Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 89 – 98

HOMOTOPY ANALYSIS METHOD FOR SOLVING KDV EQUATIONS

Hossein Jafari and M. A. Firoozjaee

Abstract. A scheme is developed for the numerical study of the Korteweg-de Vries (KdV) and the Korteweg-de Vries Burgers (KdVB) equations with initial conditions by a homotopy approach. Numerical solutions obtained by homotopy analysis method are compared with exact solution. The comparison shows that the obtained solutions are in excellent agreement.

1 Introduction

The study of nonlinear problems is of crucial importance in all areas of mathematical and physics. Some of the most interesting features of physical systems are hidden in their nonlinear behavior, and can only be studied with appropriate methods designed to tackle nonlinear problems. In the past several decades, many authors mainly had paid attention to study solutions of nonlinear equations by using various methods, Among these are Backlund transformation [4, 8], Darboux transformation [28], Inverse scattering method [12], Hirotas bilinear method [16], the tanh-function method [24], the sine—cosine method [32], the homogeneous balance method [29]. Recently an extended tanh–function method and symbolic computation are suggested in [11] for solving the new coupled modified KdV equations to obtain four kinds soliton solutions. This method has some merits in contrast with the tanh–function method. It only uses a more simple algorithm to produce an Algebraic system but also can pick up singular silton solutions with no extra effort [30, 17, 25, 10]. The Burgers equation is a special case of the KdVB equation has been found to describe various kind of phenomena such as a mathematical model of turbulence [6] and the approximate theory of flow through a shock wave traveling in viscous fluid [7]. Fletcher using the Hopf—Cole transformation [9] gave an analytic solution of the system of two dimensional Burgers equations, Several numerical methods of this equation system have been given such as algorithms based on cubic spline function technique [18], applied an explicit—implicit method [31], implicit finite–difference scheme [5]. Soliman [3] used the similarity reductions for the partial differential equations to

2010 Mathematics Subject Classification: 35A35; 65M99.
Keywords: KDVB equation; Homotopy analysis method ; Exact solution.

develop a scheme for solving the Burgers equation.far as we know that little numerical works has been done to solve the KdVB equation. Recently a numerical method is proposed for solving the KdVB equation by Zaki [33], he is used the collocation method with quintic B–spline finite element and the author [26] are use the collocation solution of the KdV equation using septic splines as element shape function.Very recently Kaya [19] is implement the Adomian decomposition method for solving the KdVB equation.

2 Basic idea of HAM

Consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [20] constructs the so-called zero–order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\,\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{2.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $u(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau), \tag{2.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $u(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $u(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{2.4}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}|_{p=0}. \tag{2.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (2.4) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{2.6}$$

which must be one of solutions of original nonlinear equation, as proved by[20]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2.2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\,\mathcal{N}[\phi(\tau;p)] = 0, \tag{2.7}$$

which is used mostly in the homotopy perturbation method[13], where as the solution obtained directly, without using Taylor series [14, 15]. According to the definition (2.5), the governing equation can be deduced from the zero-order deformation equation(2.2). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (2.2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\,\mathcal{H}(\tau)\Re_m(\vec{u}_{m-1}), \tag{2.8}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}|_{p=0}. \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{2.10}$$

It should be emphasized that $u_m(\tau)$ for $m \geqslant 1$ is governed by the linear equation (2.8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as *Mathematica*. For the convergence of the above method we refer the reader to Liao's work [20] . If Eq.(2.1) admits unique solution, then this method will produce the unique solution. If equation (2.1) does not possess unique solution, the HAM will give a solution among many other(possible)solutions.

3 Applications

In this section we apply the HAM to the KdVB equation for different cases. Consider the KdVB equation has the form [19]

$$u_t + \epsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0 \tag{3.1}$$

where ν,ϵ and μ are positive parameters. Eq. (3.1) is called the Korteweg–de Vries Burgers equation which derived by Su and Gardner [27], when the parameter $\nu = 0$, Eq. (3.1) will be the KdV equation and when the parameter $\mu = 0$, Eq. (3.1) will be Burgers equation which are solved by VIM by Abdou and Soliman [1].
In our study, we will investigate the two cases, the first one is the KdV equation (in case of $\nu = 0$) and the second one is the KdVB in case of $\epsilon = 1$.

Case 1. *For purpose of illustration of the Homotopy analysis method for solving the KdVB equation (3.1), in case of* $\nu = 0$, $\epsilon = -6$ *and* $\mu = 1$, *for the KdV equation, we start with an initial approximation:* $u_0 = u(x, 0)$ *given by*

$$u(x,0) = -2sech^2(x) \tag{3.2}$$

Then we obtained terms HAM:

$$\begin{aligned} u_1(x,t) &= 16ht\mathrm{sech}^2(x)\tanh(x) \\ u_2(x,t) &= 64h^2t^2\mathrm{sech}^4(x) - 32h^2t^2\cosh(2x)\mathrm{sech}^4(x) + 8h^2t\sinh(2x)\mathrm{sech}^4(x) \\ &\quad +16ht\tanh(x)\mathrm{sech}^2(x) \end{aligned}$$

The exact solution of $u(x,t)$ in a closed form as

$$u(x,t) = -2sech^2(x - 4t) \tag{3.3}$$

In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using nth approximations show the high degree of accuracy and in most cases u_n, the nth approximation is accurate for quite low of n $(n = 3)$. The obtained numerical results are summarized in Table 1. From these results we conclude that the method, Homotopy analysis method for KdV equation, gives remarkable accuracy in comparison with our analytical solution (3.3). The behavior of the solution obtained by homotopy analysis method and analytic solution are shown for a different values of times in Figs. 1 and 2, respectively.

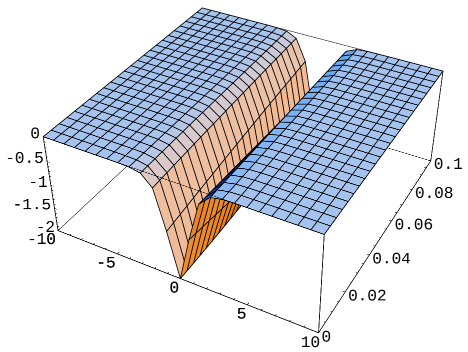

Fig.1.The behavior 4th-order of HAM versus x for different values of time

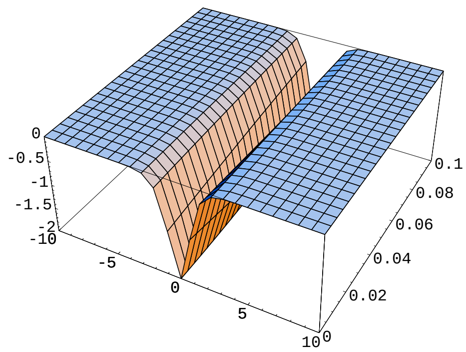

Fig.2. The behavior of the analytic solution u(x, t) versus x for different values of time.

	x	Exact solution	Numerical solution	Absolute error
t=0.01	-7.5	-0.000002259066	-0.00000225906618556596	1.78915×10^{-15}
	-2.5	-0.04914600344	-0.049146	3.49519×10^{-10}
	2.5	-0.05754985288	-0.0575498	1.92665×10^{-10}
	7.5	-0.000002651038464	-0.0000026510358499832465	4.48678×10^{-15}
t=0.02	-7.5	-0.00000208538101	-0.000002085381104550166	9.21865×10^{-15}
	-2.5	-0.04541063202	-0.0454106	5.10759×10^{-9}
	2.5	-0.06226782912	-0.0622674	1.45892×10^{-9}
	7.5	-0.000002871835528	0.00000-28718133760490534	3.7684×10^{-13}
t=0.03	-7.5	-0.000001925049378	-0.0000019250484611802373	9.16939×10^{-13}
	-2.5	-0.04195611960	-0.0419561	3.30365×10^{-8}
	2.5	-0.06736587306	-0.0673642	3.91831×10^{-8}
	7.5	-0.0000031110221	-0.0000031109292250893545	6.32451×10^{-12}
t=0.04	-7.5	-0.000001777044614	-0.0000017770436320642752	9.82231×10^{-13}
	-2.5	-0.03876179800	-0.0387616	3.20067×10^{-8}
	2.5	-0.07287344918	-0.0728687	3.11718×10^{-7}
	7.5	-0.00000337012979	-0.0000033698517163765725	3.64666×10^{-11}
t=0.05	-7.5	-0.000001640418988	-0.0000016404307329864928	1.17452×10^{-11}
	-2.5	-0.03580845352	-0.0358082	2.14132×10^{-7}
	2.5	-0.07882210802	-0.0788109	1.26521×10^{-6}
	7.5	-0.000003650817764	-0.0000036501379083592282	1.32686×10^{-10}

Case 2. *A second important case, consider the KdVB equation (3.1), $\epsilon = 1$, and in this case the initial condition will take the form [19]*

$$u(x,0) = -\frac{6\nu^2}{25\mu}1 + tanh(\frac{\nu x}{10\mu}) + \frac{1}{2}sech^2\frac{\nu x}{25\mu}. \tag{3.4}$$

We start with an initial approximation: $u_0 = u(x,0)$ and Then we obtained terms HAM:

$$
\begin{aligned}
u_1(x,t) &= \frac{39\mu^{-\frac{vx}{10\mu}}htv^5\text{sech}^5\left(\frac{vx}{10\mu}\right)}{6250\mu^3}+\frac{3\mu^{-\frac{vx}{10\mu}}htv^5\cosh\left(\frac{vx}{5\mu}\right)\text{sech}^5\left(\frac{vx}{10\mu}\right)}{6250\mu^3}\\
&-\frac{9\mu^{-\frac{vx}{10\mu}}htv^5\sinh\left(\frac{vx}{5\mu}\right)\text{sech}^5\left(\frac{vx}{10\mu}\right)}{6250\mu^3}\\
u_2(x,t) &= -\frac{192\mu^{\frac{vx}{5\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}-\frac{4704\mu^{\frac{2vx}{5\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}-\frac{15168\mu^{\frac{3vx}{5\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}\\
&+\frac{59136\mu^{\frac{4vx}{5\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}-\frac{14784\mu^{\frac{vx}{\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}+\frac{96\mu^{\frac{6vx}{5\mu}}h^2t^2v^8}{390625\mu^5\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}\\
&+\frac{39\mu^{-\frac{vx}{10\mu}}ht\text{sech}^5\left(\frac{vx}{10\mu}\right)v^5}{6250\mu^3}+\frac{3\mu^{-\frac{vx}{10\mu}}ht\cosh\left(\frac{vx}{5\mu}\right)\text{sech}^5\left(\frac{vx}{10\mu}\right)v^5}{6250\mu^3}\\
&+\frac{96\mu^{\frac{vx}{5\mu}}h^2tv^5}{3125\mu^3\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}+\frac{912\mu^{\frac{2vx}{5\mu}}h^2tv^5}{3125\mu^3\left(1+e^{\frac{vx}{5e}}\right)^8}+\frac{2112\mu^{\frac{3vx}{5\mu}}h^2tv^5}{3125\mu^3\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}\\
&+\frac{1824\mu^{\frac{4vx}{5\mu}}h^2tv^5}{3125\mu^3\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}+\frac{96\mu^{\frac{vx}{\mu}}h^2tv^5}{625\mu^3\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}-\frac{48\mu^{\frac{6vx}{5\mu}}h^2tv^5}{3125\mu^3\left(1+\mu^{\frac{vx}{5\mu}}\right)^8}\\
&-\frac{9\mu^{-\frac{vx}{10\mu}}ht\text{sech}^5\left(\frac{vx}{10\mu}\right)\sinh\left(\frac{vx}{5\mu}\right)v^5}{6250\mu^3}
\end{aligned}
$$

The exact solution of $u(x,t)$ in a closed form as

$$
u(x,t)=-\frac{6\nu^2}{25\mu}[1+tanh(\frac{\nu}{10\mu}(x+\frac{6\nu^2}{25\mu}t))]-\frac{1}{2}sech^2[(\frac{\nu}{10\mu}(x+\frac{6\nu^2}{25\mu}t))]. \quad (3.5)
$$

In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using nth approximations show the high degree of accuracy and in most cases u_n, the nth approximation is accurate for quite low of n ($n=3$). The obtained numerical results are summarized in Table 2. From these results we conclude that the Homotopy analysis method, for KdV equation, gives remarkable accuracy in comparison with our analytical solution (3.5). The behavior of the solution obtained by Homotopy analysis method and analytic solution are shown for a different values of times in Figs. 3 and 4, respectively, for $\nu=1$,$\mu=1$

	x	Exact solution	Numerical solution	Absolute error
$t = 100$	0	-0.0003605753076	-0.00036067	9.42599×10^{-8}
$\nu = 0.001$	25	-0.0004799787028	-0.000479978	2.65576×10^{-10}
$\mu = 0.001$	50	-0.0004799999989	-0.00048	1.26194×10^{-15}
	75	-0.000480000000	-0.00048	5.42101×10^{-19}
	100	0.000480000000	-0.00048	0
$t = 800$	0	-0.0003645632078	-0.000365223	6.59783×10^{-7}
$\nu = 0.001$	25	-0.0004799800782	-0.000479978	2.07112×10^{-9}
$\mu = 0.001$	50	-0.0004799999990	-0.00048	9.87581×10^{-14}
	75	-0.0004800000001	-0.00048	4.49944×10^{-18}
	100	0.00048000000000	-0.00048	0
$t = 100$	0	-0.003656898008	-0.00366482	7.92271×10^{-7}
$\nu = 0.01$	25	-0.004799804548	-0.00479978	2.57019×10^{-8}
$\mu = 0.01$	50	-0.004799999993	-0.0048	1.2269×10^{-12}
	75	-0.004799999999	-0.0048	5.46438×10^{-17}
	100	-0.0048	-0.0048	0
$t = 10$	0	-0.40098639900	-0.408521	1.34299×10^{-3}
$\nu = 1$	25	-0.4799917252	-0.479973	1.85928×10^{-5}
$\mu = 1$	50	-0.4799999996	-0.48	9.87317×10^{-10}
	75	-0.4800000000	-0.48	4.45755×10^{-14}
	100	0.4800000000	-0.48	0

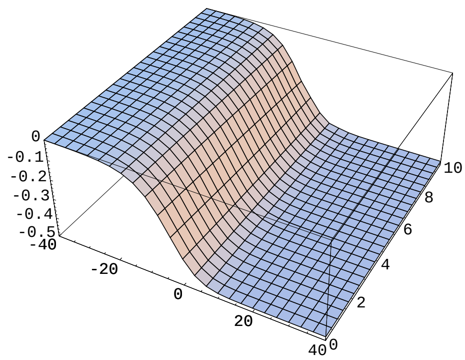

Fig.3.The behavior 3th-order of HAM versus x for different values of time

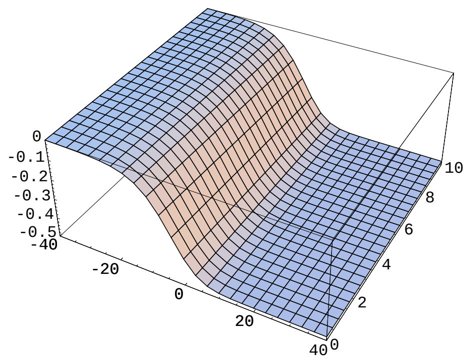

Fig.4. The behavior of the analytic solution u(x, t) versus x for different values of time.

References

[1] M. A. Abdou and A. A. Soliman, *Variational iteration method for solving Burgers and coupled Burgers equations*. J Comput Appl Math,**181**(2) (2005), 245–251. MR2146836(2006a:65139). Zbl 1072.65127.

[2] A. A. Soliman, *A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations*, Solitons and Fractals, Chaos, **29** (2006), 294–302. MR2211466. Zbl 1099.35521.

[3] A. A. Soliman, *New numerical technique for Burgers equation based on similarity reductions*. In: International conference on computational fluid dynamics, Beijing, China, October 17–20, 2000, 559–566.

[4] M. J. Ablowitz and P.A. Clarkson, *Solitons nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, **149**. Cambridge University Press, Cambridge, 1991. xii+516 pp. MR1149378(93g:35108). Zbl 0762.35001.

[5] A. R. Bahadir, *A fully implicit finite–difference scheme for two–dimensional Burgers equations*, Appl Math Comput **137** (2003), 131-137. MR1949127(2004a:65097). Zbl 1027.65111.

[6] J. M. Burger, *A mathematical model illustrating the theory of turbulence*, Adv Appl Mech, **I** (1948), 171–99.

[7] J. D. Cole, *On a quasilinear parabolic equations occurring in aerodynamics*, Q Appl Math, **9** (1951), 225-236. MR0042889 (13,178c). Zbl 0043.09902.

[8] A. Coley, et al., editors, *Backlund and Darboux transformations*. Providence, RI: American Mathematical Society, 2001. MR1870397 (2002g:37001). Zbl 0974.00040.

[9] J. D. Fletcher, *Generating exact solutions of the two-dimensional Burgers equations*, Int J Numer Meth Fluids, **3**(1983), 213–6. Zbl 0563.76082.

[10] E. G. Fan and H. Q. Zhang, *A note on the homogeneous balance method*, Phys Lett A, **246** (1998), 403–406. Zbl 1125.35308.

[11] E. Fan, *Soliton solutions for a generalized Hirota–Satsuma coupled KdV equation and a coupled MKdV equation*, Phys Lett A, **282** (2001), 18–22. MR1838205 (2002d:35177). Zbl 0984.37092.

[12] C.S. Gardner, J.M. Green , M.D. Kruska and R.M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys Rev Lett, **19** (1967), 1095–1097. Zbl 1103.35360.

[13] J. H. He, *A coupling method of homotopy technique and perturbation technique for nonlinear problems*, Int J Nonlinear Mech, **35** (2000), 37–43. MR1723761 (2000k:65103). Zbl 1091.74012.

[14] J. H. He, *Homotopy perturbation method for solving boundary value problems*, Phys Lett A, **350** (2006), 87–88. MR2199322.

[15] J. H. He, *Some asymptotic methods for strongly nonlinear equations*, Int J Mod Phys B, **20** (10) (2006), 1141–1199. MR2251264 (2007c:35151). Zbl 1102.34039.

[16] R. Hirota, *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys Rev Lett, **27** (1971), 1192–1194. Zbl 1168.35423.

[17] R. Hirota and J. Satsuma, *Soliton solutions of a coupled Korteweg-de Vries equation*, Phys Lett A, **85** (1981), 407–408. MR0632382 (82j:35128).

[18] P.C. Jain and D.N. Holla, *Numerical solution of coupled Burgers D equations*, Int J Numer Meth Eng, **12** (1978), 213-222.

[19] D. Kaya, *An application of the decomposition method for the KdVb equation*, Appl Math Comput, **152** (2004), 279–288. MR2050064. Zbl 1053.65087.

[20] S. J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, CRC Press, Boca Raton: Chapman & Hall (2004). MR2058313(2005h:65003). Zbl 1051.76001.

[21] S. J. Liao, *On the homotopy anaylsis method for nonlinear problems*, Appl Math Comput, **147** (2004),499–513.MR2012589. Zbl 1086.35005.

[22] S. J. Liao, *Comparison between the homotopy analysis method and homotopy perturbation method*, Appl Math Comput, **169** (2005), 1186–1194. MR2174713. Zbl 1082.65534.

[23] S. J. Liao, *A new branch of solutions of boundary-layer flows over an impermeable stretched plate*, Int J Heat Mass Transfer, **48** (2005), 2529–39.

[24] W. Malfeit, *Solitary wave solutions of nonlinear wave equations*, Am J Phys, **60** (1992),650–654.

[25] J. Satsuma and R. Hirota, *A coupled KdV equation is one case of the four–reduction of the KP hierarchy*. J Phys Soc Jpn, **51**(1982), 3390–3397. MR0687745(84g:58057).

[26] A. A. Soliman, *Collocation solution of the Korteweg-de Vries equation using septic splines*, Int J Comput Math, **81** (2004), 325–331. MR2174994. Zbl 1058.65113.

[27] C. H. Su and C. S. Gardner, *Derivation of the Korteweg de-Vries and Burgers equation*, J Math Phys, **10** (1969), 536-539. MR0271526 (42 #6409). Zbl 0283.35020.

[28] M. Wadati, H. Sanuki and K. Konno, *Relationships among inverse method, Backlund transformation and an infinite number of conservation laws*, Prog Theor Phys, **53** (1975), 419–436. MR0371297 (51 #7516). Zbl 1079.35506.

[29] M. L. Wang, *Exact solutions for a compound KdV-Burgers equation*, Phys Lett A, **213** (1996), 279–287. MR1390282 (96m:35289). Zbl 0972.35526.

[30] Y. T. Wu, X. G. Geng, X. B. Hu and S. M. Zhu, *A generalized Hirota–Satsuma coupled Korteweg–de Vries equation and Miura transformations*, Phys Lett A, **255** (1999), 259–264.MR1691458 (2000c:37109). Zbl 0935.37029.

[31] F. W. Wubs and E.D. de Goede, *An explicit–implicit method for a class of time-dependent partial differential equations*, Appl Numer Math, **9** (1992), 157–181. MR1147969(92j:65136). Zbl 0749.65068.

[32] C. T. Yan, *A simple transformation for nonlinear waves*, Phys Lett A, **224** (1996), 77–84. MR1427895 (97i:35161). Zbl 1037.35504.

[33] S. I. Zaki, *A quintic B-spline finite elements scheme for the KdVB equation*, Comput Meth Appl Mech Eng, **188** (2000), 121–134. Zbl 0957.65088.

Hossein Jafari
Department of Mathematics
and Computer Science,
University of Mazandaran,
Babolsar, Iran.
jafari_h@math.com, jafari@umz.ac.ir
http://www.umz.ac.ir/en/dynamic/dynamic.asp?Userid=1410

M. A. Firoozjaee
Department of Mathematics
and Computer Science,
University of Mazandaran,
Babolsar, Iran.
m64arab@math.com