Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 113 – 122

DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO LIPα CLASS AND WEIGHTED (L^r,$\xi(t)$) CLASS BY PRODUCT SUMMABILITY METHOD

Hare Krishna Nigam

Abstract. A good amount of work has been done on degree of approximation of functions belonging to $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L^r, \xi(t))$ classes using Cesàro and (generalized) Nörlund single summability methods by a number of researchers like Alexits [1], Sahney and Goel [11], Qureshi and Neha [9], Quershi [7, 8], Chandra [2], Khan [4], Leindler [5] and Rhoades [10]. But till now no work seems to have been done so far in the direction of present work. Therefore, in present paper, two quite new results on degree of approximation of functions $f \in Lip\alpha$ and $f \in W(L^r, \xi(t))$ class by (E,1)(C,1) product summability means of Fourier series have been obtained.

1 Introduction

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$f(x) \backsim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n^{th} partial sum $s_n(f; x)$.

L_∞ =norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup \{| f(x) | : x \in R\}$

L_r = norm is defined by $\|f\|_r = (\int_0^{2\pi} | f(x) |^r dx)^{\frac{1}{r}}, r \geq 1$

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial t$_n$ of order n under sup norm $\| \ \|_\infty$ is defined by Zygmund [13].

$\| t_n - f \|_\infty = \sup \{| t_n(x) - f(x) : x\}$ and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = min\|t_n - f\|_r \tag{1.2}$$

2010 Mathematics Subject Classification: 42B05, 42B08.

Keywords: Degree of approximation; $Lip\alpha$ class; $W(L^r, \xi(t))$ class of functions; (E,1) means; (C,1) means; (E,1)(C,1) product means; Fourier series; Lebesgue integral.

A function $f \in \mathrm{Lip}\alpha$ if

$$f(x+t)-f(x)=O(|\,t\,|^{\alpha})\ for\ 0<\alpha\leqq 1 \tag{1.3}$$

$f \in \mathrm{Lip}(\alpha, r)$ if

$$\left(\int_0^{2\pi} |\,f(x+t)-f(x)\,|^r\,dx\right)^{\frac{1}{r}}=O(|\,t\,|^{\alpha}),\ 0<\alpha\leqq 1\ and\ r\geqq 1 \tag{1.4}$$

([6, Definition 5.38])

Given a positive increasing function $\xi(t)$ and an integer $r \geqq 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |\,f(x+t)-f(x)\,|^r\,dx\right)^{\frac{1}{r}}=O(\xi(t)) \tag{1.5}$$

and that $f \in W(L^r, \xi(t))$ if

$$\left(\int_0^{2\pi} |\,\{f(x+t)-f(x)\}\ sin^{\beta}x\,|^r\,dx\right)^{\frac{1}{r}}=O(\xi(t)),\ \beta\geqq 0, r\geqq 1 \tag{1.6}$$

In case $\beta = 0$, we find that $W(L^r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$ and if $\xi(t) = t^{\alpha}$ then $Lip(\xi(t), r)$ class reduces to the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of it n^{th} partial sum $\{s_n\}$.

The (C,1) transform is defined as the n^{th} partial sum of (C,1) summability

$$\begin{aligned} t_n &= \frac{s_0+s_1+s_2+...+s_n}{n+1} \\ &= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s\ \ as\ n\to\infty \end{aligned} \tag{1.7}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by(C,1)method.

If

$$(E,1)=E_n^1=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k\to s \text{ as } n\to\infty \tag{1.8}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,1) to the definite number s [3].

The (E,1) transform of the (C,1) transform defines (E,1)(C,1) transform and we denote it by $(\mathrm{EC})_n^1$.

Thus if

$$(EC)_n^1=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}C_k^1\to s \tag{1.9}$$

where E_n^1 denotes the (E,1) transform of s_n and C_n^1 denotes the (C,1) transform of s_n. Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (E,1)(C,1) means or summable (E,1)(C,1) to a definite number s.

We use the following notations throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{1}{\pi\ 2^{n+1}} \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{1}{1+k} \right) \sum_{\nu=0}^{k} \frac{\sin(\upsilon + \frac{1}{2})t}{\sin \frac{t}{2}} \right\}$$

2 Main theorems

Theorem 1. *If a function f, 2π -periodic, belongs to the Lipα class, then its degree of approximation is given by*

$$\| (EC)_n^1 - f \|_\infty = O\left[\frac{1}{(n+1)^\alpha} \right], \ for\ 0 < \alpha < 1 \tag{2.1}$$

Theorem 2. *If a function f, 2π -periodic, belongs to the weighted $W(L^r, \xi(t)$ class, then its degree of approximation is given by*

$$\| (EC)_n^1 - f \|_r = O\left[\ (n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1} \right) \right] \tag{2.2}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ be a decreasing sequence,} \tag{2.3}$$

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t \mid \phi(t) \mid}{\xi(t)} \right)^r \ \sin^{\beta r} t\ dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1} \right) \tag{2.4}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} \mid \phi(t) \mid}{\xi(t)} \right)^r \ dt \ \right\}^{\frac{1}{r}} = O\left\{ (n+1)^\delta \right\} \tag{2.5}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0, \frac{1}{r} + \frac{1}{s} = 1$, $1 \le r \le \infty$ conditions (2.4)and (2.5) hold uniformly in x and $(EC)_n^1$ is (E,1)(C,1) means of the series (1.1).

3 Lemmas

For the proof of our theorems, following lemmas are required:

Lemma 3. $|\, K_n(t) \,| = O(n+1)$, *for* $0 \leqq t \leqq \frac{1}{n+1}$

Proof. For $0 \leqq t \leqq \frac{1}{n+1}$, $\sin nt \leqq n \sin t$

$$
\begin{aligned}
|\, K_n(t) \,| &= \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k} \right) \sum_{\upsilon=0}^{k} \frac{\sin(\upsilon + \frac{1}{2})t}{\sin \frac{t}{2}} \right] \right| \\
&\leqq \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k} \right) \sum_{\upsilon=0}^{k} \frac{(2\upsilon+1)\sin \frac{t}{2}}{\sin \frac{t}{2}} \right] \right| \\
&\leqq \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} (k+1) \right] \right| \\
&= O\left[\frac{(n+1)}{2^{n+1}} \sum_{k=0}^{n} \left\{ \binom{n}{k} \right\} \right] \\
&= O(n+1) \qquad\qquad \text{since} \sum_{k=0}^{n} \binom{n}{k} = (2)^n
\end{aligned}
$$

□

Lemma 4. $|\, K_n(t) \,| = O\left(\frac{1}{t}\right)$, *for* $\frac{1}{n+1} \leqq t \leqq \pi$

Proof. For $\frac{1}{n+1} \leqq t \leqq \pi$, by applying Jordan's Lemma $\sin \frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leqq 1$

$$
\begin{aligned}
|\, K_n(t) \,| &= \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k} \right) \sum_{\upsilon=0}^{k} \frac{\sin(\upsilon + \frac{1}{2})t}{\sin \frac{t}{2}} \right] \right| \\
&\leqq \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k} \right) \sum_{\upsilon=0}^{k} \frac{1}{t/\pi} \right] \right| \\
&= \frac{1}{t\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k} \right) \sum_{\upsilon=0}^{k} (1) \right] \right| \\
&= \frac{1}{t\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \right] \right| \\
&= O\left(\frac{1}{t} \right) \qquad\qquad \text{since} \sum_{k=0}^{n} \binom{n}{k} = (2)^n
\end{aligned}
$$

□

4 Proof of main theorems

4.1 Proof of Theorem 1:

Following Titchmarsh[12] and using Riemann-Lebesgue Theorem, $s_n(f;x)$ of the series (1.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Therefore using (1.1), the (C,1) transform C_n^1 of $s_n(f;x)$ is given by

$$C_n^1 - f(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi} \phi(t)\sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Now denoting (E,1)(C,1) transform of $s_n(f;x)$ by $(EC)_n^1$, we write

$$\begin{aligned}
(EC)_n^1 - f(x) &= \frac{1}{\pi\ 2^{n+1}}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\ \left(\frac{1}{k+1}\right)\left\{\sum_{\upsilon=0}^{k}\sin\left(\upsilon+\frac{1}{2}\right)t\right\}dt\ \right] \\
&= \int_0^{\pi}\phi(t)\ K_n(t)\ dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\ dt \\
&= I_{1.1} + I_{1.2}\ \text{(say)}
\end{aligned} \tag{4.1}$$

Using Lemma 3,

$$\begin{aligned}
|\ I_{1.1}\ | &\leqq \int_0^{\frac{1}{n+1}} |\ \phi(t)\ ||\ K_n(t)\ |\ dt \\
&= O(n+1)\int_0^{\frac{1}{n+1}} |\ t^{\alpha}\ |\ dt \\
&= O(n+1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{(n+1)}} \\
&= O\left[\frac{1}{(n+1)^{\alpha}}\right]
\end{aligned} \tag{4.2}$$

Now using Lemma 4, we have

$$\begin{aligned}
| I_{1.2} | &\leqq \int_{\frac{1}{n+1}}^{\pi} | \phi(t) || K_n(t) | \, dt \\
&= \int_{\frac{1}{n+1}}^{\pi} | t^{\alpha} | \, O\left(\frac{1}{t}\right) dt \\
&= O\left[\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} dt\right] \\
&= O\left[\frac{1}{(n+1)^{\alpha}}\right] \qquad (4.3)
\end{aligned}$$

Combining (4.1), (4.2) and (4.3) we get

$$\| (EC)_n^1 - f \|_{\infty} \quad = O\left[\frac{1}{(n+1)^{\alpha}}\right], \text{ for } 0 < \alpha < 1$$

This completes the proof of Theorem 1.

4.2 Proof of Theorem 2:

Following the proof of Theorem 1

$$\begin{aligned}
(EC)_n^1 - f(x) &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right] \phi(t) K_n(t) \, dt \\
&= I_{2.1} + I_{2.2} \text{ (say)} \qquad (4.4)
\end{aligned}$$

we have

$| \phi(x+t) - \phi(x) | \quad \leqq \quad | f(u+x+t) - f(u+x) | + | f(u-x-t) - f(u-x) |$

Hence, by Minkowiski's inequality,

$$[\int_0^{2\pi} | \{\phi(x+t) - \phi(x)\} \sin^{\beta} x |^r \, dx]^{\frac{1}{r}} \leqq [\int_0^{2\pi} | \{f(u+x+t) - f(u+x)\} \sin^{\beta} x |^r \, dx]^{\frac{1}{r}}$$

$$+ [\int_0^{2\pi} | \{f(u-x-t) - f(u-x)\} \sin^{\beta} x |^r \, dx]^{\frac{1}{r}} = O\{\xi(t)\}.$$

Then $f \in W(L^r, \xi(t)) \Rightarrow \phi \in W(L^r, \xi(t))$.

Using Hölder's inequality and the fact that $\phi(t) \in W(L^r, \xi(t))$ condition (2.4), $\sin t \geqq \frac{2t}{\pi}$, Lemma 3 and Second Mean Value Theorem for integrals, we have

$$\begin{aligned}
\mid I_{2.1} \mid \quad &\leqq \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t \mid \phi(t) \mid \sin^\beta t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t) \mid K_n(t) \mid}{t \ \sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{(n+1)\ \xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_\in^{\frac{1}{n+1}} \frac{dt}{t^{(1+\beta)s}}\right]^{\frac{1}{s}} \text{ for some } 0 <\in< \frac{1}{n+1} \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1 \qquad (4.5)
\end{aligned}$$

Now using Hölder's inequality, $\mid \sin t \mid \leqq 1$, $\sin t \geqq \frac{2t}{\pi}$, conditions (2.3) and (2.5), Lemma 4 and Mean Value Theorem, we have

$$\begin{aligned}
\mid I_{2.2} \mid &\leqq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta} \mid \phi(t) \mid \sin^\beta t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t) \mid K_n(t) \mid}{t^{-\delta}\ \sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\{(n+1)^\delta\}\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\{(n+1)^\delta\}\left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi(\frac{1}{y})}{y^{\delta-1-\beta}}\right\}^s \frac{dy}{y^2}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^\delta\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\frac{1}{\pi}}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^\delta\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(\beta+1-\delta)-1} - \pi^{s(\delta-1-\beta)+1}}{s(\beta+1-\delta)-1}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^\delta\ \xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right] \\
&= O\left\{(n+1)^\delta\ \xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{-\delta}\ (n+1)^{\beta+1-\frac{1}{s}}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1 \qquad (4.6)
\end{aligned}$$

Now combining (4.4), (4.5), (4.6), we get

$$\mid (EC)_n^1 - f(x) \mid_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}$$

$$\begin{aligned}
\| (EC)_n^1 - f(x) \|_r &= \left\{\int_0^{2\pi} \mid (EC)_n^1 - f(x) \mid^r dx\right\}^{\frac{1}{r}} \\
&= O\left[\left\{\int_0^{2\pi} \left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}^r\ dx\right\}^{\frac{1}{r}}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi}\ dx\right\}^{\frac{1}{r}}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}
\end{aligned}$$

This completes the proof of the Theorem 2.

5 Applications

The following corollaries can be derived from our main theorems:

Corollary 5. *If $\beta = 0$ and $\xi(t) = t^\alpha$ then the degree of approximation of a function $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$ is given by*

$$\| (EC)_n^1 - f(x) \|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}$$

Corollary 6. *If $r \to \infty$ in Corollary 5, then we have for $0 < \alpha < 1$,*

$$\| (EC)_n^1 - f(x) \|_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\}$$

Remark 7. *An independent proof of Corollary 5 can be obtained along the same line of our Theorem 2.*

Acknowledgement. The author is thankful to his parents for their encouragement and support to this work.

References

[1] G. Alexits, *Convergence problems of orthogonal series*. Translated from German by I Földer. International series of Monograms in Pure and Applied Mathematics, vol. **20**, Pergamon Press, New York-Oxford-Paris, 1961. MR0218827(36:1911).

[2] P. Chandra, *Trigonometric approximation of functions in L^p norm*, J. Math. Anal. Appl. **275** (2002), No. 1, 13-26. MR1941769(2003h:42003). Zbl 1011.42001.

[3] G. H. Hardy, *Divergent series*, first edition, Oxford University Press, 1949, 70.

[4] H. H. Khan, *On degree of approximation of functions belonging to the class* $Lip(\alpha, p)$, Indian J. Pure Appl. Math., **5** (1974), no. 2, 132-136. MR0380230(52 :1130). Zbl 0308.41010.

[5] L. Leindler, *Trigonometric approximation in L^p norm*, J. Math. Anal. Appl., **302** (2005). MR2107350(2005g:42004). Zbl 1057.42004.

[6] L. McFadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942), 168-207. MR0006379(3,295g). Zbl 0061.12106.

[7] K. Qureshi, *On the degree of approximation of a periodic function f by almost Nörlund means*, Tamkang J. Math., **12** (1981), No. 1, 35-38. MR0714492(85h:42002). Zbl 0502.42002.

[8] K. Qureshi, *On the degree of approximation of a function belonging to the class* $Lip\alpha$, Indian J. pure Appl. Math., **13** (1982), no. 8, 898-903. MR0670332(84c:41013). Zbl 0498.42002.

[9] K. Qureshi and H. K. Neha, *A class of functions and their degree of approximation*, Ganita, **41** (1990), no. 1, 37-42. MR1155949. Zbl 0856.42005.

[10] B. E. Rhaodes, *On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series*, Tamkang Journal of Mathematics, **34** (2003), no. 3, 245-247. MR2001920(2004g:41023). Zbl 1039.42001.

[11] B. N. Sahney and D. S. Goel, *On the degree of continuous functions*, Ranchi University Math. Jour., **4** (1973), 50-53. MR0352846(50:5332). Zbl 0296.41008.

[12] E. C. Titchmarsh, *The Theory of functions*, Oxford University Press, 1939, 402-403.

[13] A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. **1**, Cambridge Univ. Press, Cambridge, New York, 1959. MR0107776(21:6498). Zbl 0085.05601.

****

Hare Krishna Nigam
Department of Mathematics,
Faculty of Engineering and Technology,
Mody Institute of Technology and Science (Deemed University)
Laxmangarh, Sikar-332311,
Rajasthan, India.
e-mail: harekrishnan@yahoo.com