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FULL AVERAGING OF FUZZY IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper the substantiation of the method of full averaging for fuzzy impulsive differential inclusions is studied. We extend the similar results for impulsive differential inclusions with Hukuhara derivative [23], for fuzzy impulsive differential equations [18], and for fuzzy differential inclusions [26].

1 Introduction

In recent years the fuzzy set theory introduced by L.Zadeh [29] has emerged as an interesting and fascinating branch of pure and applied sciences [1], [3], [4], [6] - [18], [21], [22], [27], [28]. The applications of the fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering sciences.

The concept of fuzzy differential inclusion was introduced in [24], where theorems of existence and continuous dependence on parameter of classical solutions of fuzzy differential inclusions were proved. In [19, 25] the concepts of ordinary, generalized and quasisolutions of fuzzy differential inclusions were studied, the relationship between sets of such solutions was investigated. The schemes of full and partial averaging for fuzzy differential inclusions was also considered [26].

In this paper the substantiation of the method of full averaging for fuzzy impulsive differential inclusions is considered. These results generalize the similar results for impulsive differential inclusions with Hukuhara derivative [23], for fuzzy impulsive differential equations [18], and for fuzzy differential inclusions [26].

2 Main definitions

Let $conv(\mathbb{R}^n)$ be a family of all nonempty compact convex subsets of $\mathbb{R}^n$ with Hausdorff metric

$$h(A,B) = \max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

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where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Let $\mathbb{E}^n$ be a family of mappings $x : \mathbb{R}^n \to [0,1]$ satisfying the following conditions:

1) x is normal, i.e. there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$;

2) x is fuzzy convex, i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$;

3) x is upper semicontinuous, i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $||y - y_0|| < \delta,\ y \in \mathbb{R}^n$;

4) a closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is compact.

Definition 1. *The set $\{y \in \mathbb{R}^n : x(y) \geq \alpha\}$ is called an α – level $[x]^\alpha$ of a mapping $x \in \mathbb{E}^n$ for $\alpha \in (0,1]$. A closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is called a 0 - level $[x]^0$ of a mapping $x \in \mathbb{E}^n$.*

It follows from 1) – 4) that the α-level set $[x]^\alpha \in conv(R^n)$ for all $\alpha \in [0,1]$.

Let $\widehat{0}$ be a fuzzy mapping defined by

$$\widehat{0}(y) = \begin{cases} 0 \ if \ y \neq 0, \\ 1 \ if \ y = 0. \end{cases}$$

Define the metric $D : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}_+$ by the equation

$$D(x,y) = \sup_{\alpha \in [0,1]} h([x]^\alpha, [y]^\alpha).$$

Let I be an interval in $\mathbb{R}$.

Definition 2. *A mapping $f : I \to \mathbb{E}^n$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta,\ t \in I$. A mapping $f : I \to \mathbb{E}^n$ is called continuous on I if it is continuous at every point $t_0 \in I$.*

Definition 3. *[15] A mapping $f : I \to \mathbb{E}^n$ is called measurable on I if a multivalued mapping $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable for any $\alpha \in [0,1]$.*

Definition 4. *[15] An element $g \in \mathbb{E}^n$ is called an integral of $f : I \to \mathbb{E}^n$ over I if $[g]^\alpha = (A)\int\limits_I f_\alpha(t)dt$ for any $\alpha \in (0,1]$, where $(A)\int\limits_I f_\alpha(t)dt$ is the Aumann integral [2].*

Definition 5. *A mapping $f : I \to \mathbb{E}^n$ is called absolutely continuous on I if there exists an integrable mapping $g : I \to \mathbb{E}^n$ such that*

$$f(t) = f(t_0) + \int\limits_{t_0}^{t} g(s)ds,\ t_0 \in I \ for\ every\ t \in I.$$

Definition 6. *[15] A mapping $f : I \to \mathbb{E}^n$ is called differentiable at point $t_0 \in I$ if the multivalued mapping $f_\alpha(t)$ is Hukuhara differentiable at point t_0 [5] for any $\alpha \in [0,1]$ and the family $\{D_H f_\alpha(t_0) \ : \ \alpha \in [0,1]\}$ defines a fuzzy number $f'(t_0) \in \mathbb{E}^n$ (which is called a fuzzy derivative of $f(t_0)$ at point t_0). A mapping $f : I \to \mathbb{E}^n$ is called differentiable on I if it is differentiable at every point $t_0 \in I$.*

Let $comp(\mathbb{E}^n)$ $[conv(\mathbb{E}^n)]$ be a family of all subsets F of $\mathbb{E}^n$ such that the family of all $\alpha-$ levels of the elements from F is a nonempty compact [and convex] element in $comp(\mathbb{R}^n)$ (that is an element of $cc(\mathbb{R}^n)$ $(cocc(\mathbb{R}^n))$ [11]) for any $\alpha \in [0,1]$ with metric

$$\varsigma(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} D(a,b), \sup_{b\in B}\inf_{a\in A} D(a,b)\}.$$

Define also the distance from an element $x \in \mathbb{E}^n$ to a set $A \in comp(\mathbb{E}^n)$:

$$dist(x,A) = \min_{a\in A} D(x,a).$$

Consider the usual algebraic operations in $comp(\mathbb{E}^n)$:
– addition : $F + G = \{f + g : f \in F, g \in G\}$;
– multiplication by scalars : $\lambda F = \{g = \lambda f : f \in F\}$.
The following properties hold [24]:
1) if $F, G \in comp(\mathbb{E}^n)$ $[conv(\mathbb{E}^n)]$, then $F + G \in comp(\mathbb{E}^n)$ $[conv(\mathbb{E}^n)]$;
2) if $F \in comp(\mathbb{E}^n)$ $[conv(\mathbb{E}^n)]$, then $\lambda F \in comp(\mathbb{E}^n)$ $[conv(\mathbb{E}^n)]$;
3) $F + G = G + F$;
4) $F + (G + H) = (F + G) + H$;
5) there exists a null element $\{\widehat{0}\}$: $F + \{\widehat{0}\} = F$;
4) $\alpha(\beta F) = (\alpha\beta)F$;
5) $1 \cdot F = F$;
6) $\alpha(F + G) = \alpha F + \beta G$;
7) if $\alpha \geq 0,\ \beta \geq 0$ and $F \in conv(\mathbb{E}^n)$, then $(\alpha + \beta)F = \alpha F + \beta F$; otherwise $(\alpha + \beta)F \subset \alpha F + \beta F$.

Definition 7. *A mapping $F : I \to comp(\mathbb{E}^n)$ is called a fuzzy multivalued mapping.*

Definition 8. *A fuzzy multivalued mapping $F : I \to comp(\mathbb{E}^n)$ is called measurable on I if the set $\{t \in I \ : \ F(t) \bigcap G \neq \emptyset\}$ is measurable for every $G \in comp(\mathbb{E}^n)$.*

Definition 9. *A fuzzy multivalued mapping $F : I \to comp(\mathbb{E}^n)$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that $\varsigma(F(t), F(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta,\ t \in I$. A fuzzy multivalued mapping $f : I \to \mathbb{E}^n$ is called continuous on I if it is continuous at every point $t_0 \in I$.*

Definition 10. *A mapping $f : I \to \mathbb{E}^n$ is called a selector of a fuzzy multivalued mapping $F : I \to comp(\mathbb{E}^n)$ if $f(t) \in F(t)$ for almost every $t \in I$.*

Obviously a selector $f(t)$ always exists as the set $F(t)$ is not empty for all $t \in I$. Define an integral of $F : I \to comp(\mathbb{E}^n)$ over I :

$$\int\limits_I F(t)dt = \left\{ \int\limits_I f(t)dt \ : \ f(t) \in F(t) \text{ almost everywhere on } I \right\}.$$

Consider a fuzzy differential inclusion

$$x' \in F(t,x), \quad x(t_0) = x_0, \tag{2.1}$$

where $t \in I$ is time; $x \in G \subset \mathbb{E}^n$ is a phase variable; the initial conditions $t_0 \in I, x_0 \in G$; a fuzzy multivalued mapping $F : I \times G \to comp(\mathbb{E}^n)$.

Definition 11. *An absolutely continuous fuzzy mapping $x(t)$, $x(t_0) = x_0$, is called an ordinary solution of differential inclusion (2.1) if*

1) $x(t) \in G$ for all $t \in I$;

2) $x'(t) \in F(t, x(t))$ almost everywhere on I.

Theorem 12. *Let the fuzzy multivalued mapping $F : I \times G \to comp(\mathbb{E}^n)$ satisfy the following conditions:*

1) $F(\cdot, x)$ is measurable on I;

2) $F(t, \cdot)$ satisfies the Lipschitz condition with the constant $k > 0$, i.e. for all $(t,x), (t,y) \in I \times G$ the inequality holds

$$\varsigma(F(t,x), F(t,y)) \le kD(x,y);$$

3) there exists an absolutely continuous fuzzy mapping $y(t)$, $y(t_0) = y_0$, such that $D(y(t), x_0) \le b$ and $dist(y'(t), F(t, y(t))) \le \eta(t)$ for almost all $t : |t - t_0| \le a$, where the function $\eta(t)$ is Lebesque summable.

Then on the interval $[t_0, t_0 + \sigma]$ there exists a solution $x(t)$ of fuzzy differential inclusion (2.1) such that $D(x(t), y(t)) \le r(t)$, where

$$r(t) = r_0 e^{k(t-t_0)} + \int\limits_{t_0}^{t} e^{k(t-s)} \eta(s) ds, \; r_0 = D(x_0, y_0),$$

$$\sigma = \min \left\{ a, \frac{b}{M} \right\}, \;\; M = \max_{(t,x) \in I \times G} d(F(t,x), \{\widehat{0}\}).$$

The proof is similar to [24].

3 Main Results

Consider the substantiation of the full averaging method on the finite interval for fuzzy impulsive differential inclusion

$$x' \in \varepsilon F(t,x),\ t \neq \tau_i,\ x(0)=x_0, \tag{3.1}$$
$$\Delta x|_{t=\tau_i} \in \varepsilon I_i(x).$$

If for any $t \geq 0, x \in G$ there exists a limit

$$\bar{F}(x) = \lim_{T\to\infty}\left(\frac{1}{T}\int_t^{t+T} F(t,x)dt + \frac{1}{T}\sum_{t\leq\tau_i<t+T} I_i(x)\right), \tag{3.2}$$

then in the correspondence to inclusion (3.1) we will set the following averaged inclusion

$$y' \in \varepsilon \bar{F}(y),\ y(0)=x_0. \tag{3.3}$$

Theorem 13. *Let in the domain $Q = \{t \geq 0,\ x \in G \subset \mathbb{E}^n\}$ the following hold:*

1) the fuzzy multivalued mappings $F : Q \to conv(\mathbb{E}^n)$, $I_i : G \to conv(\mathbb{E}^n)$ are continuous, uniformly bounded by M and satisfy the Lipschitz condition in x with constant λ;

2) uniformly with respect to $t \geq 0$ and $x \in D$ limit (3.2) exists and

$$\frac{1}{T}\, i(t,t+T) \leq d < \infty,$$

where $i(t,t+T)$ is the quantity of points of the sequence τ_i on the interval $(t,t+T]$;

3) for any $x_0 \in G' \subset G$ and $t \geq 0$ the solutions of inclusion (3.3) together with a $\rho-$ neighborhood belong to the domain G.

Then for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta,L) > 0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) for any solution $y(t)$ of inclusion (3.3) there exists a solution $x(t)$ of inclusion (3.1) such that

$$D(x(t),y(t)) < \eta; \tag{3.4}$$

2) for any solution $x(t)$ of inclusion (3.1) there exists a solution $y(t)$ of inclusion (3.3) such that inequality (3.4) holds.

Proof. From conditions 1), 2) it follows that the fuzzy multivalued mapping $\bar{F} : D \to conv(\mathbb{E}^n)$ is uniformly bounded by $M_1 = M(1+d)$ and satisfies the Lipschitz condition with constant $\lambda_1 = \lambda(1+d)$.

Namely

$$|\bar{F}(x)| = \varsigma(\bar{F}(x),\{\hat{0}\}) \leq \varsigma\left(\bar{F}(x), \frac{1}{T}\int_t^{t+T} F(t,x)dt + \frac{1}{T}\sum_{t\leq\tau_i<t+T} I_i(x)\right) +$$

$$+\varsigma\left(\frac{1}{T}\int_t^{t+T}F(t,x)dt+\frac{1}{T}\sum_{t\le\tau_i<t+T}I_i(x),\{\hat{0}\}\right)<\alpha+\frac{1}{T}\int_t^{t+T}|F(s,x)|ds+$$

$$+\frac{1}{T}\sum_{t\le\tau_i<t+T}|I_i(x)|<\alpha+M+dM=\alpha+M(1+d);$$

$$\varsigma(\bar{F}(x_1),\bar{F}(x_2))\le\varsigma\left(\bar{F}(x_1),\frac{1}{T}\int_t^{t+T}F(t,x_1)dt+\frac{1}{T}\sum_{t\le\tau_i<t+T}I_i(x_1)\right)+$$

$$+\varsigma\left(\frac{1}{T}\int_t^{t+T}F(t,x_1)dt+\frac{1}{T}\sum_{t\le\tau_i<t+T}I_i(x_1),\frac{1}{T}\int_t^{t+T}F(t,x_2)dt+\frac{1}{T}\sum_{t\le\tau_i<t+T}I_i(x_2)\right)+$$

$$+\varsigma\left(\frac{1}{T}\int_t^{t+T}F(t,x_2)dt+\frac{1}{T}\sum_{t\le\tau_i<t+T}I_i(x_2),\bar{F}(x_2)\right)<$$

$$<2\alpha+\frac{1}{T}\int_t^{t+T}\varsigma(F(s,x_1),F(s,x_2))ds+\frac{1}{T}\sum_{t\le\tau_i<t+T}\varsigma(I_i(x_1),I_i(x_2))\le$$

$$\le 2\alpha+\lambda h(x_1,x_2)+\lambda d h(x_1,x_2)=2\alpha+\lambda(1+d)D(x_1,x_2),$$

where α can be done arbitrary small by choosing T. Hence

$$|\bar{F}(x)|\le M(1+d),\ \ \varsigma(\bar{F}(x_1),\bar{F}(x_2))\le\lambda(1+d)D(x_1,x_2).$$

Let us proof the first statement of the theorem. Let $y(t)$ be a solution of inclusion (3.3). Divide the interval $[0,L\varepsilon^{-1}]$ on the partial intervals with a step $\gamma(\varepsilon)$ such that $\gamma(\varepsilon)\to\infty$ and $\varepsilon\gamma(\varepsilon)\to 0$ as $\varepsilon\to 0$. Then there exists a measurable selection $v(t)\in\bar{F}(y(t))$ such that

$$y(t)=y(t_j)+\varepsilon\int_{t_j}^{t}v(s)ds,\ \ t\in[t_j,t_{j+1}],\ \ y(0)=x_0, \tag{3.5}$$

where $t_j=j\gamma(\varepsilon),\ j=\overline{0,m},\ m\gamma(\varepsilon)\le L\varepsilon^{-1}<(m+1)\gamma(\varepsilon)$.

Consider the mapping

$$y^1(t)=y^1(t_j)+\varepsilon v_j(t-t_j),\ \ t\in[t_j,t_{j+1}],\ \ y^1(0)=x_0, \tag{3.6}$$

where $v_j\in\mathbb{E}^n$ satisfies the condition

$$D\left(\gamma(\varepsilon)v_j,\int_{t_j}^{t_{j+1}}v(s)ds\right)=\min_{v\in\bar{F}(y^1(t_j))}D\left(\gamma(\varepsilon)v,\int_{t_j}^{t_{j+1}}v(s)ds\right). \tag{3.7}$$

Obviously, v_j exists in view of the compactness of the set $\bar{F}(y^1(t_j))$ and the continuity of the function being minimized.

Denote by $\delta_j = D(y(t_j), y^1(t_j))$. For $t \in [t_j, t_{j+1}]$ using (3.5) and (3.6), we have

$$D(y(t), y(t_j)) \le M_1 \varepsilon \gamma(\varepsilon), \quad D(y^1(t), y^1(t_j)) \le M_1 \varepsilon \gamma(\varepsilon). \tag{3.8}$$

Therefore the following inequalities hold for $t \in [t_j, t_{j+1}]$:

$$\begin{aligned} D(y(t), y^1(t_j)) &\le D(y(t_j), y^1(t_j)) + D(y(t), y(t_j)) \le \delta_j + \varepsilon M_1 (t - t_j), \\ \varsigma(\bar{F}(y(t)), \bar{F}(y^1(t_j)) &\le \lambda_1 D(y(t), y^1(t_j)) \le \lambda_1(\delta_j + \varepsilon M_1(t - t_j)). \end{aligned} \tag{3.9}$$

From (3.7) and (3.9) it follows that

$$\begin{aligned} D\left(\int_{t_j}^{t_{j+1}} v(s)ds, \gamma(\varepsilon)v_j\right) &\le \int_{t_j}^{t_{j+1}} \varsigma(\bar{F}(y(s)), \bar{F}(y^1(t_j)))ds \le \\ &\le \lambda_1 \left(\delta_j \gamma(\varepsilon) + \varepsilon M_1 \frac{\gamma^2(\varepsilon)}{2}\right). \end{aligned} \tag{3.10}$$

By (3.5) and (3.6) we obtain

$$\delta_{j+1} \le \delta_j + \varepsilon \lambda_1 \left(\delta_j \gamma(\varepsilon) + \varepsilon M_1 \frac{\gamma^2(\varepsilon)}{2}\right) = (1 + \lambda_1 \varepsilon \gamma(\varepsilon))\delta_j + \lambda_1 M_1 \frac{\varepsilon^2 \gamma^2(\varepsilon)}{2}. \tag{3.11}$$

Since $\delta_0 = 0$ from inequality (3.11) we get

$$\delta_1 \le \lambda_1 M_1 \frac{\varepsilon^2 \gamma^2(\varepsilon)}{2},$$

$$\delta_2 \le (1 + \lambda_1 \varepsilon \gamma(\varepsilon))\delta_1 + \lambda_1 M_1 \frac{\varepsilon^2 \gamma^2(\varepsilon)}{2} \le \lambda_1 M_1 \frac{\varepsilon^2 \gamma^2(\varepsilon)}{2}((1 + \lambda_1 \varepsilon \gamma(\varepsilon)) + 1).$$

By induction

$$\begin{aligned} \delta_{j+1} &\le \lambda_1 M_1 \frac{\varepsilon^2 \gamma^2(\varepsilon)}{2}((1 + \lambda_1 \varepsilon \gamma(\varepsilon))^i + (1 + \lambda_1 \varepsilon \gamma(\varepsilon))^{i-1} + ... + 1) = \\ &= \frac{M_1 \varepsilon \gamma(\varepsilon)}{2}\left((1 + \lambda_1 \varepsilon \gamma(\varepsilon))^{i+1} - 1\right) \le \frac{M_1 \varepsilon \gamma(\varepsilon)}{2}\left((1 + \lambda_1 \varepsilon \gamma(\varepsilon))^{\frac{L}{\varepsilon \gamma(\varepsilon)}} - 1\right) \le \\ &\le \frac{M_1 \varepsilon \gamma(\varepsilon)}{2}(e^{\lambda_1 L} - 1). \end{aligned} \tag{3.12}$$

So in view of inequalities (3.8) we derive the estimation:

$$\begin{aligned} D(y(t), y^1(t)) &\le D(y(t), y(t_j)) + D(y(t_j), y^1(t_j)) + D(y^1(t_j), y^1(t)) \le \\ &\le 2M_1 \varepsilon \gamma(\varepsilon) + \frac{M_1 \varepsilon \gamma(\varepsilon)}{2}(e^{\lambda_1 L} - 1) \le \frac{M_1 \varepsilon \gamma(\varepsilon)}{2}(e^{\lambda_1 L} + 3). \end{aligned} \tag{3.13}$$

It follows from condition 2) of the theorem that for any $\eta_1 > 0$ there exists $\varepsilon_1(\eta_1) > 0$ such that for $\varepsilon \le \varepsilon_1(\eta_1)$ the inequality holds

$$\varsigma\left(\bar{F}(y^1(t_j)), \frac{1}{\gamma(\varepsilon)}\int_{t_j}^{t_{j+1}} F(s, y^1(t_j))ds + \frac{1}{\gamma(\varepsilon)}\sum_{t_j\le\tau_i<t_{j+1}} I_i(y^1(t_j))\right) < \eta_1. \quad (3.14)$$

Hence, there exist a measurable selection $u_j(t) \in F(t, y^1(t_j))$ and $p_{ij} \in I_i(y^1(t_j))$ such that

$$D\left(v_j, \frac{1}{\gamma(\varepsilon)}\left(\int_{t_j}^{t_{j+1}} u_j(s)ds + \sum_{t_j\le\tau_i<t_{j+1}} p_{ij}\right)\right) < \eta_1. \quad (3.15)$$

Consider the mapping

$$x^1(t) = x^1(t_j) + \varepsilon\int_{t_j}^{t} u_j(s)ds + \varepsilon\sum_{t_j\le\tau_i<t} p_{ij},\ t \in (t_j, t_{j+1}],\ x^1(0) = x_0. \quad (3.16)$$

Since $x^1(0) = y^1(0)$ it follows from (3.6), (3.16) and (3.15) that for $j = \overline{1, m}$

$$D(x^1(t_j), y^1(t_j)) \le D(x^1(t_{j-1}), y^1(t_{j-1})) + \eta_1\varepsilon\gamma(\varepsilon) \le \ldots \le j\eta_1\varepsilon\gamma(\varepsilon) \le L\eta_1. \quad (3.17)$$

As for $t \in (t_j, t_{j+1}]$ we have

$$D(x^1(t), x^1(t_j)) \le M(1 + d)\varepsilon\gamma(\varepsilon) = M_1\varepsilon\gamma(\varepsilon),$$

taking into account inequality (3.8), we get

$$D(x^1(t), y^1(t)) \le L\eta_1 + 2M_1\varepsilon\gamma(\varepsilon), \quad (3.18)$$

$$D(x^1(t), y^1(t_j)) \le L\eta_1 + M_1\varepsilon\gamma(\varepsilon).$$

Let us show that there exists a solution $x(t)$ of inclusion (3.1) that is sufficiently close to $x^1(t)$.

Let $\theta_1, \ldots, \theta_p$ be the moments of impulses τ_i, that get into the interval $(t_j, t_{j+1}]$. For convenience denote by $\theta_0 = t_j$, $\theta_{p+1} = t_{j+1}$. Let $\mu_k^+ = D(x^1(\theta_k + 0), x(\theta_k + 0))$, $\mu_k^- = D(x^1(\theta_k), x(\theta_k))$, $k = \overline{0, p+1}$.

Using the Lipschitz condition, we have

$$dist\left(x^{1\prime}(t), \varepsilon F(t, x^1(t))\right) \le \varsigma\left(\varepsilon F(t, y^1(t_j)), \varepsilon F(t, x^1(t))\right) \le$$

$$\le \varepsilon\lambda D(x^1(t), y^1(t_j)) \le \varepsilon\lambda(M_1\varepsilon\gamma(\varepsilon) + L\eta_1) = \eta^*,$$

$$dist\left(\Delta x^1|_{t=\theta_k}, \varepsilon I_i(x^1(\theta_k))\right) \le \varsigma\left(\varepsilon I_i(y^1(t_j)), \varepsilon I_i(x^1(\theta_k))\right) \le$$

$$\le \varepsilon\lambda D(y^1(t_j), x^1(\theta_k)) \le \varepsilon\lambda(M_1\varepsilon\gamma(\varepsilon) + L\eta_1) = \eta^*.$$

According to Theorem 12 there exists a solution $x(t)$ of inclusion (3.1) such that for $t \in (\theta_k, \theta_{k+1}]$ the estimate holds

$$D(x(t), x^1(t)) \leq \mu_k^+ e^{\varepsilon\lambda(t-\theta_k)} + \varepsilon \int_{\theta_k}^{t} e^{\varepsilon\lambda(t-s)}\eta^* ds.$$

Denote by $\gamma_k = \theta_{k+1} - \theta_k \leq \gamma(\varepsilon),\ \gamma_0 + ... + \gamma_p = \gamma(\varepsilon)$. Then

$$\mu_{k+1}^- \leq \mu_k^+ e^{\varepsilon\lambda\gamma_k} + \frac{\eta^*}{\lambda}\left(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1\right). \tag{3.19}$$

When getting over the impulse point we have

$$\begin{aligned} \mu_{k+1}^+ &\leq \mu_{k+1}^- + \varepsilon\varsigma\left(I_i(y^1(t_j)), I_i(x(\theta_{k+1}))\right) \leq \\ &\leq \mu_{k+1}^- + \varepsilon\varsigma\left(I_i(x^1(\theta_{k+1})), I_i(x(\theta_{k+1}))\right) + \varepsilon\varsigma\left(I_i(y^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \leq \\ &\leq \mu_{k+1}^- + \varepsilon\lambda\mu_{k+1}^- + \varepsilon\varsigma\left(I_i(y^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \leq \\ &\leq (1+\varepsilon\lambda)\mu_{k+1}^- + \eta^*. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) it follows that

$$\mu_{k+1}^+ \leq (1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_k}\mu_k^+ + \beta, \quad \beta = \frac{\eta^*}{\lambda}(1+\varepsilon\lambda)\left(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1\right) + \eta^*.$$

Hence,

$$\begin{aligned} \mu_1^+ &\leq (1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma_0}\mu_0^+ + \beta \leq (1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta, \\ \mu_2^+ &\leq (1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_1}\mu_1^+ + \beta \leq (1+\varepsilon\lambda)^2 e^{\varepsilon\lambda(\gamma_0+\gamma_1)}\mu_0^+ + \\ &+\beta(1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_1} + \beta \leq (1+\varepsilon\lambda)^2 e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta\left((1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma(\varepsilon)} + 1\right), \end{aligned}$$

etc.

$$\begin{aligned} \mu_{k+1}^+ &\leq (1+\varepsilon\lambda)^{k+1}e^{\varepsilon\lambda\gamma(\varepsilon)}\mu_0^+ + \beta\left(e^{\lambda\varepsilon\gamma(\varepsilon)}((1+\varepsilon\lambda)^k + ... + (1+\varepsilon\lambda)) + 1\right) = \\ &= (1+\varepsilon\lambda)^{k+1}e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{(1+\varepsilon\lambda)^k - 1}{\varepsilon\lambda}(1+\varepsilon\lambda) + 1\right) \leq \\ &\leq e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}\mu_0^+ + \eta^*\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1) + 1\right)\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{e^{\lambda d\varepsilon\gamma(\varepsilon)} - 1}{\varepsilon\lambda}(1+\varepsilon\lambda) + 1\right) = \\ &= \alpha\mu_0^+ + \beta_1, \end{aligned}$$

where

$$\alpha = e^{\lambda\varepsilon\gamma(\varepsilon)(1+d)},$$

$$\beta_1 = (M_1\varepsilon\gamma(\varepsilon)+L\eta_1)\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)}-1)+1\right)\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda\right).$$

So

$$\delta^+_{j+1} = D(x(t_{j+1}), x^1(t_{j+1})) \le \alpha\delta^+_j + \beta_1.$$

We obtain the sequence of inequalities

$$\delta^+_0 = 0,\ \delta^+_1 \le \beta_1,\ \delta^+_2 \le \alpha\beta_1 + \beta_1 = (\alpha+1)\beta_1, ...,$$

$$\delta^+_{j+1} \le (\alpha^j + ... + 1)\beta_1 = \frac{\alpha^{j+1}-1}{\alpha-1}\beta_1 \le$$

$$\le \frac{e^{\lambda L(1+d)}-1}{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}-1}(M_1\varepsilon\gamma(\varepsilon)+L\eta_1)\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)}-1)+1\right)\times$$

$$\times\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda\right).$$

Since

$$\lim_{\varepsilon\downarrow 0}\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)}-1)+1\right) = 1$$

and

$$\lim_{\varepsilon\downarrow 0}\frac{e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda}{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}-1} = \lim_{\varepsilon\to 0}\frac{e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{e^{\lambda d\varepsilon\gamma(\varepsilon)}-1}{\lambda\varepsilon\gamma(\varepsilon)}+\frac{1}{\gamma(\varepsilon)}}{\frac{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}-1}{\lambda\varepsilon\gamma(\varepsilon)}} = \frac{d}{1+d},$$

one has that

$$\delta^+_{j+1} \le C(M_1\varepsilon\gamma(\varepsilon)+L\eta_1)$$

for $\varepsilon \le \varepsilon_2$.

Therefore for $t \in (t_j, t_{j+1}]$ the inequality holds

$$D(x(t), x^1(t)) \le D(x(t), x(t_j)) + D(x(t_j), x^1(t_j))) + D(x^1(t), x^1(t_j)) \le$$

$$\le M(1+d)\varepsilon\gamma(\varepsilon)+M_1\varepsilon\gamma(\varepsilon)+C(M_1\varepsilon\gamma(\varepsilon)+L\eta_1) = M_1(2+C)\varepsilon\gamma(\varepsilon)+CL\eta_1. \quad (3.21)$$

In view of inequalities (3.13),(3.18), and (3.21) we get that $D(x(t), y(t))$ can be done less than any preassigned η by means of choosing $\varepsilon \le \varepsilon_0$ and η_1.

The second statement of the theorem is proved similarly.

If the fuzzy multivalued mappings $F(t, x)$ and $I_i(x)$ are periodic in t, one derives better estimation.

Theorem 14. *Let in the domain $Q = \{t \ge 0,\ x \in G \subset \mathbb{E}^n\}$ the following hold:*

1) the fuzzy multivalued mappings $F : Q \to conv(\mathbb{E}^n)$, $I_i : G \to conv(\mathbb{E}^n)$ are continuous, uniformly bounded by M and satisfy the Lipschitz condition in x with constant λ;

2) the fuzzy multivalued mapping $F(t,x)$ is $2\pi-$ periodic in t and there exists such $d \in \mathbb{N}$ that for all $i \in \mathbb{N}$ the equalities hold $\tau_{i+d} = \tau_i + 2\pi,\ I_{i+d}(x) \equiv I_i(x)$;

3) for any $x_0 \in G' \subset G$ and $t \geq 0$ the solutions of inclusion (3.3) together with a $\rho-$ neighborhood belong to the domain G.

Then for any $L > 0$ there exist $\varepsilon^0(L) > 0$ and $C(L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) for any solution $y(t)$ of inclusion (3.3) there exists a solution $x(t)$ of inclusion (3.1) such that

$$D(x(t), y(t)) \leq C\varepsilon; \tag{3.22}$$

2) for any solution $x(t)$ of inclusion (3.1) there exists a solution $\xi(t)$ of inclusion (3.3) such that inequality (3.22) holds.

Proof. Using condition 2) of the theorem, we obtain that

$$\bar{F}(x) = \frac{1}{2\pi}\int_0^{2\pi} F(s,x)ds + \frac{1}{2\pi}\sum_{0 \leq \tau_i < 2\pi} I_i(x). \tag{3.23}$$

It follows from condition 1) of the theorem that $\bar{F} : D \to conv(\mathbb{E}^n)$ is uniformly bounded by $M_1 = M(1+d)$ and satisfies the Lipschitz condition with constant $\lambda_1 = \lambda(1+d)$.

We are going to prove the first statement of the theorem. Let $y(t)$ be a solution of inclusion (3.3). Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with a step 2π by the points $t_j = 2\pi j,\ j = \overline{0,m}$, where $m :\ t_m \leq L\varepsilon^{-1} < t_{m+1}$. Then there exists a measurable selector $v(t)$ of the fuzzy multivalued mapping $\bar{F}(y(t))$ such that

$$y(t) = y(t_j) + \varepsilon\int_{t_j}^{t} v(s)ds,\ t \in [t_j, t_{j+1}],\ y(0) = x_0. \tag{3.24}$$

Consider the mapping

$$y^1(t) = y^1(t_j) + \varepsilon v_j(t - t_j),\ t \in [t_j, t_{j+1}],\ y^1(0) = x_0, \tag{3.25}$$

where $v_j \in \mathbb{E}^n$ satisfies the condition

$$D\left(2\pi v_j, \int_{t_j}^{t_{j+1}} v(s)ds\right) = \min_{v \in \bar{F}(y^1(t_j))} D\left(2\pi v, \int_{t_j}^{t_{j+1}} v(s)ds\right). \tag{3.26}$$

Obviously, v_j exists in view of the compactness of the set $\bar{F}(y^1(t_j))$ and the continuity of the function being minimized.

Denote by $\delta_j = D(y(t_j), y^1(t_j))$. For $t \in [t_j, t_{j+1}]$ using (3.24) and (3.25), we have

$$D(y(t), y(t_j)) \leq 2\pi M_1\varepsilon, \quad D(y^1(t), y^1(t_j)) \leq 2\pi M_1\varepsilon. \tag{3.27}$$

Hence for $t \in [t_j, t_{j+1}]$ the following inequalities hold:

$$D(y(t), y^1(t_j)) \le D(y(t_j), y^1(t_j)) + D(y(t), y(t_j)) \le \delta_j + \varepsilon M_1(t - t_j),$$

$$\varsigma(\bar{F}(y(t)), \bar{F}(y^1(t_j)) \le \lambda_1 D(y(t), y^1(t_j)) \le \lambda_1(\delta_j + \varepsilon M_1(t - t_j)). \tag{3.28}$$

From (3.26) and (3.28) it follows that

$$D\left(\int_{t_j}^{t_{j+1}} v(s)ds, 2\pi v_j\right) \le \int_{t_j}^{t_{j+1}} \varsigma(\bar{F}(y(s)), \bar{F}(y^1(t_j)))ds \le \lambda_1 \left(2\pi\delta_j + 2\pi^2 M_1 \varepsilon\right). \tag{3.29}$$

Considering (3.24) and (3.25) we obtain

$$\delta_{j+1} \le \delta_j + \varepsilon\lambda_1 \left(2\pi\delta_j + 2\pi^2 M_1\varepsilon\right) = (1 + 2\pi\lambda_1\varepsilon)\delta_j + 2\pi^2\lambda_1 M_1\varepsilon^2. \tag{3.30}$$

From inequality (3.30) taking into account $\delta_0 = 0$, we get

$$\delta_1 \le 2\pi^2\lambda_1 M_1\varepsilon^2,$$

$$\delta_2 \le (1 + 2\pi\lambda_1\varepsilon)\delta_1 + 2\pi^2\lambda_1 M_1\varepsilon^2 \le 2\pi^2\lambda_1 M_1\varepsilon^2((1 + 2\pi\lambda_1\varepsilon) + 1),$$

etc.

$$\delta_{j+1} \le 2\pi^2\lambda_1 M_1\varepsilon^2((1 + 2\pi\lambda_1\varepsilon)^i + (1 + 2\pi\lambda_1\varepsilon)^{i-1} + ... + 1) =$$

$$= \pi M_1\varepsilon\left((1 + 2\pi\lambda_1\varepsilon)^{i+1} - 1\right) \le \pi M_1\varepsilon\left((1 + 2\pi\lambda_1\varepsilon)^{\frac{L}{2\pi\varepsilon}} - 1\right) \le \pi M_1\varepsilon(e^{\lambda_1 L} - 1). \tag{3.31}$$

In view of inequalities (3.8) one has that:

$$D(y(t), y^1(t)) \le D(y(t), y(t_j)) + D(y(t_j), y^1(t_j)) + D(y^1(t_j), y^1(t)) \le$$

$$\le 4\pi M_1\varepsilon + \pi M_1\varepsilon(e^{\lambda_1 L} - 1) \le \pi M_1\varepsilon(e^{\lambda_1 L} + 3). \tag{3.32}$$

Furthermore,

$$\bar{F}(y^1(t_j)) = \frac{1}{2\pi}\int_{t_j}^{t_{j+1}} F(s, y^1(t_j))ds + \frac{1}{2\pi}\sum_{t_j \le \tau_i < t_{j+1}} I_i(y^1(t_j)). \tag{3.33}$$

and there a exist $p_{ij} \in I_i(y^1(t_j))$ and a measurable selection $u_j(t) \in F(t, y^1(t_j))$ such that

$$v_j = \frac{1}{2\pi}\left(\int_{t_j}^{t_{j+1}} u_j(s)ds + \sum_{t_j \le \tau_i < t_{j+1}} p_{ij}\right). \tag{3.34}$$

Consider the mapping

$$x^1(t) = x^1(t_j) + \varepsilon\int_{t_j}^{t} u_j(s)ds + \varepsilon\sum_{t_j \le \tau_i < t} p_{ij},\ t \in (t_j, t_{j+1}],\ x^1(0) = x_0. \tag{3.35}$$

From (3.25), (3.35), and (3.34) using $x^1(0) = y^1(0)$, it follows that for $j = \overline{1, m}$

$$x^1(t_j) = y^1(t_j),\ D(x^1(t), x^1(t_j)) \le 2\pi M_1 \varepsilon,\ D(x^1(t), y^1(t)) \le 4\pi M_1 \varepsilon. \tag{3.36}$$

Let us show that there exists a solution $x(t)$ of inclusion (3.1) that is sufficiently close to $x^1(t)$.

Let $\theta_1, ..., \theta_p$ be the moments of impulses τ_i, that get into the interval $(t_j, t_{j+1}]$. For convenience denote by $\theta_0 = t_j,\ \theta_{p+1} = t_{j+1}$. Let $\mu_k^+ = D(x^1(\theta_k + 0), x(\theta_k + 0))$, $\mu_k^- = D(x^1(\theta_k), x(\theta_k))$, $k = \overline{0, p+1}$.

Using the Lipschitz condition, we have

$$dist\left(x^{1'}(t), \varepsilon F(t, x^1(t))\right) \le \varsigma\left(\varepsilon F(t, y^1(t_j)), \varepsilon F(t, x^1(t))\right) \le$$
$$\le \varepsilon\lambda D(x^1(t), x^1(t_j)) \le 2\pi\lambda M_1 \varepsilon^2 = \eta^*,$$
$$dist\left(\Delta x^1|_{t=\theta_k}, \varepsilon I_i(x^1(\theta_k))\right) \le \varsigma\left(\varepsilon I_i(y^1(t_j)), \varepsilon I_i(x^1(\theta_k))\right) \le$$
$$\le \varepsilon\lambda D(x^1(t_j), x^1(\theta_k)) \le 2\pi\lambda M_1 \varepsilon^2 = \eta^*.$$

According to Theorem 12 we obtain that there exists a solution $x(t)$ of inclusion (3.1) such that for $t \in (\theta_k, \theta_{k+1}]$ the following estimation holds

$$D(x(t), x^1(t)) \le \mu_k^+ e^{\varepsilon\lambda(t-\theta_k)} + \varepsilon \int_{\theta_k}^t e^{\varepsilon\lambda(t-s)} \eta^* ds.$$

Denote by $\gamma_k = \theta_{k+1} - \theta_k \le 2\pi,\ \gamma_0 + ... + \gamma_p = 2\pi$. Then

$$\mu_{k+1}^- \le \mu_k^+ e^{\varepsilon\lambda\gamma_k} + \frac{\eta^*}{\lambda}\left(e^{2\pi\lambda\varepsilon} - 1\right). \tag{3.37}$$

When getting over the impulse point we have

$$\mu_{k+1}^+ \le \mu_{k+1}^- + \varepsilon\varsigma\left(I_i(y^1(t_j)), I_i(x(\theta_{k+1}))\right) \le$$
$$\le \mu_{k+1}^- + \varepsilon\varsigma\left(I_i(x^1(\theta_{k+1})), I_i(x(\theta_{k+1}))\right) + \varepsilon\varsigma\left(I_i(x^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \le$$
$$\le \mu_{k+1}^- + \varepsilon\lambda\mu_{k+1}^- + \varepsilon\varsigma\left(I_i(x^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \le$$
$$\le (1 + \varepsilon\lambda)\mu_{k+1}^- + \eta^*. \tag{3.38}$$

From (3.19) and (3.20) it follows that

$$\mu_{k+1}^+ \le (1 + \varepsilon\lambda)e^{\varepsilon\lambda\gamma_k}\mu_k^+ + \beta, \quad \beta = \frac{\eta^*}{\lambda}(1 + \varepsilon\lambda)\left(e^{2\pi\lambda\varepsilon} - 1\right) + \eta^*.$$

Hence,

$$\mu_1^+ \le (1 + \varepsilon\lambda)e^{\lambda\varepsilon\gamma_0}\mu_0^+ + \beta \le (1 + \varepsilon\lambda)e^{2\pi\lambda\varepsilon}\mu_0^+ + \beta,$$
$$\mu_2^+ \le (1 + \varepsilon\lambda)e^{\varepsilon\lambda\gamma_1}\mu_1^+ + \beta \le (1 + \varepsilon\lambda)^2 e^{\varepsilon\lambda(\gamma_0+\gamma_1)}\mu_0^+ +$$

$$+\beta(1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_1}+\beta \leq (1+\varepsilon\lambda)^2 e^{2\pi\lambda\varepsilon}\mu_0^+ + \beta\left((1+\varepsilon\lambda)e^{2\pi\lambda\varepsilon}+1\right),$$

etc.

$$\mu_{k+1}^+ \leq (1+\varepsilon\lambda)^{k+1}e^{2\pi\lambda\varepsilon}\mu_0^+ + \beta\left(e^{2\pi\lambda\varepsilon}((1+\varepsilon\lambda)^k+...+(1+\varepsilon\lambda))+1\right) =$$

$$= (1+\varepsilon\lambda)^{k+1}e^{2\pi\lambda\varepsilon}\mu_0^+ + \beta\left(e^{2\pi\lambda\varepsilon}\frac{(1+\varepsilon\lambda)^k-1}{\varepsilon\lambda}(1+\varepsilon\lambda)+1\right) \leq$$

$$\leq e^{2\pi\lambda(1+d)\varepsilon}\mu_0^+ + \eta^*\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{2\pi\lambda\varepsilon}-1)+1\right)\left(e^{2\pi\lambda\varepsilon}\frac{e^{\lambda d\varepsilon 2\pi}-1}{\varepsilon\lambda}(1+\varepsilon\lambda)+1\right) =$$

$$= \alpha\mu_0^+ + \beta_1,$$

where

$$\alpha = e^{2\pi\lambda(1+d)\varepsilon},$$

$$\beta_1 = 2\pi M_1\varepsilon\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{2\pi\lambda\varepsilon}-1)+1\right)\left(e^{2\pi\lambda\varepsilon}\left(e^{2\pi\lambda d\varepsilon}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda\right).$$

So

$$\delta_{j+1}^+ = D(x(t_{j+1}), x^1(t_{j+1})) \leq \alpha\delta_j^+ + \beta_1.$$

We obtain the sequence of inequalities

$$\delta_0^+ = 0,\ \delta_1^+ \leq \beta_1,\ \delta_2^+ \leq \alpha\beta_1+\beta_1 = (\alpha+1)\beta_1, ...,$$

$$\delta_{j+1}^+ \leq (\alpha^j+...+1)\beta_1 = \frac{\alpha^{j+1}-1}{\alpha-1}\beta_1 \leq$$

$$\leq 2\pi M_1\frac{e^{\lambda L(1+d)}-1}{e^{2\pi\lambda(1+d)\varepsilon}-1}\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{2\pi\lambda\varepsilon}-1)+1\right)\left(e^{2\pi\lambda\varepsilon}\left(e^{2\pi\lambda d\varepsilon}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda\right)\varepsilon.$$

As

$$\lim_{\varepsilon\to 0}\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{2\pi\lambda\varepsilon}-1)+1\right) = 1$$

and

$$\lim_{\varepsilon\to 0}\frac{e^{2\pi\lambda\varepsilon}\left(e^{2\pi\lambda d\varepsilon}-1\right)(1+\varepsilon\lambda)+\varepsilon\lambda}{e^{2\pi\lambda(1+d)\varepsilon}-1} = \lim_{\varepsilon\to 0}\frac{e^{2\pi\lambda\varepsilon}\frac{e^{2\pi\lambda d\varepsilon}-1}{\lambda\varepsilon}+1}{\frac{e^{2\pi\lambda(1+d)\varepsilon}-1}{\lambda\varepsilon}} = \frac{2d\pi+1}{2(1+d)\pi},$$

then

$$\delta_{j+1}^+ \leq C_0\varepsilon$$

for $\varepsilon \leq \varepsilon_2$.

Therefore for $t \in (t_j, t_{j+1}]$ the inequality holds

$$D(x(t), x^1(t)) \leq D(x(t), x(t_j)) + D(x(t_j), x^1(t_j))) + D(x^1(t), x^1(t_j)) \leq$$

$$\leq 2\pi M(1+d)\varepsilon + 2\pi M_1\varepsilon + C_0\varepsilon = (4\pi M_1 + C_0)\varepsilon. \tag{3.39}$$

In view of inequalities (3.32),(3.36), and (3.39) we get that

$$D(x(t), y(t)) \leq C_1\varepsilon, \tag{3.40}$$

$C_1 = \pi M_1(e^{\lambda_1 L} + 3) + 4\pi M_1 + C_0$.

The first part of the theorem is proved.

Taking any solution $x(t)$ of inclusion (3.1) and making the calculations similar to the previous, it is possible to find a solution $y(t)$ of inclusion (3.3) such that inequality similar to (3.40) with some constant C_2 is fair. Choosing $C = \max(C_1, C_2)$ we will receive the justice of all statements of the theorem.

4 Conclusion

It is also possible to use the partial averaging of fuzzy impulsive differential inclusions, i.e. to average only some summands or factors. Such variant of the averaging method also leads to the simplification of the initial inclusion and happens to be useful when the average of some functions does not exist or their presence in the system does not complicate its research.

References

[1] J.-P. Aubin, *Fuzzy differential inclusions*, Probl. Control Inf. Theory **19** (1) (1990), 55-67. MR1050534 (90m:34031). Zbl 0718.93039.

[2] R.J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965) 1-12. MR0185073 (32 #2543). Zbl 0163.06301.

[3] V.A. Baidosov, *Differential inclusions with fuzzy right-hand side*, Soviet Mathematics **40** (3) (1990), 567-569. MR1037651.

[4] V.A. Baidosov, *Fuzzy differential inclusions*, J. of Appl. Math. and Mechan. **54** (1) (1990), 8-13. MR1060244 (91d:34015).

[5] M. Hukuhara, *Integration des applications mesurables dont la valeur est un compact convexe*, Func. Ekvacioj **10** (1967), 205-223. MR0226503 (37 #2092). Zbl 0161.24701.

[6] E. Hullermeier, *An approach to modelling and simulation of uncertain dynamical system*, Internat. J. Uncertain. Fuzziness Knowledge - Based Systems **7** (1997), 117-137. MR1444079 (98a:93004).

[7] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems **24** (3) (1987), 301-317. MR0919058 (88j:34008). Zbl 0646.34019 .

[8] O. Kaleva, *The Cauchy problem for fuzzy differential equations*, Fuzzy Sets and Systems **35** (1990), 389-396. MR1055256 (91f:34004). Zbl 0696.34005 .

[9] O. Kaleva, *The Peano theorem for fuzzy differential equations revisited*, Fuzzy Sets and Systems **98** (1998), 147-148. MR1640147 (2000i:34011a). Zbl 0930.34003 .

[10] O. Kaleva, *A note on fuzzy differential equations*, Nonlinear Anal. **64** (2006), 895-900. MR2196799 (2006j:26022). Zbl 1100.34500 .

[11] T.A. Komleva, V.A. Plotnikov, *Differential inclusions with the Hukuhara derivative*, Nonlinear Oscil. **10** (2) (2007), 229-245. MR2369813(2008j:34009).

[12] V. Lakshmikantham, T. Granna Bhaskar, J. Vasundhara Devi, *Theory of set differential equations in metric spaces*, Cambridge Scientific Publishers, Cambridge, 2006. MR2438229 (2009h:34002). Zbl 1156.34003 .

[13] V. Lakshmikantham, R. Mohapatra, *Theory of fuzzy differential equations anf inclusions*, Taylor - Francis, London, 2003. MR2052737 (2004m:34004).

[14] V. Laksmikantham, A.A. Tolstonogov, *Existence and interrelation between set and fuzzy differential equations*, Nonlinear Anal. **55** (2003), 255-268. MR2007473 (2004h:34024). Zbl 1035.34064 .

[15] J.Y. Park, H.K. Han, *Existence and uniqueness theorem for a solution of fuzzy differential equations*, Internat. J. Math. and Math. Sci. **22** (2) (1999), 271-279. MR1695269 (2000d:34006). Zbl 0963.34054.

[16] J.Y. Park, H.K. Han, *Fuzzy differential equations*, Fuzzy Sets and Systems **110** (2000), 69-77. MR1748109 (2001b:34014). Zbl 0946.34055 .

[17] A.V. Plotnikov, *Averaging of differential inclusions with Hukuhara derivative*, Ukr. Math. J. **41** (1) (1989) 121-125. MR0986725 (90d:34041).

[18] A.V. Plotnikov, N.V. Skripnik, *Differential equations with "clear" and fuzzy multivalued right-hand side: Asymptotical methods*, Astroprint, Odessa, 2009.

[19] A.V. Plotnikov, N.V. Skripnik, *The generalized solutions of the fuzzy differential inclusions*, Int. J. Pure Appl. Math. **56** (2) (2009), 165-172. MR2569863. Zbl pre05652604 .

[20] V.A. Plotnikov, A.V. Plotnikov, A.N. Vityuk, *Differential equations with a multivalued right-hand side: Asymptotic methods*, Astroprint, Odessa, 1999. MR1738934 (2001k:34022).

[21] M.L. Puri, D.A. Ralescu, *Differentielle d'une fonction floue*, C.R. Acad. Sc. Paris **293-I** (1981), 237-239. MR0636972 (82m:58006). Zbl 0489.46038.

[22] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Sets and Systems 24 (1987) 319-330. MR0919059 (88j:34010). Zbl 0643.34005.

[23] N.V. Skripnik, *Averaging of impulsive differential inclusions with the Hukuhara derivative*, Nonlin. Oscil. **10** (3) (2007), 416-432. MR2394929 (2008k:34047).

[24] N.V. Skripnik, *Existence of classic solutions of the fuzzy differential inclusions*, Ukr. Math. Bull. **5** (2008), 244-257. MR2559837.

[25] N.V. Skripnik, *Kvazisolutions of the fuzzy differential inclusions*, Thesis of reports of IX Crimean intern. math. school "Method of Lyapunov functions and its applications", Alushta (2008), 152.

[26] N.V. Skripnik, *The full averaging of fuzzy differential inclusions*, Iranian J. Optimization **1** (3) (2009), 302-317.

[27] S.J. Song, C.X. Wu, *Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations*, Fuzzy Sets and Systems **111** (2000), 55-67. MR1748108 (2001a:34012). Zbl 0946.34054 .

[28] D. Vorobiev, S. Seikkala, *Towards the theory of fuzzy differential equations*, Fuzzy Sets and Systems **125** (2002), 231-237. MR1880339 (2002k:34028). Zbl 1003.34046.

[29] L. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353. MR0219427 (36 #2509). Zbl 0139.24606.

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