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UNCERTAINTY FUNCTIONAL DIFFERENTIAL EQUATIONS FOR FINANCE

Iuliana Carmen Bărbăcioru

Abstract. In this paper, we prove a local existence and uniqueness result for uncertain functional differential equation driven by canonical process.

1 Introduction

Differential equations with delay (or memory), known also as functional differential equations, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential equations with delay encompasses a large variety of differential equations. Differential equations with delay play an important role in an increasing number of system models in biology, engineering, physics and other sciences. There exists an extensive literature dealing with functional differential equations and their applications. We refer to the monographs [1], and references therein. In order to deal with general uncertainty, "self-duality" plus "countable subadditivity" is much more important than "continuity" and "semicontinuity". For this reason, Liu [3] founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms. Uncertainty theory provides the commonness of probability theory, credibility theory and chance theory.

Uncertain differential equation, proposed by Liu [3], is a type of differential equation driven by uncertain process [3]. Stochastic differential equation, fuzzy differential equation and hybrid differential equation are special cases of uncertain differential equation.

The aim of this paper is to prove the existence and uniqueness theorem for uncertain functional differential equation driven by uncertain process.

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2 Preliminaries

Let be Ω a nonempty set, and let $\mathcal{P}$ be a σ-algebra over. Each element $A \in \mathcal{P}$ is called an event. A mapping $\mathcal{M} : \mathcal{P} \to [0, 1]$ is called uncertain measure if it satisfies the following axioms [3]:

(A1) (Normality) $\mathcal{M}(\Omega) = 1$

(A2) (Monotonicity) $\mathcal{M}(A) \leq \mathcal{M}(B)$ whenever $A \subset B$

(A3) (Self-Duality) $\mathcal{M}(A) + \mathcal{M}(A^C) = 1$ for any event A. Here $A^C = \{\omega \in \Omega; \omega \notin A\}$

(A4) (Countable Subadditivity) For every countable sequence of events A_i, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}(A_i)$$

The triplet $(\Omega, \mathcal{P}, \mathcal{M})$ is called a uncertainty space.

Example 1. *Probability, credibility [2] and chance measures [3] are instances of uncertain measure.*

Example 2. *Let Pr be a probability measure, Cr a credibility measure [3], and a a number in $[0, 1]$. Then*

$$\mathcal{M}(A) = a \cdot Pr(A) + (1 - a)\, Cr(A)$$

is an uncertain measure.

A uncertain variable is defined as a measurable function $\xi : (\Omega, \mathcal{P}, \mathcal{M}) \to \mathbb{R}$, that is for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\omega \in \Omega \mid \xi(\omega) \in B\}$$

is an event. We denote by $\cup(\Omega)$ the space of fuzzy variables.

An uncertain variable ξ on the uncertainty space $(\Omega, \mathcal{P}, \mathcal{M})$ is said to be continuous if $\mathcal{M}\{\xi = x\}$ is a continuous function of x. Also, we say that $\xi_1 = \xi_2$ if $\xi_1(\omega) = \xi_2(\omega)$ for almost all $\omega \in \Omega$.

Example 3. *Random variable, fuzzy variable and hybrid variable are instances of uncertain variable [4].*

Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}(\xi \geq r)dr - \int_{-\infty}^{0} \mathcal{M}(\xi \leq r)dr$$

provided that at least one of the two integrals is finite. The variance of ξ is defined by

$$V[\xi] = E\left[(\xi - E[\xi])^2\right]$$

Let T be an index set and $(\Omega, \mathcal{P}, \mathcal{M})$ be an uncertainty space. A uncertain process is a family of uncertain variables $X : T \to \cup(\Omega)$ such that for each $t \in T$ and any Borel set B, the set

$$\{\xi \in B\} = \{\omega \in \Omega \mid X(t)(\omega) \in B\}$$

is an event.

An uncertain process X is said to have independent increments if

$$X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_k) - X(t_{k-1})$$

are independent uncertain variables for any times $t_0 < t_1 < ... < t_k$ that is,

$$E\left[\sum_{i=1}^{k} X(t_i)\right] = \sum_{i=1}^{k} E[X(t_i)]$$

A uncertain process X is said to have stationary increments if, for any given $t > 0$, the $X(s+t) - X(t)$ are identically distributed uncertain variables for all $s > 0$.

An uncertain process X is said to be continuous if the function $t \to X(t)(\omega)$ is continuous for almost all $\omega \in \Omega$.

An uncertain process W is said to be a canonical process if

(i) $W(0) = 0$ and W is continuous,

(ii) $W(t)$ has stationary and independent increments,

(iii) $W(1)$ is an uncertain variable with expected value 0 and variance 1.

In fact, standard Brownian motion and standard Liu C process are instances of canonical process.

3 Uncertain functional differential equation

Let X be a uncertain process. For a positive number r, we denote by C_r the space $C\left(\left[-r,0\right],\mathbb{R}\right)$. Also we denote by

$$D\left(\varphi,\psi\right)=\sup_{t\in[-r,0]}\left|\varphi(t)-\psi(t)\right|$$

the metric on the space C_r. For each $t>0$ we denote by $X_t\in C_r$ uncertain process defined by

$$X_t\left(s\right)=X\left(t+s\right),\quad s\in\left[-r,0\right]$$

X_t is called a uncertain process with delay (or memory) of the uncertain process X at moment $t\geq 0$.

Lemma 4. *If $F:[0,\infty)\times C_r\to\mathbb{R}$ is a jointly continuous function and X is continuous uncertain process, then the function $t\longmapsto F\left(t,X_t\right):[0,\infty)\to\mathbb{R}$ is also continuous.*

Proof. Let us fixed $(\tau,\varphi)\in[0,\infty)\times C_r$ and $\varepsilon>0$. Since $F:[0,\infty)\times C_r\to\mathbb{R}$ is jointly continuous, there exists $\delta_1>0$ such that, for every $(t,\psi)\in[0,\infty)\times C_r$ with $|t-\tau|+D\left(\varphi,\psi\right)<\delta_1$, we have that $\left|F\left(t,\psi\right)-F\left(\tau,\varphi\right)\right|<\varepsilon$. On the other hand, since X is continuous, then it is uniformly continuous on the compact interval $I_1=\left[\max\left\{-r,\tau-r-\delta_1\right\},\tau+\delta_1\right]$. Hence, there exists $\delta_2>0$ such that, for every $t_1,t_2\in I_1$ with $|t_1-t_2|<\delta_2$, we have that $|X(t_1)-X(t_2)|<\frac{\delta_1}{2}$. Next, since for every $s\in[-r,0]$ we have that $\tau+s\in I_1$ and $t+s\in I_1$ if $|t-\tau|<\frac{\delta_1}{2}$ then, by the fact that $|(t+s)-(\tau+s)|<\delta_2$, it follows that

$$D\left(X_t,X_\tau\right)=\sup_{s\in[-r,0]}\left|X_t(s)-X_\tau(s)\right|=\sup_{s\in[-r,0]}\left|X_t(t+s)-X_\tau(\tau+s)\right|\leq\frac{\delta_1}{2}$$

Therefore, $|t-\tau|+D\left(X_t,X_\tau\right)<\delta_1$ and hence, since F is jointly continuous, we have $|F(t,u_t)-F(\tau,u_\tau)|<\varepsilon$. This implies that the function $t\longmapsto F\left(t,u_t\right)$ is continuous. □

Suppose that W is a canonical process and $F,G:[\tau,b)\times C_r\to\mathbb{R}$ are some given function.

We consider the following uncertain functional differential equation

$$\begin{cases} dX(t)=F(t,X_t)dt+G(t,X_t)dW(t), & t\geq\tau\\ X(t)=\varphi\left(t-\tau\right), & \tau-r\leq t\leq\tau\end{cases}\tag{3.1}$$

By solution of uncertain functional differential equation (3.1) on some interval $[\tau,b)$ we mean a continuous function $X:[\tau-\sigma,b)\to\mathbb{R}$, that satisfies (3.1). We remark that $X : [\tau - \sigma, b) \to \mathbb{R}$ is a solution for (3.1) if

$$X(t) = \begin{cases} \varphi(t-\tau), & \text{for } \tau - r \le t \le \tau \\ \varphi(0) + \int\limits_{\tau}^{t} F(s, X_s)ds + \int\limits_{\tau}^{t} G(s, X_s)dW(s), & \text{for } \tau \le t < b \end{cases}$$

Example 5. *(Liu's Stock Model). It was assumed that stock price follows geometric Brownian motion, and stochastic financial mathematics was then founded based on this assumption. Liu [3] presented an alternative assumption that stock price follows uncertain process (see, [3]). Based on this assumption, we obtain a basic stock model with delay for uncertain financial market in which the bond price X and the stock price Y follows*

$$\begin{cases} dX(t) = r(t)X_t dt \\ dY(t) = e(t)Y_t dt + \sigma(t) Y_t dW(t) \end{cases}$$

where $r(t)$ is the riskless interest rate, $e(t)$ is the stock drift, $\sigma(t)$ is the stock diffusion, and $W(t)$ is a canonical process.

Theorem 6. *Suppose that $f : [0, \infty) \times C_r \to \mathbb{R}$ is a locally Lipschitz continuous function, that is, for all $a, b \in [0, \infty)$, there exists $L > 0$ such that*

$$|f(t, \varphi) - f(t, \psi)| \le LD(\varphi, \psi), \quad a \le t \le b, \quad \varphi, \psi \in B_\rho$$

where $B_\rho = \{\varphi \in C_\sigma; D(\varphi, 0) \le \rho\}$. Then there exists $T > 0$ such that the uncertain functional differential equation:

$$\begin{cases} dX(t) = f(t, X_t)dt + dW(t), & t \ge \tau \\ X(t) = \varphi(t-\tau), & \tau - r \le t \le \tau \end{cases} \tag{3.2}$$

has a unique solution on $[\tau - r, T]$.

Proof. Let $\rho > 0$ be any positive number. From the fact that f is locally Lipschitz, there exists $L > 0$ such that

$$|f(t, \varphi) - f(t, \psi)| \le LD(\varphi, \psi), \quad \tau \le t \le h, \quad \varphi, \psi \in B_\rho \tag{3.3}$$

for some $h > \tau$. Also, there exists $K > 0$ such that for $|f(t, \varphi)| \le K$ for $(t, \varphi) \in [\tau, h] \times B_{2\rho}$.

Since $W(t)$ is a continuous canonical process, there exists $M > 0$ such that $\max\limits_{0 \le s \le t} |W(s)| \le M$ for all times $0 \le t \le a$, where $a > 0$ is a fixed real number. Let $T := \min\left\{h, \dfrac{\rho}{K+M}\right\}$.Next, we consider the set E of all functions $u \in C([\tau - r, T], \mathbb{R})$ such that $u(t) = \varphi(t - \tau)$ on $[\tau - r, T]$ and $|u(t)| \le 2\rho$ on $[\tau, T]$.

Further, we observe that if $v \in E$ then we can define a continuous function $w : [\tau - r, T) \to \mathbb{R}$ by

$$w(t) = \begin{cases} \varphi(t-\tau), & \text{for } \tau - r \le t \le \tau \\ \varphi(0) + \int_{\tau}^{t} f(s, v_s) ds + W(t), & \text{for } \tau \le t < T \end{cases}$$

Then for $t \in [\tau, T]$ we have

$$\begin{aligned} |w(t)| &\le |\varphi(0)| + \left| \int_0^t f(s, X_s) ds + W(t) \right| \le \rho + \int_{\tau}^{t} |f(s, X_s)| \, ds + |W(t)| \\ &\le \rho + (K + M) T \le 2\rho \end{aligned}$$

and so $w \in E$.

To solve (3.2) we shell apply the method of successive approximations, constructing a sequence of continuous functions $X^m : [\tau - r, T] \to E$ starting with the initial continuous function

$$X^0(t) = \begin{cases} \varphi(t-\tau), & \text{for } \tau - r \le t \le \tau \\ \varphi(0), & \text{for } \tau \le t < T \end{cases}$$

Clearly, $\left| DX^0(t) \right| \le \rho$ on $[\tau, T]$. Next, we define

$$X^{m+1}(t) = \begin{cases} \varphi(t-\tau), & \text{for } \tau - r \le t \le \tau \\ \varphi(0) + \int_{\tau}^{t} f(s, X_s^m) ds + W(t), & \text{for } \tau \le t < T \end{cases} \tag{3.4}$$

if $m = 0, 1, \dots$. Then for $t \in [\tau, T]$, we have

$$\left| X^1(t) - X^0(t) \right| \le \left| \int_{\tau}^{t} f(s, X_s^m) ds \right| \le \int_{\tau}^{t} |f(s, X_s^m)| \, ds \le K(t - \tau).$$

By (3.3) and (3.4), we find that

$$\begin{aligned}\left|X^{m+1}(t)-X^{m}(t)\right| &\leq \left|\int_{\tau}^{t} f(s,X_s^m)ds-\int_{\tau}^{t} f(s,X_s^{m-1})ds\right| \\ &\leq \int_{\tau}^{t}\left|f(s,X_s^m)-f(s,X_s^{m-1})\right|ds \leq \int_{\tau}^{t} LD\left(X_s^m-X_s^{m-1}\right)ds \\ &\leq \int_{\tau}^{t} L\sup_{\theta\in[-r,0]}\left|X_s^m(\theta)-X_s^{m-1}(\theta)\right|ds \\ &\leq \int_{\tau}^{t} L\sup_{\theta\in[-r,0]}\left|X^m(\theta+s)-X^{m-1}(\theta+s)\right|ds \\ &\leq \int_{\tau}^{t} L\sup_{\zeta\in[s-r,s]}\left|X^m(\zeta)-X^{m-1}(\zeta)\right|ds\end{aligned}$$

$t\in[\tau,T]$. In particular,

$$\left|X^2(t)-X^1(t)\right|\leq L\int_{\tau}^{t} K(s-\tau)\,ds\leq \frac{K}{L}\frac{[L(t-\tau)]^2}{2!},\quad t\in[\tau,T].$$

Further, if we assume that

$$\left|X^m(t)-X^{m-1}(t)\right|\leq \frac{K}{L}\frac{[L(t-\tau)]^m}{m!},\quad t\in[\tau,T] \tag{3.5}$$

then we have

$$\left|X^{m+1}(t)-X^m(t)\right|\leq L\int_{\tau}^{t}\frac{K}{L}\frac{[L(t-\tau)]^m}{m!}ds=\frac{K}{L}\frac{[L(t-\tau)]^{m+1}}{(m+1)!},\quad t\in[\tau,T]$$

If follows by mathematical induction that (3.5) holds for any $m\geq 1$. Consequently, the series $\sum_{m=1}^{\infty}\left|X^m(t)-X^{m-1}(t)\right|$ is uniformly convergent on $[\tau,T]$ and so is the sequence $\{X^m\}_{m\geq 0}$.

It follows that there exists a continuous function $X:[\tau,T]\to\mathbb{R}$ i such that $|X^m(t)-X(t)|\to 0$ as $m\to\infty$. Since

$$|f(s,X_s^m)-f(s,X_s)|\leq LD(X_s^m,X_s)\leq\sup_{\tau\leq t\leq T}|X^m(t)-X(t)|$$

we deduce that $|f(s,X_s^m)-f(s,X_s)|\to 0$ uniformly on as $m\to\infty$.

Therefore, since

$$\left|\int_\tau^t f(s,X_s^m)ds-\int_\tau^t f(s,X_s)ds\right|\le\int_\tau^t |f(s,X_s^m)-f(s,X_s)|\,ds$$

it follows that $\lim\limits_{m\to\infty}\int_\tau^t f(s,X_s^m)ds=\int_\tau^t f(s,u_s)ds,\ \ t\in[\tau,T]$.

Extending X to $[\tau-r,\tau]$ in the usual way by $X(t)=\varphi(t-\tau)$ for $t\in[\tau-r,\tau]$ then by (3.4) we obtain that

$$X(t)=\begin{cases}\varphi(t-\tau), & \text{for } \tau-r\le t\le\tau\\ \varphi(0)+\int_\tau^t f(s,X_s)ds+W(t), & \text{for } \tau\le t\le T\end{cases}$$

and so $X(t)$ is a solution for (3.2).

To prove the uniqueness, assume that X and Y are solution of (3.2). Then for every $t\in[\tau,T]$ we have

$$\begin{aligned}|X(t)-Y(t)| &= \left|\int_\tau^t f(s,X_s)ds-\int_\tau^t f(s,Y_s)ds\right|\le\int_\tau^t |f(s,X_s)-f(s,Y_s)|\,ds\\ &\le \int_\tau^t LD(X_s-Y_s)\,ds\le L\int_\tau^t \sup_{\zeta\in[s-r,s]}|X(\zeta)-Y(\zeta)|\,ds\end{aligned}$$

If we let $\zeta(s)=\sup\limits_{\zeta\in[s-r,s]}|X(\zeta)-Y(\zeta)|,\ s\in[\tau,T]$ by Gronwall's lemma we obtain that $X(t)=Y(t)$ on $[\tau,T]$. This proves the uniqueness of the solution of (3.2). □

Theorem 7. *Assume that the function $f:[0,\infty)\times C_r\to\mathbb{R}$ is continuous and locally Lipschitz. If $[\tau,\varphi],(\tau,\psi)\in[0,\infty)\times C_r$ and $X(\varphi):[\tau-r,T_1)\to\mathbb{R}$ and $X(\psi):[\tau-r,T_2)\to\mathbb{R}$ are unique solutions of (3.2) with $X(t)=\varphi(t-\tau)$ and $Y(t)=\psi(t-\tau)$ on $[\tau-r,\tau]$. Then*

$$|X(\varphi)(t)-Y(\psi)(t)|\le D(\varphi,\psi)\,e^{L(t-\tau)}\quad \textit{for all } t\in[\tau,T) \tag{3.6}$$

where $T=\min\{T_1,T_2\}$.

Proof. On $[\tau,T)$ solution $X(\varphi)$ satisfies the relation

$$X(t)=\begin{cases}\varphi(t-\tau), & \text{for } t\in[\tau-r,\tau]\\ \varphi(0)+\int_\tau^t F(s,X_s(\varphi))ds, & \text{for } t\in[\tau,T)\end{cases}$$

and solution $X(\psi)$ satisfies the same relation but with ψ in place of φ. Then, for $t \in [\tau, T)$, we have

$$\begin{aligned} |X(\varphi)(t) - Y(\psi)(t)| &\leq |\varphi(0) - \psi(0)| + \int_{\tau}^{t} |F(s, X_s(\varphi) - F(s, X_s(\psi)| \, ds \\ &\leq D(\varphi, \psi) + L \int_{\tau}^{t} D[X_s(\varphi) - X_s(\psi)] \, ds \\ &\leq D(\varphi, \psi) + L \int_{\tau}^{t} \max_{\theta \in [\tau - r, s]} |X(\varphi)(\theta) - X(\psi)(\theta)| \end{aligned}$$

If we let $w(t) = \sup\limits_{\theta \in [\tau - r, s]} |X(\varphi)(\theta), X(\psi)(\theta)|$, $\tau \leq s \leq t$, then we have

$$w(t) \leq D(\varphi, \psi) + L \int_{\tau}^{t} w(s) ds, \quad t \in [\tau, T)$$

and Gronwall's inequality gives

$$w(t) \leq D(\varphi, \psi) e^{L(t-\tau)} \quad , t \in [\tau, T)$$

implying that (3.6) holds. □

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Iuliana Carmen Bărbăcioru
University Constantin Brâncuşi of Târgu Jiu,
Str. Geneva, Nr. 3, 210136 Târgu Jiu, Romania.
e-mail: bcicarmen@utgjiu.ro
http://www.utgjiu.ro/math/cbarbacioru/