-metrics are still valid. Moreover, regarding to Example 16 we have

Theorem 23. *The compact Hausdorff topological space $([0,1]^\Omega, \mathfrak{t}_p)$, where $\mathfrak{t}_p$ is the product topology, is $\bar{\mathcal{F}}$-metrizable.*

Proof. Clearly, $([0,1]^\Omega, \mathfrak{t}_p)$ is Hausdorff and Tychonoff's Theorem guarantees that it is compact. Set $X = [0,1]^\Omega$ and define $\delta : X \times X \to [0,1]^\Omega$ by $\delta(f,g)(\omega) = |f(\omega) - g(\omega)|$. Then δ is an $\bar{\mathcal{F}}$-metric and by the same argument as in Example 16 we can show that $\mathfrak{t}_p = \tau_\delta^{\mathcal{R}k}$, where $\mathcal{R}$ is the weak allowance subset of $[0,1]^\Omega$. $\square$

The following result can be expected, whose proof is the same as its ordinary version.

Theorem 24. *For any $\mathcal{R}$-extended $\bar{\mathcal{A}}$-metric space (X, δ) there exists a unique (up to an isometry) complete $\mathcal{R}$-extended $\bar{\mathcal{A}}$-metric space $(\tilde{X}, \tilde{\delta})$ and an isometry φ of X onto a dense subspace of $\tilde{X}$.*

5   Application to topology

As an application of the concept of $\mathcal{F}$-metrics, we prove in this section that each normal topological space is Π-metrizable. Prior to that, let us state a simple result.

Proposition 25. *Each $\mathcal{R}$-extended $\bar{\mathcal{F}}$-metric space (X, δ) is Hausdorff.*

Proof. Let $x, y \in X$ and $x \neq y$. Then $\delta(x,y) \neq 0$. We claim that there is an $r_0 \in \mathcal{R}$ such that $\delta(x,y) \not\triangleleft r_0$. Otherwise for fix $r \in \mathcal{R}$ we would have $\delta(x,y) \triangleleft \lambda r$ for all $\lambda \in \mathbb{R}^{++}$ which implies $\delta(x,y) = 0$ as a contradiction. Now for $r = \frac{r_0}{2}$ we have $N_r^\delta(x) \cap N_r^\delta(y)$ is the empty set, because $z \in N_r^\delta(x) \cap N_r^\delta(y)$ implies that $\delta(x,y) \leq \delta(x,z) + \delta(z,y) \triangleleft 2r = r_0$ which is impossible. $\square$

Theorem 26. *Each normal topological space (X, τ) is $\bar{\mathcal{F}}$-metrizable, in the sense that there is a compact Hausdorff topological space Ω, a $\overline{\mathcal{C}(\Omega)}$-metric δ and an allowance set $\mathcal{R}$ with respect to δ such that $\tau_\delta^{\mathcal{R}k} = \tau$.*

Proof. Since X is normal, by Urysohn's Lemma, for each $U, V \in \tau$ with τ-cl$(U) \subseteq V$ there is a continuous mapping $f_{U,V} : X \to [0,1]$ with the property that $f_{U,V}(x) = 0$ for each $x \in \tau$-cl(U) and $f_{U,V}(x) = 1$ for each $x \in X \setminus V$. Now let

$$\Omega = \{(U,V) : U, V \in \tau \text{ and } \tau-\text{cl(U)} \subseteq \text{V}\}.$$

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If we consider the product topology on $2^\Omega = \{0,1\}^\Omega$ then 2^Ω will be a compact Hausdorff space in which Ω can be embedded by $\omega \hookrightarrow \chi_{\{\omega\}}$ as a closed subspace. Consequently, Ω can be regarded as a compact Hausdorff topological space. A net $\{(U_\gamma, V_\gamma)\}_{\gamma \in \Gamma}$ converges to (U, V) if and only if for each $\omega \in \Omega$ there is a $\gamma_\omega \in \Gamma$ such that $\chi_{U_\gamma}(\omega) = \chi_U(\omega)$ and $\chi_{V_\gamma}(\omega) = \chi_V(\omega)$ for all $\gamma \geq \gamma_\omega$.

Define $\varphi : X \to [0,1]^\Omega$ by

$$\varphi(x)(\omega) = f_{U,V}(x),$$

for each $\omega = (U, V) \in \Omega$. Then φ is one to one. To see this, let $x, y \in X$ and $x \neq y$. Then there are $U, V \in \tau$ such that $x \in U \subseteq \tau\text{-cl}(U) \subseteq V \subseteq X \setminus \{y\}$. For $\omega = (U, V)$, we have

$$\varphi(x)(\omega) = f_{U,V}(x) = 0 \neq 1 = f_{U,V}(y) = \varphi(y)(\omega).$$

Thus $\varphi(x) \neq \varphi(y)$.

Moreover, φ is continuous. For each $\omega = (U, V) \in \Omega$ we have $\pi_\omega(\varphi(x)) = f_{U,V}(x)$, where $\pi_\omega : [0,1]^\Omega \to [0,1]$ is the projection defined by $\pi_\omega(f) = f(\omega), (f \in [0,1]^\Omega)$. This implies that the mappings $\pi_\omega \circ \varphi = f_{U,V}$'s are continuous and so is φ, by the definition of the product topology.

Now let V be a τ-open subset of X and $x \in V$. Thus there is a $U \in \tau$ such that $x \in U \subseteq \tau\text{-cl}(U) \subseteq V$. Let $\omega = (U, V)$. We then have

$$\varphi(x) \in \{f \in [0,1]^\Omega : f(\omega) < \frac{1}{2}\} \cap \varphi(X) \subseteq \varphi(V) \quad (*)$$

since $\varphi(x)(\omega) = f_{U,V}(x) = 0 < \frac{1}{2}$, and if $\varphi(z)(\omega) < \frac{1}{2}$ then $\varphi(z)(\omega) \neq 1$ so that $z \notin X \setminus V$. This implies that $z \in V$ and hence $\varphi(z) \in \varphi(V)$.

Now $(*)$ shows that x is an interior point of $\varphi(V)$ with respect to the relative topology on $\varphi(X)$ induced by τ_p. Hence $\varphi(V)$ is a relatively $\mathfrak{t}_p$-open subset of $\varphi(X)$. Thus $\varphi : X \to \varphi(X) \subseteq [0,1]^\Omega$ is a homeomorphism. Since $[0,1]^\Omega$ is $\bar{\mathcal{F}}$-metrizable, we can therefore deduce that so is X. Note that the suitable allowance set is $\mathcal{R} = \mathcal{R}_w$. $\qquad \square$

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