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DYNAMIC SHORTFALL CONSTRAINTS FOR OPTIMAL PORTFOLIOS

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Abstract. We consider a portfolio problem when a Tail Conditional Expectation constraint is imposed. The financial market is composed of n risky assets driven by geometric Brownian motion and one risk-free asset. The Tail Conditional Expectation is calculated for short intervals of time and imposed as risk constraint dynamically. The method of Lagrange multipliers is combined with the Hamilton-Jacobi-Bellman equation to insert the constraint into the resolution framework. A numerical method is applied to obtain an approximate solution to the problem. We find that the imposition of the Tail Conditional Expectation constraint when risky assets evolve following a log-normal distribution, curbs investment in the risky assets and diverts the wealth to consumption.

1 Introduction

In recent years, particular stress has been laid on the substitution of variance as a risk measure in the standard Markowitz [12] mean-variance problem. Since it makes no distinction between positive and negative deviations from the mean, variance is a good measure of risk only for distributions that are (approximately) symmetric around the mean such as the normal distribution or more generally, elliptical distributions (McNeil, Frey and Embrechts [13]). However, in most cases such as in portfolios containing options, as well as credit portfolios, we are dealing with wealth distributions that are highly skewed. It is thus more reasonable to consider asymmetric risk measures since individuals are typically loss averse. In this regard, Value-at-Risk (VaR), a downside risk measure (Jorion [10]), has emerged as the industry standard with regulatory authorities enforcing its use.

Despite its widespread acceptance, VaR is known to possess unappealing features. Artzner et al. [2] proposed an axiomatic foundation for risk measures, by identifying four properties that a reasonable risk measure should satisfy and providing a charact-

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erization of the risk measures satisfying these properties, which they called coherent risk measures. Going by these axioms, VaR is not coherent. Tail Conditional Expectation (TCE), on the other hand, for an underlying continuous distribution, is one of such so-called coherent risk measures (Rockafellar and Uryasev [17]).

In a defined contribution pension plan, a pensioner with income drawdown option (Gerrard et al. [7]) retires and compulsorily has to purchase an annuity within a certain period of time after retirement. In the interim, the accumulated capital is dynamically allocated while the pensioner withdraws periodic amounts of money to provide for daily life in accordance with restrictions imposed by the scheme's rules or by legislation. In particular we assume that an individual who retires at time t_0 acquires control of a fund of size v_0 which is invested in a market that consists of risky and riskless assets. At age T the entire fund must be invested in an annuity. The retiree has to find optimal investment and consumption choices between time t_0 and time T, the future date at which he is obliged to annuitize.

Our focus in this paper is the dynamic portfolio (by dynamic portfolio strategy we mean portfolio re-balancing as well as re-calculation of TCE at short intervals of time within the investment horizon. This is in contrast to the static (one-period) model of Markowitz [12] whereby the portfolio once chosen, is never revised) and consumption choice of a trader subject to a risk limit specified in terms of TCE. In the existing literature, investment and consumption strategies are often studied in separate problems. Here, we consider both in the same problem formulation. We apply the TCE constraint while maximizing the agent's utility over consumption throughout the investment horizon, and over terminal wealth. This problem has not yet received adequate attention in the existing literature. One exception is Pirvu [15] who considers a similar problem with a constraint to VaR instead of TCE. We show through numerical simulations by applying an algorithm similar to that in Yiu [18] that the introduction of a TCE constraint reduces investment in risky assets and increases consumption (Cuoco et al. [3]). Putschögl and Sass [16] use Expected Shortfall instead of TCE and find explicit solutions for logarithmic utility. Gabih et al. [5, 6] study the problem for a static risk constraint which is imposed at terminal trading time only.

The rest of this paper is structured as follows. In Section 2, we model the financial market and describe the portfolio dynamics. Section 3 derives the Value-at-Risk and Tail Conditional Expectation constraints, while Section 4 makes precise the optimal control problem to be solved. Section 5 develops the solution of the problem by using the Lagrange technique to combine the Hamilton-Jacobi-Bellman (HJB) equation and the TCE constraint. In Section 6, a numerical algorithm is presented to obtain an approximate solution to the TCE-constrained problem. Section 7 presents some simulations and Section 8 concludes the paper.

2 The Model

We consider a standard Black-Scholes type market (see, e.g., Korn [11]) for relevant definitions) consisting of one risk-free bond and n risky stocks. The financial market is continuous-time with a finite time horizon [0,T].

Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a filtration that is a non-decreasing family $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-fields of $\mathcal{F}$

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \qquad 0 \le s < t < T.$$

It is assumed throughout this paper that all inequalities as well as equalities hold **P**- almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. The price of the risk-free asset (bond) S^0 is supposed to evolve according to

$$dS_t^0 = rS_t^0 dt, \;\; S_0^0 = s^0, \tag{2.1}$$

where r denotes the risk-free interest rate. For the risky assets (stocks), for which the prices will be denoted by $S_t = (S_t^1, \dots, S_t^n)$ for some $n \in \mathbb{N}$, the basic evolution model is that of a log-normal diffusion process:

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^{k} \sigma^{ij} dW_t^j, \qquad t \in [0,T], \qquad S_0^i = s^i, \quad i = 1, \dots, n, \tag{2.2}$$

where, for some $k \in \mathbb{N}$, $W_t = [W_t^1, \dots, W_t^k]'$, with the symbol ($'$) standing for transpose, is a k-dimensional standard Wiener process, i.e., a vector of k independent one-dimensional Wiener processes. The n-vector $\mu = (\mu^1, \dots, \mu^n)'$, contains the expected instantaneous rates of return and the $n\times k$-matrix $\sigma = \sigma^{ij}$, $(i = 1, \dots, n,\ j = 1, \dots, k)$ measures the instantaneous sensitivities of the risky asset prices with respect to exogenous shocks so that the $(n \times n)$-matrix $\sigma\sigma'$ contains the variance and covariance rates of instantaneous rates of return. An agent invests according to an investment strategy that can be described by the $(n+1)$-dimensional, $\mathcal{F}_t$-predictable process

$$x_t = (x_t^0, x_t^1, \dots, x_t^n), \tag{2.3}$$

where x_t^i $(i = 1, \dots, n)$ denotes the number of shares of asset i held in the portfolio at time t ($i = 0$ refers to the bond). The process x describes an investor's portfolio as carried forward through time. The value of the investor's wealth at time t is then

$$V_t^x = x_t^0 S_t^0 + \sum_{i=1}^{n} x_t^i S_t^i, \tag{2.4}$$

where $x_t^i S_t^i$ represents the amount invested in asset i at time t.
Equivalently, one may consider the vector

$$\theta_t = (\theta_t^1, \dots, \theta_t^n), \qquad \theta_t^i = \frac{x_t^i S_t^i}{V_t^x}, \quad (i = 1, \dots, n),$$

with θ_t^i denoting the fraction of wealth invested in the risky asset i at time t, whereby the remaining fraction $1 - \sum_{j=1}^n \theta_t^i$ of the agent's wealth is invested in the risk-free asset. Let also c_t be the instantaneous consumption rate. It is assumed that $\theta_t^1, \ldots, \theta_t^n$ and c_t are $\mathcal{F}_t$- adapted control processes. That is, θ_t^i and c_t are non-anticipative functions. The corresponding portfolio value process reads

$$dV_t^{\theta,c} = V_t^{\theta,c} \left[\left(1 - \sum_{i=1}^n \theta_t^i \right) \frac{dS_t^0}{S_t^0} + \sum_{i=1}^n \theta_t^i \frac{dS_t^i}{S_t^i} \right] - c_t dt, \qquad V_0^{\theta,c} = v_0. \tag{2.5}$$

Since μ, σ and r are constant it is enough for (2.5) to be well defined that we require $\int_0^T |c_t| + \sum_{1=1}^n (\theta_t^i)^2 dt < \infty$. Control processes satisfying these conditions and $V_t^{\theta,c} \geq 0$ for all $t \in [0,T]$ will be called admissible. By $A(v_t, t)$ we denote the corresponding class of *admissible controls* (θ_t, c_t) for portfolios starting at time t with capital $v_t = V_t^{\theta,c}$.

To have a better exposition, we adopt a matrix expression: denote $\sigma = [\sigma^{i,j}]$, $\theta_t = [\theta_t^1, \ldots, \theta_t^n]'$, $\mu = [\mu^1, \ldots, \mu^n]'$, $1_n = [1, \ldots, 1]'$ and $W_t = [W_t^1, \ldots, W_t^k]'$, so that σ is an $n \times k$ matrix, $\mu - r1_n$ and θ_t are n-dimensional column vectors and W_t is a k-dimensional column vector. Hence equation (2.5) can be rewritten as

$$dV_t^{\theta,c} = V_t^{\theta,c} \left[\left(r + \theta_t'(\mu - r1_n) \right) dt + \theta_t' \sigma_t dW_t \right] - c_t dt, \qquad V_0^{\theta,c} = v_0. \tag{2.6}$$

We have adopted an incomplete market asset pricing setting of He and Pearson [8]. To eliminate redundant assets, we assume that σ is of full row rank, that is, $\sigma\sigma'$ is an invertible matrix.

3 Tail Conditional Expectation

Here we start by defining Value-at-risk since the subsequent definition of Tail Conditional Expectation (TCE) will depend on it.

Definition 1. ***(Value-at-Risk)***
Given some probability level $\alpha \in (0,1)$, a time t wealth benchmark Υ_t and horizon Δt , the Value-at-Risk (VaR_t^α) of time t wealth V_t at the confidence level $1-\alpha$ is given by the smallest number L such that the probability that the loss $G_{t+\Delta t} := \Upsilon_t - V_{t+\Delta t}^{\theta,c}$ exceeds L is no larger than α.

$$VaR_t^\alpha = \inf \{ L : \mathbf{P}(G_{t+\Delta t} \geq L | \mathcal{F}_t) \leq \alpha \} := -Q_t^\alpha, \tag{3.1}$$

where

$$Q_t^\alpha = \sup \left\{ L \in \mathbb{R} : \mathbf{P}((V_{t+\Delta t}^{\theta,c} - \Upsilon_t) \leq L | \mathcal{F}_t) \leq \alpha \right\} \tag{3.2}$$

is the quantile of the projected wealth surplus at the horizon $t + \Delta t$.

VaR_t^α is therefore the loss of wealth with respect to a benchmark Υ_t at the horizon Δt which could be exceeded only with a small conditional probability α if the current portfolio θ_t were kept unchanged. Typical values for the probability level α are $\alpha = 0.05$ or $\alpha = 0.01$. In market risk management the time horizon Δt is usually one or ten days.

Proposition 2. ***(Computation of Value-at-Risk)***
We have

$$VaR_t^\alpha = -Q_t^\alpha = V_t^{\theta,c} \exp\Big[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \Big(\theta_t'(\mu - r1_n) + r - \frac{c_t}{V_t^{\theta,c}} - \frac{1}{2}\|\theta_t'\sigma\|^2\Big)\Delta t\Big] - \Upsilon_{t+\Delta t}, \quad (3.3)$$

where $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the normal distribution and the inverse distribution functions respectively, and $\|\cdot\|$ stands for the Euclidean norm.

We refer to [1] for the proof.

Tail Conditional Expectation is closely related to the Value-at-Risk concept, but overcomes some of the conceptual deficiencies of Value-at-Risk (Rockafellar and Uryasev [17]). In particular, it is a coherent risk measure for continuous distributions (Artzner et al. [2]).

Definition 3. ***(Tail Conditional Expectation)***
Consider the loss distribution $G_{t+\Delta t} := \Upsilon_t - V_{t+\Delta t}^{\theta,c}$ represented by a continuous distribution function $F_{G_{t+\Delta t}}$ with $\int_{\mathbb{R}} |G_{t+\Delta t}| dF(G_{t+\Delta t}) < \infty$. Then the TCE_t^α at confidence level $1-\alpha$ is defined as

$$\begin{aligned} TCE_t^\alpha &= \mathbb{E}_t\left\{G_{t+\Delta t} | G_{t+\Delta t} \geq VaR_t^\alpha\right\}, \\ &= \frac{1}{\alpha}\mathbb{E}_t\left\{(\Upsilon_t - V_{t+\Delta t}^{\theta,c}) I(\Upsilon_t - V_{t+\Delta t}^{\theta,c} \geq -Q_t^\alpha) | \mathcal{F}_t\right\}, \end{aligned}$$

where I(A) is the indicator function of the set A.

In other words, the Tail Conditional Expectation of wealth V_t at time t is the conditional expected value of the loss exceeding $-Q_t^\alpha$. Again, given the log-normal distribution of asset returns, the TCE_t^α can be explicitly computed as can be seen in the following proposition.

Proposition 4. ***(Computation of Tail Conditional Expectation)***
We have

$$TCE_t^\alpha = \frac{1}{\alpha}\Big(\alpha\Upsilon_t - V_t^{\theta,c} \exp\Big((\theta_t'(\mu - r1_n) + r - \frac{c_t}{V_t^{\theta,c}})\Delta t\Big)\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\Big),$$

Under the Black-Scholes model (μ, σ constant) and for fixed θ_t, c_t the conditional distribution of $G_{t+\Delta t}$ given $\mathcal{F}_t$ is continuous (since it is lognormal).

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions.*

We refer to [1] for the proof.

4 Statement of the Problem

We seek the optimal asset and consumption allocation that maximizes (over all admissible $\{\theta_t, c_t\}$) the expected discounted utility of terminal wealth at time T and consumption over the entire horizon $[0, T]$, for a risk averse investor who limits his risk by imposing an upper bound on the TCE.

The choice of this problem is motivated by the income drawdown option in defined contribution pension schemes. As mentioned in the introduction, such an option allows the member who retires not to convert the accumulated capital into annuity immediately at retirement, but to defer the purchase of the annuity until a certain point in time after retirement. The period of time can be limited to time T. Usually, freedom is given for a fixed number of years after retirement and at a certain age the annuity is bought.

Here, we consider the income drawdown option (Gerrard et al. [7]) and investigate, by means of stochastic optimal control techniques, what should be the optimal investment and consumption allocation of the fund after retirement until the purchase of the annuity. The reason the pensioner chooses the drawdown option is the hope of being able to invest the accumulated capital at retirement and increase its value in order to buy a better annuity in the future than the one he otherwise could have bought at retirement.

In mathematical terms the final stochastic optimal control problem with TCE constraint is

$$\max_{\{\theta,\ c\}\in A(v_0,0)} \mathbb{E}_{0,v_0}\left\{\int_0^T e^{-\rho s}U^1(c_s)ds + e^{-\rho T}U^2(V_T^{\theta,c})\right\}, \tag{4.1}$$

subject to the wealth dynamics

$$dV_t^{\theta,c} = \left[V_t^{\theta,c}(\theta_t'(\mu - r1_n) + r)\right]dt - c_t dt + V_t^{\theta,c}\theta_t'\sigma dW_t,\ V_0^{\theta,c} = v_0$$

and the TCE constraint

$$TCE_t^\alpha \le \varepsilon(v,t),\ \forall\ t \in [0, T - \Delta t), \tag{4.2}$$

where for fixed $\Delta t > 0$

$$\begin{aligned} TCE_t^\alpha &= TCE_t^\alpha(V_t^{\theta,c}, \theta_t, c_t) \\ &= \frac{1}{\alpha}\Big(\alpha\Upsilon_t - V_t^{\theta,c}\exp\Big((\theta_t'(\mu - r1_n) + r - \frac{c_t}{V_t^{\theta,c}})\Delta t\Big) \\ &\quad \cdot\Phi\Big(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\Big)\Big). \end{aligned} \tag{4.3}$$

Here $\mathbb{E}_{t,v}$ denotes the expectation operator at time t, given $V_t^{\theta,c} = v$ (and given the chosen consumption and investment strategies), U^1 and U^2 are twice differentiable, increasing, concave utility functions, $\varepsilon(v,t)$ is an upper bound on TCE and $\rho > 0$ is the rate at which utility of consumption and utility of terminal wealth are discounted. We let

$$U(x) = U^1(x) = U^2(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma \in (0,\infty)\backslash\{1\}$. This falls in the category of power utility functions, also known as Constant Relative Risk Aversion (CRRA) utility functions. For logarithmic utility $U(x) = \log x$, which corresponds to the limit for $\gamma \to 1$, the optimization problem can be tackled directly, see e.g. Pirvu [15, Section 4].

5 Optimality Conditions

In applying the dynamic programming approach we solve the HJB equation associated with the utility maximization problem (4.1). Defining the value function

$$J(v,t) = \sup_{\{\theta,\ c\}\in A(v,t)} \mathbb{E}_{t,v}\left\{\int_t^T e^{-\rho s}U(c_s)ds + e^{-\rho T}U(V_T^{\theta,c})\right\}, \tag{5.1}$$

following Fleming and Rishel [4]) we deduce the corresponding HJB equation

$$\rho J(v,t) = \max_{c\geq 0,\ \theta\in\mathbb{R}^n}\Big\{U(c) + J_t(v,t) + J_v(v,t)\left(v[\theta'(\mu - r1_n) + r] - c\right) + \frac{1}{2}J_{vv}(v,t)v^2\theta'\sigma\sigma'\theta\Big\}, \tag{5.2}$$

subject to the terminal condition

$$J(v,T) = e^{-\rho T}U(v),$$

where subscripts on J denote partial derivatives and $v = V_t^{\theta,c}$ the wealth realization at time t.

One of the main tools of stochastic control theory consists of verification theorems, i.e., theorems stating that a sufficiently regular solution of the HJB equation coincides with the value function and that an optimal portfolio process (θ^{opt}, c^{opt}), in the context of stochastic control theory denoted as an optimal control, can be constructed by looking at the values that yield the supremum in equation (5.2). For these technical theorems and proofs we refer again to the book by Fleming and Rishel [4]. We therefore solve equation (5.2), upon the assumption that these verification theorems are valid.

See Korn [11] for a definition and properties of a power or CRRA utility function.

In solving the HJB equation (5.2), the static optimization problem

$$\max_{c\geq 0,\ \theta\in\mathbb{R}^n}\Big\{U(c)+J_v(v,t)\Big(v[\theta'(\mu-r1_n)+r]-c\Big)+\frac{1}{2}J_{vv}(v,t)v^2\theta'\sigma\sigma'\theta\Big\}, \tag{5.3}$$

subject to the TCE constraint (4.3) can be tackled separately to reduce the HJB equation (5.2) to a nonlinear partial differential equation of J only.

We introduce the Lagrange function $\mathcal{L}\,(\theta,c,\lambda)=\mathcal{L}\,(\theta(v,t),c(v,t),\lambda(v,t))$ as

$$\begin{aligned}\mathcal{L}\,(\theta,c,\lambda)=J_v(v,t)\left(v\left[\theta'(\mu-r1_n)+r\right]-c\right)+\frac{1}{2}v^2\theta'\sigma\sigma'\theta J_{vv}(v,t)\\ +U(c)-\lambda(v,t)\left(\alpha TCE_t^{\alpha}(v,\theta,c)-\varepsilon_1\right),\end{aligned} \tag{5.4}$$

where λ is the Lagrange multiplier, $\varepsilon_1=\varepsilon\cdot\alpha$ and TCE_t^{α} is given in (4.3). The first-order necessary conditions with respect to θ, c and λ respectively of the static optimization of (5.4) are given by

$$\begin{aligned}0=\nabla_\theta\mathcal{L}=vJ_v(\mu-r1_n)+\frac{1}{2}J_{vv}v^2\sigma\sigma'\theta\\ +\lambda v\Big[(\mu-r1_n)\Delta t\exp\Big((\theta'(\mu-r1_n)+r-\frac{c}{v})\Delta t\Big)\cdot\Phi\left(\Phi^{-1}(\alpha)-\|\theta'\sigma\|\sqrt{\Delta t}\right)\\ -\exp\Big((\theta'(\mu-r1_n)+r-\frac{c}{v})\Delta t\Big)\cdot\frac{\sqrt{\Delta t}}{2}\frac{\sigma\sigma'\theta}{\|\theta'\sigma\|}\\ \cdot\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}(\Phi^{-1}(\alpha)-\|\theta'\sigma\|\sqrt{\Delta t})^2\Big)\Big],\end{aligned} \tag{5.5}$$

$$\begin{aligned}0=\frac{\partial\mathcal{L}}{\partial c}=-J_v+U_c-\lambda\Delta t\cdot\exp\Big((\theta'(\mu-r1_n)+r-\frac{c}{v})\Delta t\Big)\\ \cdot\Phi\left(\Phi^{-1}(\alpha)-\|\theta'\sigma\|\sqrt{\Delta t}\right),\end{aligned} \tag{5.6}$$

where U_c is the first-order derivative of U with respect to c and

$$\begin{aligned}0=\frac{\partial\mathcal{L}}{\partial\lambda}=H(v,t):=-\alpha\Upsilon_t+v\exp\Big((\theta'(\mu-r1_n)+r-\frac{c}{v})\Delta t\Big)\\ \cdot\Phi\Big(\Phi^{-1}(\alpha)-\|\theta'\sigma\|\sqrt{\Delta t}\Big)+\varepsilon_1,\end{aligned} \tag{5.7}$$

while the complimentary slackness condition is given as

$$\lambda(v,t)H(v,t)=0\quad\text{and}\quad\lambda(v,t)\geq 0. \tag{5.8}$$

Simultaneous resolution of these first-order conditions yields the optimal solutions θ^{opt}, c^{opt} and λ^{opt}. Substituting these into (5.2) gives the partial differential equation

$$-\rho J(v,t)+\frac{(c^{opt}(v,t))^{1-\gamma}}{1-\gamma}+J_t(v,t)+J_v(v,t)\left(v[(\theta^{opt}(v,t))'(\mu-r1_n)+r]\right.$$
$$\left.-c^{opt}(v,t)\right)+\frac{1}{2}J_{vv}(v,t)v^2(\theta^{opt}(v,t))'\sigma\sigma'\theta^{opt}(v,t)=0, \quad (5.9)$$

with terminal condition

$$J(v,T)=e^{-\rho T}\frac{v^{1-\gamma}}{1-\gamma},$$

which can then be solved for the optimal value function $J^{opt}(v,t)$. Because of the non-linearity in θ^{opt} and c^{opt}, the first-order conditions together with the HJB equation are a non-linear system. So the partial differential equation (5.9) has no analytic solution and numerical methods such as Newton's method or Sequential Quadratic Programming (SQP) (see, e.g., Nocedal and Wright [14]) are required to solve for $\theta^{opt}(v,t)$, $c^{opt}(v,t)$, $\lambda^{opt}(v,t)$ and $J^{opt}(v,t)$ iteratively.

6 Numerical Solution

We use an iterative algorithm similar to that of Yiu [18] which yields a $\mathcal{C}^{2,1}$ approximation $\widehat{J}$ of the exact solution J. The pair $(\widehat{\theta}_t,\widehat{c}_t)$ is the investment strategy related to $\widehat{J}$.

When the optimal solution strictly satisfies the TCE constraint (4.3), the Lagrange multiplier $\lambda(v,t)$ is zero. If the constraint is active, the multiplier is positive.

First, we divide the domain of resolution into a grid of $n_v\times n_t$ mesh points. Iterations are indexed by k.

1. For each point (t,v), with $t\in\{0,\Delta t,\ldots,n_t\Delta t\}$, $v\in\{0,\Delta v,\ldots,n_v\Delta v\}$, we compute the value function $\widehat{J}^{k=0}=J(v,t)$ and the optimal strategy $(\theta_t^{opt},c_t^{opt})$ of the unconstrained problem. All Lagrange multipliers are set to zero, $\lambda_{t,v}^{k=0}=0$. This solution is the starting point of the algorithm.

2. For all points of the grid, the constraint is checked. If the constraint is not active ($TCE_t^\alpha<\varepsilon$), the multiplier is zero $\lambda_{t,v}^{k+1}=0$ and $(\theta_t^{k+1},c_t^{k+1})$ is the solution of a similar equation to that of the unconstrained case,

$$\lambda_{t,v}^{k+1}=0,\quad \theta_t^{k+1}=-\frac{\widehat{J}_v}{v\widehat{J}_{vv}}(\mu-r1_n)(\sigma^T\sigma)^{-1},\quad \widehat{U}_c(c_t^{k+1})=\widehat{J}_v.$$

   If the TCE_t^α constraint is active, ($TCE_t^\alpha\geq\varepsilon$), we solve a nonlinear system in $\lambda_{t,v}^{k+1}$, $\widehat{\theta}_t^{j+1}$ and $\widehat{c}_t^{j+1}$. This nonlinear system is composed of the first-order necessary conditions of the static optimization problem (5.4). That system is numerically solved by the sequential quadratic programming method (Nocedal and Wright [14]).

3. The last stage consists in the calculation of the value function $\widehat{J}^{k+1}$ according to the investment/consumption strategy $(\widehat{\theta}_t^{k+1}, \widehat{c}_t^{k+1})$ as described below this algorithm.

4. Return to step 2 with $k = k+1$ until the error at time t from wealth level v, $\epsilon_{t,v}$, satisfies $|\epsilon_{t,v}| < \delta$ with some small $\delta > 0$, where

$$\epsilon_{t,v} = \hat{J}_t - \rho \hat{J}(v,t) + \hat{J}_v \left(v[(\hat{\theta}_t^{opt})'(\mu - r1_n) + r] - c_t^{opt} \right) + \frac{1}{2} v^2 \|(\hat{\theta}_t^{\,opt})'\sigma\|^2 \hat{J}_{vv} + U(c_t^{opt}).$$

For the numerical solution of the partial differential equation (5.9), to obtain the value function, we use the trial function

$$J(v,t) = f(t)\frac{v^{1-\gamma}}{1-\gamma}, \qquad f(T) = e^{-\rho T},$$

such that

$$J_t = f'(t)\frac{1}{1-\gamma}v^{1-\gamma}, \quad J_v = f(t)v^{-\gamma} \quad \text{and} \quad J_{vv} = -\gamma f(t) v^{-(\gamma+1)}.$$

Substituting these derivatives into (5.9) and dividing by $v^{1-\gamma}$, we derive the ordinary differential equation

$$f'(t) = -\kappa(\theta^{opt}(v,t), c^{opt}(v,t), v) f(t) - B(c^{opt}(v,t), v), \tag{6.1}$$

whereby

$$\begin{aligned} \kappa(\theta^{opt}(v,t), c^{opt}(v,t), v) \;\; = \;\; & (1-\gamma)\bigg(\frac{-\rho}{1-\gamma} + \left[(\theta^{opt}(v,t))'(\mu - r1_n) + \gamma \right] \\ & -c^{opt}(v,t)v^{-1} - \frac{\gamma}{2}(\theta^{opt}(v,t))'\sigma\sigma'\theta^{opt}(v,t) \bigg) \end{aligned}$$

and

$$B(c^{opt}(v,t), v) \;\; = \;\; (c^{opt}(v,t))^{1-\gamma} v^{\gamma - 1},$$

with terminal condition $f(T) = e^{-\rho T}$. The function f in equation (6.1) is computed numerically by the Euler-Cauchy method (see Isaacson and Keller [9]).

Parameter	**Value**
Stock (S^1)	$\mu_1 = 4\%$, $\sigma^{11} = 5\%$, $\sigma^{12} = 5\%$
Stock (S^2)	$\mu_2 = 6\%$, $\sigma^{21} = 5\%$, $\sigma^{22} = 20\%$
Bond (S^0)	$r = 3\%$
Investment horizon	$t \in [0, 1]$
State of wealth	$v \in [0, 20]$
Shortfall probability	$\alpha = 1\%$
Value-at-Risk horizon	$\Delta t = \frac{1}{48} \approx 7$ days
No. of wealth mesh points	$N_v = 81$
Mesh size for wealth	$\Delta v = \frac{20}{80} = 0.25$
Utility function	$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma = 0.9$
discount rate	$\rho = 0.03$

Table 1: Parameters for the consumption and investment portfolio optimization problem.

Wealth benchmark, Υ_t	**Bound**, ε
Conditional expectation	0.3
Money market	1.0

Table 2: Bounds and benchmarks for the TCE-constrained problem.

7 Simulations

We have implemented the above algorithm to illustrate the optimal portfolio of the preceding section with examples. To this end, we have written a program in MATLAB to carry out the procedure. We assume that $n = 2$. That is, the market is composed of two risky stocks and a risk-free bond. Table 1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. We consider the Tail Conditional Expectation of the wealth surplus $V_t - \Upsilon_t$ with respect to the benchmark Υ_t such that it satisfies

$$TCE_t^{\alpha}(V_{t+\Delta t}^{\theta,c} - \Upsilon_t) \leq \varepsilon,$$

where ε comes from Table 2. That is, the TCE is re-evaluated at each discrete time step (TCE horizon) Δt and kept below the upper bound ε, by making use of conditioning information. Here, in the first scenario, the shortfall benchmark is taken to be the conditional expected wealth $\Upsilon_t = \mathbb{E}_t\{V_{t+\Delta t}\}$, given as

$$\Upsilon_t = \mathbb{E}_t\{V_{t+\Delta t}\} = V_t \exp\left[\left(\theta_t'(\mu - r1_n) + r - \frac{c_t}{V_t}\right)\Delta t\right], \tag{7.1}$$

while, in the second scenario, it is the investment in the risk-free bond $\Upsilon_t = V_t e^{r\Delta t}$.

Figures 1 and 2 show in the right panel the amount of wealth invested in the risky assets with and without the TCE constraint, plotted against the possible wealth realization at different times. The left panel shows the value function. In

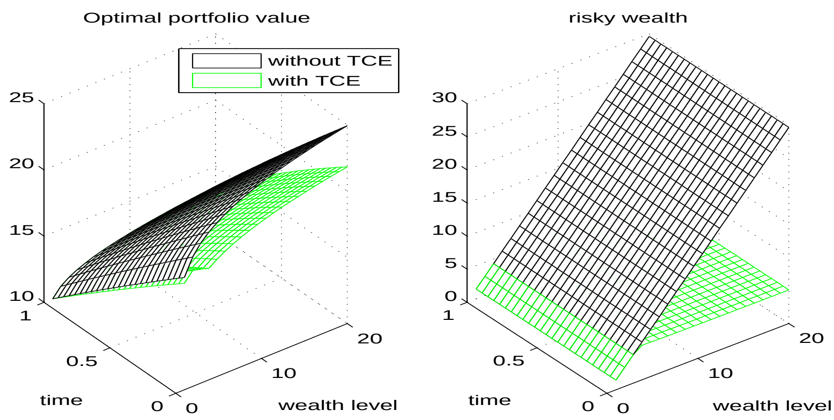

Figure 1: Optimal portfolio value and risky wealth when benchmark is the conditional expected wealth plotted against wealth at various times of the investment horizon. In green, TCE $\leq \varepsilon = 0.3$.

Figure 1, the shortfall benchmark is taken to be the conditional expected wealth while in Figure 2 it is the investment in the risk-free bond. As can be observed

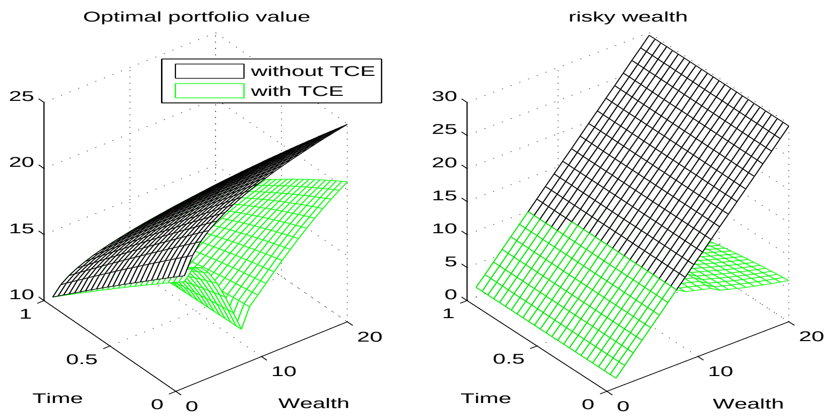

Figure 2: Effect of the TCE constraint when benchmark is investment in the bond.

from the images, as the wealth level increases, so does the investment in risky assets. This results from the property of constant relative risk aversion of the utility function. A good control over the investment in the risky assets has been achieved and the proportions invested in the risky assets are reduced in order to fulfill the TCE constraint. In particular, when the constraint is not active, the optimal portfolio follows the unconstrained solution; as the portfolio value increases, the TCE constraint becomes active and allocates less to the risky assets. Figure 3 reveals to

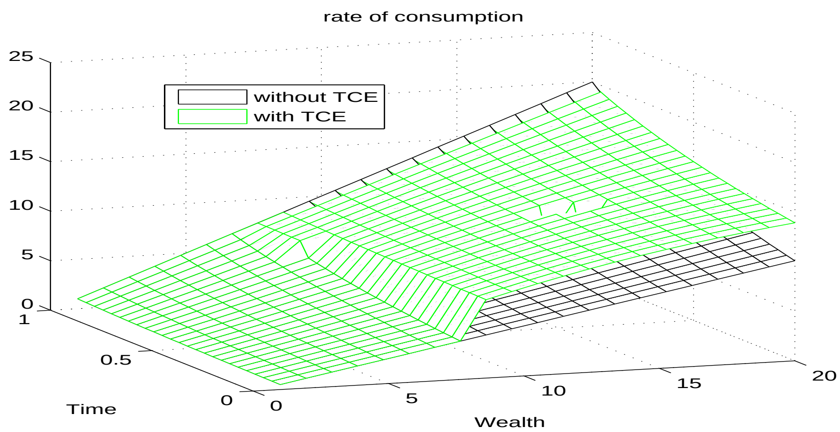

Figure 3: Effect of the TCE constraint on consumption when benchmark is investment in the bond.

us that the local minimum (around wealth level 10) observed in the left panel of Figure 2 comes as a result of a sudden increase in the consumption rate once the constraint becomes active. The left panel of Figure 1 suggests that this increase in consumption is more subtle when we take as wealth benchmark, the conditional expected wealth.

The value function of the constrained problem is identical to that of the unconstrained one when the Lagrange multipliers are null, whereas it is inferior when the constraint is active.

8 Concluding Remarks

Using a CRRA utility function, we have investigated how a bound imposed on TCE affects the optimal portfolio choice and consumption. In so doing, we have used dynamic wealth benchmarks - conditional expected wealth and investment in risk-free bonds, whereby the TCE was re-evaluated at short intervals along the investment horizon. We deduce from our observations that the constraint reduces risky investment. Moreover, part of the wealth hitherto invested in risky assets is diverted to consumption when the constraint is tight.

Akume [1], Chapter six obtains similar results with constrained VaR and concludes for the log-normal diffusion model that TCE and VaR effect similar risk controls when bounded.

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