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FIXED POINTS FOR MULTIVALUED CONTRACTIONS ON A METRIC SPACE

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Abstract. The purpose of this paper is to prove a fixed point theorem for multivalued operators and a fixed point theorem for multivalued weakly Picard operators in the terms of τ-distance.

1 Introduction

In 2001 T. Suzuki (see [14]) introduced the concept of τ-distance on a metric space which is a generalization of both w-distance and Tataru's distance. He gave some examples of τ-distance and improve the generalization of Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle and the Takahashi's nonconvex minimization theorem, see [14]. Also, some fixed point theorems for multivalued operators on a complete metric space endowed with a τ-distance were established in T. Suzuki [15].

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by I. A. Rus, A. Petruşel and A. Sântămărian, see [11].

Consider (X,d) be a complete metric space. Let $T : X \to P(X)$ be a multivalued mapping. Define function $f : X \to \mathbb{R}$ as $f(x) = D(x, T(x))$. For a positive constant $b \in (0, 1)$ define the set $I_b^x \subset X$ as:

$$I_b^x = \{y \in T(x) \mid bd(x, y) \leq D(x, T(x))\}.$$

In 2006 Y. Feng and S. Liu proved the following theorem, see [1].

Let (X,d) be a complete metric space $T : X \to P_{cl}(X)$ be a multivalued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying $D(y, T(y)) \leq cd(x, y)$. Then T has a fixed point in X provided $c < b$ and f is lower semicontinuous on X.

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In [2] and [3] we find some generalizations of this theorem for a metric spaces endowed with a w-distance.

The purpose of this paper is to extend the above fixed point result for multivalued operators from [1] in terms of a τ-distance on a complete metric space.

2 Notation and basic notions

Let (X, d) be a complete metric space. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of* X;

$D : P(X) \times P(X) \to \mathbb{R}_+, D(Z,Y) = inf\{d(x,y) : x \in Z\ , y \in Y\}$, $Z \subset X$ - *the gap between two nonempty sets.*

First we define the concept of **L-space**.

Definition 1. *Let X be a nonempty set and $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an **L-space** if the following conditions are satisfied:*

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By the definition an element of $c(X)$ is convergent and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we can write $x_n \to x$ as $n \to \infty$.

We will denote an L-space by $(X, \to)$.

Let us give some examples of L-spaces, see [8].

Example 2. ***(L-structures on Banach spaces)*** *Let X be a Banach space. We denote by $\to$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \to)$, $(X, \rightharpoonup)$ are L-spaces.*

Example 3. ***(L-structures on function spaces)*** *Let X and Y be two metric spaces. Let $\mathbb{M}(X,Y)$ the set of all operators from X to Y. We denote by $\stackrel{p}{\to}$ the point convergence on $\mathbb{M}(X,Y)$, by $\stackrel{unif}{\to}$ the uniform convergence and by $\stackrel{cont}{\to}$ the convergence with continuity. Then $(\mathbb{M}(X,Y), \stackrel{p}{\to})$, $(\mathbb{M}(X,Y), \stackrel{unif}{\to})$ and $(\mathbb{M}(X,Y), \stackrel{cont}{\to})$ are L-spaces.*

Definition 4. *Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a **multivalued weakly Picard operator** (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:*

1. *$x_0 = x$, $x_1 = y$;*

2. *$x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;*

3. *the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.*

Let us give some examples of MWP operators, see [8],[11].

Example 5. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a Reich type multivalued operator, i.e. there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)),$$

for all $x, y \in X$. Then T is a MWP operator.

Example 6. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a closed multifunction for which there exists $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + \beta < 1$ such that $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$. Then T is a MWP operator.*

Example 7. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ for which there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$H(T_1(x), T_2(y)) \leq \alpha[D(x, T_1(x)) + D(y, T_2(y))],$$

for each $x, y \in X$. Then T_1 and T_2 are a MWP operators.

The concept of τ-distance was introduced by T. Suzuki (see[1]) as follows.

Definition 8. *Let (X,d) be a metric space, $\tau : X \times X \to [0, \infty)$ is called $\tau-$distance on X if there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and the following are satisfied :*

(τ_1) $\tau(x, z) \leq \tau(x, y) + \tau(y, z)$, for any $x, y, z \in X$;

(τ_2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its the second variable;

(τ_3) $\lim_n x_n = x$ and $\lim_n sup\{\eta(z_n, \tau(z_n, x_m)) : m \geq n\} = 0$ imply $\tau(w, x) \leq \liminf_n(\tau(w, x_n))$ for all $w \in X$;

(τ_4) $\lim_n sup\{\tau(x_n, y_m)) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(τ_5) $\lim_n \eta(z_n, \tau(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, \tau(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

We may replace (τ_2) by the following $(\tau_2)'$:

$(\tau_2)'$ $inf\{\eta(x, t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in the second variable.

Let us give some examples of τ-distance (see[14]).

Example 9. *Let (X, d) be a metric space. Then the metric "d" is a τ-distance on X.*

Example 10. *Let (X,d) be a metric space and $w : X \times X \to \mathbb{R}_+$ be a w-distance on X. Then w is also a τ-distance on X.*

Example 11. *Let (X,d) be a metric space and τ be a τ-distance on X, let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\int_0^\infty \frac{1}{1+h(r)}dr = \infty$, and let $z_0 \in X$ be fixed. Then a function $q : X \times X \to \mathbb{R}_+$ defined by:*

$$q(x,y) = \int_{\tau(z_0,x)}^{\tau(z_0,x)+\tau(x,y)} \frac{dr}{1+h(r)}, \text{ for all } x,y \in X$$

is a τ-distance.

For the proof of the main result we need of the definition of the τ-Cauchy sequence and the following lemmas (see [15]).

Definition 12. *Let (X,d) be a metric space and let τ be a τ-distance on X. Then a sequence $\{x_n\}$ in X is called $\tau - Cauchy$ if there exists a function $\eta : X \times [0,\infty) \to [0,\infty)$ satisfying (τ_2)-(τ_5) and a sequence $\{z_n\}$ in X such that $\lim_n sup\{\eta(z_n, \tau(z_n, x_m)) : m \geq n\} = 0$.*

A crucial results in order to obtain fixed point theorems by using τ-distance are the following lemmas.

Lemma 13. *Let (X,d) be a metric space and let τ be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $\lim_n sup\{\tau(x_n,x_m) : m > n\} = 0$, then $\{x_n\}$ is a τ-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X satisfies $\lim_n \tau(x_n,y_n) = 0$, then $\{y_n\}$ is also a τ-Cauchy sequence and $\lim_n d(x_n,y_n) = 0$.*

Lemma 14. *Let (X,d) be a metric space and let τ be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $\lim_n \tau(z,x_n) = 0$ for $z \in X$ then $\{x_n\}$ is a τ-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X also satisfies $\lim_n \tau(z,y_n) = 0$, then $\lim_n d(x_n,y_n) = 0$. In particular, for $x,y,z \in X$, $\tau(z,x) = 0$ and $\tau(z,y) = 0$ imply $x = y$.*

Lemma 15. *Let (X,d) be a metric space and let τ be a τ-distance on X. If $\{x_n\}$ is a τ-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_n sup\{\tau(x_n,y_m) : m > n\} = 0$, then $\{y_n\}$ is a τ-Cauchy sequence and $\lim_n d(x_n,y_n) = 0$.*

3 Main results

Definition 16. *Let $T : X \to P(X)$ be a multivalued operator, $\tau : X \times X \to [0,\infty)$ be a τ-distance on X. Define the function $f : X \to \mathbb{R}$ as $f(x) = D_\tau(x,T(x))$, where $D_\tau(x,T(x)) = inf\{\tau(x,y) \mid y \in T(x)\}$.*

For a positive constant $b \in (0,1)$ define the set $I^x_{b,\tau} \subset X$ as follows:

$$I^x_{b,\tau} = \{y \in T(x) \mid b\tau(x,y) \leq D_\tau(x,T(x))\}.$$

We will present now a fixed point theorem for multivalued operators on a complete metric space endowed with a τ-distance.

Theorem 17. *Let (X, d) be a complete metric space, $T : X \to P_{cl}(X)$ a multivalued operator, $\tau : X \times X \to [0, \infty)$ be a τ-distance on X and $b \in (0, 1)$.*
Suppose that

(i) there exists $c \in (0, b)$, such that for any $x \in X$ there is $y \in I^x_{b,\tau}$ satisfying $D_\tau(y, T(y)) \leq c\tau(x, y)$;

(ii) f is lower semicontinuous, where f is previous defined.

Then T has a fixed point in X. Moreover, if $T(z) = z$, then $\tau(z, z) = 0$.

Proof. For any initial point $x_0 \in X$, there is $x_1 \in I^{x_0}_{b,\tau}$ such that:

$$D_\tau(x_1, T(x_1)) \leq c\tau(x_0, x_1).$$

For any $x_1 \in X$ there is $x_2 \in I^{x_1}_{b,\tau}$ such that:

$$D_\tau(x_2, T(x_2)) \leq c\tau(x_1, x_2).$$

We obtain an iterative sequence $\{x_n\}_{n=0}^\infty$ where $x_{n+1} \in I^{x_n}_{b,\tau}$ and

$$D_\tau(x_{n+1}, T(x_{n+1})) \leq c\tau(x_n, x_{n+1}), forn = 0, 1, 2, \ldots.$$

We will verify that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence.
Indeed:

$$D_\tau(x_{n+1}, T(x_{n+1})) \leq c\tau(x_n, x_{n+1}), n = 0, 1, 2, \ldots \tag{3.1}$$

Since $x_{n+1} \in I^{x_n}_{b,\tau}$ we obtain:

$$b\tau(x_n, x_{n+1}) \leq D_\tau(x_n, T(x_n)), n = 0, 1, 2, \ldots \tag{3.2}$$

By (3.2) it follows : $\tau(x_n, x_{n+1}) \leq \frac{1}{b} D_\tau(x_n, T(x_n))$, $n = 0, 1, 2, \ldots$
Using (3.1)we obtain :

$$D_\tau(x_{n+1}, T(x_{n+1})) \leq c\frac{1}{b} D_\tau(x_n, T(x_n)), n = 0, 1, 2, \ldots \tag{3.3}$$

By (3.1) we have :

$$D_\tau(x_n, T(x_n)) \leq c\tau(x_{n-1}, x_n), n = 0, 1, 2, \ldots$$

$$D_\tau(x_{n+1}, T(x_{n+1})) \leq c\tau(x_n, x_{n+1}), n = 0, 1, 2, \ldots$$

We replace in (3.3) and we obtain :

$$c\tau(x_n, x_{n+1}) \leq \frac{c}{b} c\tau(x_{n-1}, x_n), n = 0, 1, 2, \ldots$$

If we divide by c we obtain :

$$\tau(x_n, x_{n+1}) \leq \frac{c}{b}\tau(x_{n-1}, x_n), n = 0, 1, 2, \dots$$

then:

$$\tau(x_{n+1}, x_{n+2}) \leq \frac{c}{b}\tau(x_n, x_{n+1}), n = 0, 1, 2, \dots \tag{3.4}$$

By using an induction argument it is easy to see that:

$$\tau(x_n, x_{n+1}) \leq \frac{c^n}{b^n}\tau(x_0, x_1), n = 0, 1, 2, \dots$$

$$D_\tau(x_n, T(x_n)) \leq \frac{c^n}{b^n}D_\tau(x_0, T(x_0)), n = 0, 1, 2, \dots$$

Then for $m, n \in \mathbb{N}$, $m > n$, $a = \frac{c}{b}$ and using the previous inequalities we have:

$$\tau(x_n, x_m) \leq \tau(x_n, x_{n+1}) + \tau(x_{n+1}, x_{n+2}) + \dots + \tau(x_{m-1}, x_m) \leq$$

$$\leq a^n\tau(x_0, x_1) + a^{n+1}\tau(x_0, x_1) + \dots + a^{m-1}\tau(x_0, x_1) \leq$$

$$\leq \frac{a^n}{1-a}\tau(x_0, x_1)$$

For $n \to \infty$ we have $a^n \to 0$, because $a = \frac{c}{b}$ with $c < b$. Thus

$$\lim_{n\to\infty} \tau(x_n, x_m) = 0, \text{ with } m > n.$$

Using Lemma 13 it follows that the sequence $\{x_n\}_{n=0}^{\infty}$ is a τ-Cauchy sequence and from the Lemma 15 result that this sequence is a Cauchy sequence. Since X is a complete space, then the sequence $\{x_n\}_{n=0}^{\infty}$ converge to some point $z \in X$. The sequence

$$\{f(x_n)\}_{n=0}^{\infty} = \{D_\tau(x_n, T(x_n))\}_{n=0}^{\infty}$$

is decreasing and from the above construction it converges to 0.
Since f is lower semicontinuous we have:

$$0 \leq f(z) \leq \lim_{n\to\infty} f(x_n) = 0$$

Thus $f(z) = 0$.

Since $f(z) = 0$, then there exists a sequence $(w_n) \in T(z)$ such that $\tau(z, w_n) \to 0$. Therefore,

$$0 \leq \lim_n \sup\{\tau(x_n, w_m) : m > n\} \leq \lim_n \sup\{\tau(x_n, z) + \tau(z, w_m) : m > n\} = 0.$$

Thus, by Lemma 15 we obtain that $w_n \to z$ and since $T(z)$ is a closed set we conclude that

$$z \in T(z).$$

Since $T(z) = z$, hence $D_\tau(z, T(z)) = \tau(z, z) \geq 0$ and since by hypothesis $I^z_{b,\tau} \neq \emptyset$ we know that $b\tau(z, z) \leq \tau(z, z)$ but $0 < b < 1$, therefore $\tau(z, z) = 0$. □

The second main result of this paper is a fixed point theorem for MWP operators on metric spaces endowed with a τ-distance.

Corollary 18. *Let (X,d) be a complete metric space, $T : X \to P_{cl}(X)$ a multivalued operator, $\tau : X \times X \to [0,\infty)$ be a τ-distance on X and $b \in (0,1)$. Suppose that there exists $c \in (0,1)$, with $c < b$, such that for any $x \in X$ there is $y \in I^x_{b,\tau}$ satisfying $D_\tau(x,T(x)) \le c\tau(x,y)$. Then T is a MWP operator.*

Proof. As in the proof of the previous theorem we construct inductively a sequence $\{x_n\}_{n=0}^{\infty}$ such that:

1. $x_{n+1} \in T(x_n)$, for every $n \in \mathbb{N}$;
2. $\tau(x_n, x_{n+1}) \le \frac{c^n}{b^n}\tau(x_0,x_1)$, for every $n \in \mathbb{N}$;
3. $D_\tau(x_n, T(x_n)) \le \frac{c^n}{b^n} D_\tau(x_0, T(x_0))$, for every $n \in \mathbb{N}$.

Thus $x_0 = x$, $x_1 = y$ and $x_{n+1} \in T(x_n)$. For $m,n \in \mathbb{N}$ with $m > n$ and $a = \frac{c}{b}$ we have the inequality

$$\tau(x_n, x_m) \le \frac{a^n}{1-a}\tau(x_0,x_1).$$

Using Lemma 13 it follows that the sequence $\{x_n\}_{n=0}^{\infty}$ is a τ-Cauchy sequence. From Lemma 15 we have that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (X,d) is a complete metric space then there exists $z \in X$ such that $x_n \xrightarrow{d} z$.

For $n \in \mathbb{N}$, from (τ3) we get that:

$$\tau(x_n, z) \le \lim_{m\to\infty} \inf \tau(x_n, x_m) \le \frac{a^n}{1-a}\tau(x_0,x_1).$$

By hypothesis, for $a = \frac{c}{b}$, we also have $w_n \in T(z)$ such that

$$\tau(x_n, w_n) \le a\tau(x_{n-1}, z) \text{ for every } n \in \mathbb{N}.$$

So, we have

$$\lim_{n\to\infty} \sup \tau(x_n, w_n) \le \lim_{n\to\infty} \sup a\tau(x_{n-1}, z) \le \lim_{n\to\infty} \frac{a^n}{1-a}\tau(x_0,x_1) = 0.$$

By Lemma 13 result that w_n converge to z. Since $T(z)$ is closed we obtain that $z \in T(z)$. Then T is MWP operator. □

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