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ENCLOSING ROOTS OF POLYNOMIAL EQUATIONS AND THEIR APPLICATIONS TO ITERATIVE PROCESSES

Ioannis K. Argyros and Saïd Hilout

Abstract. We introduce a special class of real recurrent polynomials f_n $(n \geq 1)$ of degree n, with unique positive roots s_n, which are decreasing as n increases. The first root s_1, as well as the last one denoted by s_∞ are expressed in closed form, and enclose all s_n $(n > 1)$.

This technique is also used to find weaker than before [5] sufficient convergence conditions for some popular iterative processes converging to solutions of equations.

1 Introduction

We introduce a special class of recurrent polynomials f_n $(n \geq 1)$ of degree n with real coefficients.

Then, we find sufficient conditions under which each polynomial f_n has a unique positive root s_n, such that $s_{n+1} \leq s_n$ $(n \geq 1)$. The first root s_1, as well as the last one denoted by s_∞ are expressed in simple closed form.

Two applications are provided. In the first one, we show how to use s_1 and s_∞ to locate any s_n belonging in $(s_\infty, s_1]$ $(n \geq 1)$.

In the second one, using this technique on Newton's method (3.1), we show that the famous for its simplicity and clarity Newton–Kantorovich condition (3.2) for solving equations can always replaced by a weaker one (2.16).

Moreover, the ratio of the quadratic convergence of Newton's method $2\ q_0$ (see, (2.16), (2.24), and (3.2)) under our approach is smaller than $2\ q_K$ given in [5].

2 Locating roots of polynomials

We need the main result on locating roots of polynomials.

Theorem 1. *Let $a > 0$, $b > 0$, and $c < 0$ be given constants. Define polynomials f_n $(n \geq 1)$, g on $[0, +\infty)$ by:*

$$f_n(s) = b\ s^n + a\ s^{n-1} + b\ (s^{n-1} + s^{n-2} + \cdots + 1) + c, \tag{2.1}$$

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and

$$g(s)=b\ s^2+a\ s-a. \tag{2.2}$$

Set

$$d=\frac{2\ a}{a+\sqrt{a^2+4\ a\ b}}. \tag{2.3}$$

Assume:

$$d \leq 1+\frac{b}{c}, \tag{2.4}$$

and

$$a+b+c<0. \tag{2.5}$$

Then, each polynomial f_n ($n \geq 1$) has a unique positive root s_n.
Moreover, the following estimates hold for all $n \geq 1$:

$$1+\frac{b}{c} \leq s^\star \leq s_{n+1} \leq s_n, \tag{2.6}$$

and

$$f_n(d) \leq 0, \tag{2.7}$$

where,

$$s^\star=\lim_{n\longrightarrow\infty} s_n.$$

Proof. Each polynomial f_n has a unique positive root s_n ($n \geq 1$), by the Descarte's rule of signs.

Polynomial f_n can be written for $s \in [0,1)$:

$$f_n(s)=b\ s^n+a\ s^{n-1}+b\ \frac{1-s^n}{1-s}+c. \tag{2.8}$$

By letting $n \longrightarrow \infty$, we get:

$$f_\infty(s)=\lim_{n\longrightarrow\infty} f_n(s)=\frac{b}{1-s}+c. \tag{2.9}$$

Function f_∞ has a unique positive root denoted by s_∞, and given by:

$$s_\infty=1+\frac{b}{c}<1. \tag{2.10}$$

Function f_∞ is also increasing, since

$$f'_\infty(s)=\frac{b}{(1-s)^2}>0. \tag{2.11}$$

Furthermore, we shall show estimates (2.6), and (2.7) hold.

We need the relationship between two consecutive polynomials f_n's ($n \geq 1$):

$$\begin{aligned} f_{n+1}(s) &= b\ s^{n+1} + a\ s^n + b\ (s^n + \cdots + 1) + c \\ &= b\ s^n + a\ s^{n-1} + b\ (s^{n-1} + \cdots + 1) + c + \\ &\quad a\ s^n - a\ s^{n-1} + b\ s^{n+1} \\ &= f_n(s) + s^{n-1}\ (b\ s^2 + a\ s - a) \\ &= f_n(s) + g(s)\ s^{n-1}. \end{aligned} \tag{2.12}$$

Note that d is the unique positive root of function g.
Then, using (2.12), we obtain for all $n \geq 1$:

$$f_n(d) = f_{n-1}(d) = \cdots = f_1(d).$$

Let i be any fixed but arbitrary natural number. Then, we get:

$$f_i(d) = \lim_{n \longrightarrow \infty} f_n(d) = f_\infty(d) \leq f_\infty(s_\infty) = 0, \tag{2.13}$$

since, function f_∞ is increasing, and $d \leq s_\infty$ by hypothesis (2.4). It follows from the definition of the zeros s_n and (2.13) that

$$d \leq s_n \qquad \text{for all } n \geq 1. \tag{2.14}$$

Polynomials f_n are increasing which together with (2.14) imply

$$f_n(d) \leq f_n(s_n) = 0.$$

In particular

$$f_\infty(d) = \lim_{n \longrightarrow \infty} f_n(d) \leq 0.$$

Hence, estimate (2.7) holds.
We then get from (2.12), and (2.14):

$$f_{n+1}(s_{n+1}) = f_n(s_{n+1}) + g(s_{n+1})\ s_{n+1}^{n-1}\ \eta$$

or

$$f_n(s_{n+1}) \leq 0, \tag{2.15}$$

since $f_{n+1}(s_{n+1})=0$, and $g(s_{n+1})\ s_{n+1}^{n-1}\ \eta \geq 0$, which imply

$$s_{n+1} \leq s_n \qquad (n \geq 1).$$

Sequence $\{s_n\}$ is non–increasing, bounded below by zero, and as such it converges to $s^\star$.

We shall show $s_\infty \le s^\star$. Using (2.12), we have:

$$\begin{array}{rcl} f_{i+1}(s_i) & = & f_i(s_i) + g(s_i)\ s_i^{i-1}\ \eta \\ & = & g(s_i)\ s_i^{i-1}\ \eta \ge 0, \end{array}$$

so,

$$f_{i+2}(s_i){=}f_{i+1}(s_i) + g(s_i)\ s_i^{i}\ \eta \ge 0.$$

If, $f_{i+m}(s_i) \ge 0$, $m \ge 0$, then

$$f_{i+m+1}(s_i){=}f_{i+m}(s_i) + g(s_i)\ s_i^{i+m-1}\ \eta \ge 0.$$

Hence, by the definition of function f_∞, we get:

$$f_\infty(s_n) \ge 0 \qquad \text{for all } n \ge 1.$$

But we also have $f_\infty(0){=}b + c < 0$. That is $s_\infty \le s_n$ for all $n \ge 1$, and consequently $s_\infty \le s^\star$.

That completes the proof of Theorem 1. □

Set $a{=}L\ \eta$, $b{=}2\ L_0\ \eta$, and $c{=}-2$, in Theorem 1.

It is simple algebra to show that conditions (2.4), and (2.5) reduce to (2.16) in the majorizing lemma that follows:

Lemma 2. *Assume there exist constants $L_0 \ge 0$, $L \ge 0$, and $\eta \ge 0$, with $L_0 \le L$, such that:*

$$q_0{=}\overline{L}\ \eta \begin{cases} \le \dfrac{1}{2} & \text{if} \quad L_0 \ne 0 \\ < \dfrac{1}{2} & \text{if} \quad L_0{=}0 \end{cases}, \tag{2.16}$$

where,

$$\overline{L}{=}\frac{1}{8}\left(L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}\right). \tag{2.17}$$

Then, sequence $\{t_k\}$ $(k \ge 0)$ given by

$$t_0{=}0, \quad t_1{=}\eta, \quad t_{k+1}{=}t_k + \frac{L\ (t_k - t_{k-1})^2}{2\ (1 - L_0\ t_k)} \qquad (k \ge 1), \tag{2.18}$$

is nondecreasing, bounded above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star}{=}\frac{2\ \eta}{2-\delta}, \tag{2.19}$$

$$\delta{=}\frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \ne 0. \tag{2.20}$$

Moreover the following estimates hold:

$$L_0 \; t^\star \leq 1, \tag{2.21}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2} \; (t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \; \eta, \quad (k \geq 1), \tag{2.22}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k \; (2 \; q_0)^{2^k-1} \; \eta, \quad (k \geq 0), \tag{2.23}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \; \frac{(2 \; q_0)^{2^k-1} \; \eta}{1 - (2 \; q_0)^{2^k}}, \quad (2 \; q_0 < 1), \quad (k \geq 0). \tag{2.24}$$

Proof. We shall show using induction on k that for all $k \geq 0$:

$$L \; (t_{k+1} - t_k) + \delta \; L_0 \; t_{k+1} < \delta, \tag{2.25}$$

$$0 < t_{k+1} - t_k, \tag{2.26}$$

$$L_0 \; t_{k+1} < 1, \tag{2.27}$$

and

$$0 < t_{k+1} < t^{\star\star}. \tag{2.28}$$

Estimates (2.25)–(2.28) hold true for k=0 by the initial condition t_1=η, and hypothesis (2.16). It then follows from (2.18) that

$$0 < t_2 - t_1 \leq \frac{\delta}{2} \; (t_1 - t_0) \quad \text{and} \quad t_2 \leq \eta + \frac{\delta}{2} \; \eta = \frac{2+\delta}{2} \; \eta < t^{\star\star}.$$

Let us assume estimates (2.25)–(2.28) hold true for all integer values n: $n \leq k$ $(n \geq 0)$.

We also get

$$\begin{aligned} t_{k+1} &\leq t_k + \frac{\delta}{2} \; (t_k - t_{k-1}) \\ &\leq t_{k-1} + \frac{\delta}{2} \; (t_{k-1} - t_{k-2}) + \frac{\delta}{2} \; (t_k - t_{k-1}) \\ &\leq \eta + \left(\frac{\delta}{2}\right) \; \eta + \cdots + \left(\frac{\delta}{2}\right)^k \; \eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{k+1}}{1 - \frac{\delta}{2}} \; \eta \\ &< \frac{2 \; \eta}{2 - \delta} = t^{\star\star}. \end{aligned} \tag{2.29}$$

We have:

$$L\ (t_{k+1}-t_k)+\delta\ L_0\ t_{k+1}\leq L\ \left(\frac{\delta}{2}\right)^k\ \eta+L_0\ \delta\ \frac{1-\left(\frac{\delta}{2}\right)^{k+1}}{1-\frac{\delta}{2}}\ \eta. \tag{2.30}$$

In view of estimate (2.30), (2.25) holds, if

$$\left\{L\ \left(\frac{\delta}{2}\right)^n+\delta\ L_0\ \frac{1-\left(\frac{\delta}{2}\right)^{n+1}}{1-\frac{\delta}{2}}\right\}\ \eta\leq\delta. \tag{2.31}$$

Estimate (2.31) motivates us to define for $s=\frac{\delta}{2}$, the sequence $\{f_n\}$ of polynomials on $[0,+\infty)$ by

$$f_n(s)=\Big(L\ s^{n-1}+2\ L_0\ (1+s+s^2+\cdots+s^n)\Big)\ \eta-2. \tag{2.32}$$

In view of Theorem 1, the induction for (2.25)–(2.28) is completed.

Hence, sequence $\{t_n\}$ is non–decreasing, bounded above by $t^{\star\star}$, and as such that it converges to its unique least upper bound $t^\star$. The induction is completed for (2.21), and (2.22).

If L_0=0, then (2.21) holds trivially. In this case, for $L>0$, an induction argument shows that

$$t_{k+1}-t_k=\frac{2}{L}\ (2\ q_0)^{2^k}\qquad(k\geq 0),$$

and therefore

$$t_{k+1}=t_1+(t_2-t_1)+\cdots+(t_{k+1}-t_k)=\frac{2}{L}\ \sum_{m=0}^{k}(2\ q_0)^{2^m},$$

and

$$t^\star=\lim_{k\to\infty}t_k=\frac{2}{L}\ \sum_{k=0}^{\infty}(2\ q_0)^{2^k}.$$

Clearly, this series converges, since $k\leq 2^k$, $2\ q_0<1$, and is bounded above by the number

$$\frac{2}{L}\ \sum_{k=0}^{\infty}(2\ q_0)^k=\frac{4}{L\ (2-L\ \eta)}.$$

If L=0, then, since, $0\leq L_0\leq L$, we deduce: L_0=0, and $t^\star=t_k=\eta$ $(k\geq 1)$.

In the rest of the proof, we assume that $L_0>0$.

In order for us to later complete the induction for (2.23), we first need to show the estimate:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+1}}{1-\frac{\delta}{2}}\,\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right) \qquad (k \geq 1). \tag{2.33}$$

For k=1, (2.33) becomes

$$\left(1+\frac{\delta}{2}\right)\eta \leq \frac{4\,\overline{L}-L}{4\,\overline{L}\,L_0}$$

or

$$\left(1+\frac{2\,L}{L+\sqrt{L^2+8\,L_0\,L}}\right)\eta \leq \frac{4\,L_0-L+\sqrt{L^2+8\,L_0\,L}}{L_0\,(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})}$$

In view of (2.16), it suffices to show:

$$\frac{L_0\,(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})\,(3\,L+\sqrt{L^2+8\,L_0\,L})}{(L+\sqrt{L^2+8\,L_0\,L})\,(4\,L_0-L+\sqrt{L^2+8\,L_0\,L})} \leq 2\,\overline{L},$$

which is true as equality.

Let us now assume estimate (2.33) is true for all integers smaller or equal to k. We must show (2.33) holds for k replaced by $k+1$:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+2}}{1-\frac{\delta}{2}}\,\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right) \qquad (k \geq 1).$$

or

$$\left(1+\frac{\delta}{2}+\left(\frac{\delta}{2}\right)^2+\cdots+\left(\frac{\delta}{2}\right)^{k+1}\right)\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right). \tag{2.34}$$

By the induction hypothesis to show (2.34), it suffices

$$\frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right)+\left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right)$$

or

$$\left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(\left(\frac{\delta}{2}\right)^{k-1}-\left(\frac{\delta}{2}\right)^{k}\right)\frac{L}{4\,\overline{L}}$$

or

$$\delta^2\,\eta \leq \frac{L\,(2-\delta)}{2\,\overline{L}\,L_0}.$$

In view of (2.16) it suffices to show

$$\frac{2\ \overline{L}\ L_0\ \delta^2}{L\ (2-\delta)} \leq 2\ \overline{L},$$

which holds as equality by the choice of δ given by (2.20).

That completes the induction for estimates (2.33).

We shall show (2.23) using induction on $k \geq 0$: estimate (2.23) is true for k=0 by (2.16), (2.18), and (2.20). The inductive argument later in the proof requires the second base case of k=1. In order for us to show estimate (2.23) for k=1, since $t_2 - t_1 = \dfrac{L\ (t_1 - t_0)^2}{2\ (1 - L_0\ t_1)}$, it suffices:

$$\frac{L\ \eta^2}{2\ (1 - L_0\,\eta)} \leq \delta\ \overline{L}\ \eta^2$$

or

$$\frac{L}{1 - L_0\,\eta} \leq \frac{8\ \overline{L}\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} \quad (\eta \neq 0)$$

or

$$\eta \leq \frac{1}{L_0}\left(1 - \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{8\ \overline{L}}\right) \quad (L_0 \neq 0,\ L \neq 0).$$

But by (2.16)

$$\eta \leq \frac{4}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}}.$$

It then suffices to show

$$\frac{4}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}} \leq \frac{1}{L_0}\left(1 - \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{8\ \overline{L}}\right)$$

or

$$\frac{L + \sqrt{L^2 + 8\ L_0\ L}}{8\ \overline{L}} \leq 1 - \frac{4\ L_0}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}}$$

or

$$\frac{L + \sqrt{L^2 + 8\ L_0\ L}}{8\ \overline{L}} \leq \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}},$$

which is true as equality by (2.17).

Let us assume (2.34) holds for all integers smaller or equal to k. We shall show (2.34) holds for k replaced by $k+1$.

Using (2.18), and the induction hypothesis, we have in turn

$$\begin{aligned}
t_{k+2} - t_{k+1} &= \frac{L}{2\,(1 - L_0\, t_{k+1})}\,(t_{k+1} - t_k)^2 \\
&\leq \frac{L}{2\,(1 - L_0\, t_{k+1})}\left(\left(\frac{\delta}{2}\right)^k (2\, q_0)^{2^k - 1}\,\eta\right)^2 \\
&\leq \frac{L}{2\,(1 - L_0\, t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\, q_0)^{-1}\,\eta\right)\left(\left(\frac{\delta}{2}\right)^{k+1} (2\, q_0)^{2^{k+1} - 1}\,\eta\right) \\
&\leq \left(\frac{\delta}{2}\right)^{k+1} (2\, q_0)^{2^{k+1} - 1}\,\eta,
\end{aligned}$$

since,

$$\frac{L}{2\,(1 - L_0\, t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\, q_0)^{-1}\,\eta\right) \leq 1, \qquad (k \geq 1). \tag{2.35}$$

Indeed, we can show instead of (2.35):

$$t_{k+1} \leq \frac{1}{L_0}\left(1 - \left(\frac{\delta}{2}\right)^{k-1} \frac{L}{4\,\overline{L}}\right),$$

which is true by (2.33), since, in view of (2.22), and the induction hypothesis:

$$\begin{aligned}
t_{k+1} &\leq t_k + \frac{\delta}{2}\,(t_k - t_{k-1}) \\
&\leq t_1 + \frac{\delta}{2}\,(t_1 - t_0) + \cdots + \frac{\delta}{2}\,(t_k - t_{k-1}) \\
&\leq \eta + \left(\frac{\delta}{2}\right)\eta + \cdots + \left(\frac{\delta}{2}\right)^k \eta \\
&= \frac{1 - \left(\frac{\delta}{2}\right)^{k+1}}{1 - \frac{\delta}{2}}\,\eta \\
&\leq \frac{1}{L_0}\left(1 - \left(\frac{\delta}{2}\right)^{k-1} \frac{L}{4\,\overline{L}}\right).
\end{aligned}$$

That completes the induction for estimate (2.23).

In view of (2.23), we obtain in turn for $2\, q_0 < 1$, and $j \geq k$:

$$\begin{aligned} t_{j+1} - t_k &= (t_{j+1} - t_j) + (t_j - t_{j-1}) + \cdots + (t_{k+1} - t_k) \\ &\leq \left(\left(\frac{\delta}{2}\right)^j (2\, q_0)^{2^j - 1} + \left(\frac{\delta}{2}\right)^{j-1} (2\, q_0)^{2^{j-1}-1} + \cdots + \left(\frac{\delta}{2}\right)^k (2\, q_0)^{2^k - 1} \right) \eta \\ &\leq \left(1 + (2\, q_0)^{2^k} + \left((2\, q_0)^{2^k}\right)^2 + \cdots \right) \left(\frac{\delta}{2}\right)^k (2\, q_0)^{2^k - 1}\, \eta \\ &= \left(\frac{\delta}{2}\right)^k \frac{(2\, q_0)^{2^k - 1}\, \eta}{1 - (2\, q_0)^{2^k}}. \end{aligned} \tag{2.36}$$

Estimate (2.24) follows from (2.36) by letting $j \longrightarrow \infty$.
That completes the proof of Lemma 2. □

3 Applications and examples

As a first application, we show how to locate a root of a polynomial f_n $(n \geq 2)$, using, say e.g. s_{n-1}, and s_∞.

Application 3. *Let $a{=}b{=}1$, $c{=}-3$, and $n{=}2$. We obtain using (2.1)–(2.3), and (2.10):*

$$f_1(s){=}s - 1, \qquad f_2(s){=}s^2 + 2\, s - 2,$$

$$s_1{=}1, \qquad s_\infty{=}\frac{2}{3}, \qquad d{=}.618033989.$$

Conditions (2.4), and (2.5) become:

$$.618033989 < \frac{2}{3},$$

and

$$-1 < 0.$$

Hence, the conclusions of Theorem 1 hold. In particular, we know $s_2 \in (s_\infty, s_1)$. Actual direct computation justifies the theoretical claim, since

$$s_2{=}\sqrt{3} - 1{=}.732050808 \in \left(\frac{2}{3}, 1\right).$$

Application 4. *As a second application, we show how to use Theorem 1 to derive sufficient convergence conditions for scalar majorizing sequences of certain popular iterative methods such that as Newton's method:*

$$x_{n+1}{=}x_n - F'(x_n)^{-1}\, F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{3.1}$$

where, F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Let us consider the famous Kantorovich hypotheses for solving nonlinear equations [5]:

($\mathcal{K}$):
$x_0 \in \mathcal{D}$, *with* $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$,

$$\parallel F'(x_0)^{-1}\ F(x_0) \parallel \leq \eta,$$

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(y)) \parallel \leq L\ \ \parallel x - y \parallel \quad \text{for all} \quad x, y \in \mathcal{D},$$

$$q_K = L\ \eta \leq \frac{1}{2}, \tag{3.2}$$

$$\overline{U}(x_0, r) = \{x \in \mathcal{X}\ : \parallel x - x_0 \parallel \leq r\} \subseteq \mathcal{D},$$

for

$$r = \frac{1 - \sqrt{1 - 2\ q_K}}{\ell}.$$

Under the ($\mathcal{K}$) hypotheses, the Newton–Kantorovich method converges quadratically (if $2\ q_K < 1$) to a unique solution $x^\star \in \overline{U}(x_0, r)$ of equation $F(x)=0$.

Moreover, scalar iteration $\{v_n\}$ ($n \geq 0$), given by

$$v_0 = 0, \quad v_1 = \eta, \quad v_{n+2} = v_{n+1} + \frac{L\ (v_{n+1} - v_n)^2}{2\ (1 - L\ v_{n+1})},$$

is a majorizing sequence for $\{x_n\}$ in the sense that for all $n \geq 0$:

$$\parallel x_{n+1} - x_n \parallel \leq v_{n+1} - v_n$$

and

$$\parallel x_n - x^\star \parallel \leq r - v_n.$$

Let us consider our hypotheses:

($\mathcal{A}$):
$x_0 \in \mathcal{D}$, *with* $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$,

$$\parallel F'(x_0)^{-1}\ F(x_0) \parallel \leq \eta,$$

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(y)) \parallel \leq L\ \ \parallel x - y \parallel,$$

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \parallel \leq L_0\ \ \parallel x - x_0 \parallel \quad \text{for all} \quad x, y \in \mathcal{D},$$

$$q_0 \leq \frac{1}{2},$$

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D} \quad (\text{or } \overline{U}(x_0, t^{\star\star}) \subseteq \mathcal{D}).$$

Note that the center–Lipschitz condition is not an additional hypothesis, since, in practice the computation of L *requires that of* L_0.

The scalar iteration $\{t_n\}$ *is a finer majorizing sequence for* $\{x_n\}$ *than* $\{v_n\}$ *provided that* $L_0 < L$ *[1], [3], [5].*

Remark 5. *In general*

$$L_0 \le L$$

holds, and $\frac{L}{L_0}$ *can be arbitrarily large [1]–[4].*

Condition (2.16) coincides with the Newton–Kantorovich hypothesis (3.2).

if $L{=}L_0$. *Otherwise (2.16) is weaker than (3.2). Moreover the ratio* $2\ q_0$ *is also smaller than* $2\ q_K$.

That is, (2.16) can always replace (3.2) in the Newton–Kantorovich theorem [5].

Hence, the applicability of Newton's method has been extended.

Example 6. *Define the scalar function* F *by* $F(x){=}c_0\ x + c_1 + c_2\ \sin e^{c_3\ x}$, $x_0{=}0$, *where* c_i, $i{=}1,2,3$ *are given parameters. Then it can easily be seen that for* c_3 *large and* c_2 *sufficiently small,* $\frac{L}{L_0}$ *can be arbitrarily large. That is (2.16) may be satisfied but not (3.2).*

Example 7. *Let* $\mathcal{X}{=}\mathcal{Y}{=}\mathbb{R}^2$, *be equipped with the max–norm, and*

$$x_0{=}(1,1)^T, \quad U_0{=}\{x : \| x - x_0 \| \le 1 - \beta\}, \quad \beta \in \left[0, \frac{1}{2}\right).$$

Define function F *on* U_0 *by*

$$F(x){=}(w^3 - \beta,\ z^3 - \beta), \qquad x{=}(w,z)^T. \tag{3.3}$$

The Fréchet–derivative of operator F *is given by*

$$F'(x){=}\left[\begin{array}{cc} 3\ w^2 & 0 \\ 0 & 3\ z^2 \end{array}\right]. \tag{3.4}$$

Using our hypotheses, we get:

$$\eta{=}\frac{1}{3}\ (1-\beta), \quad L_0{=}3-\beta, \quad \textit{and} \quad L{=}2\ \ (2-\beta).$$

The Kantorovich condition (3.2) is violated, since

$$\frac{4}{3}\ (1-\beta)\ (2-\beta) > 1 \quad \text{for all} \quad \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (3.1) converges to $x^\star{=}(\sqrt[3]{\beta}, \sqrt[3]{\beta})^T$, *starting at* x_0.

However, our condition (2.16) is true for all $\beta \in I{=}\left[.450339002, \frac{1}{2}\right)$.

Hence, our results apply to solve equation (3.3) for all $\beta \in I$.

Other applications where $L_0 < L$ can be found in [1], [3].

Remark 8. *Define scalar sequence* $\{\alpha_n\}$ *by*

$$\alpha_0{=}0, \quad \alpha_1{=}\eta, \quad \alpha_{n+2}{=}\alpha_n + \frac{L_1\ (\alpha_{n+1}-\alpha_n)^2}{2\ (1-L_0\ \alpha_{n+1})} \qquad (n \geq 0), \tag{3.5}$$

where,

$$L_1{=}\begin{cases} L_0, & \text{if} \quad n{=}0 \\ L, & \text{if} \quad n > 0 \end{cases}.$$

It follows from (2.18), and (3.5) that $\{\alpha_n\}$ *converges under the hypotheses of Lemma 2. Note that, under the* $(\mathcal{K})$ *hypotheses,* $\{\alpha_n\}$ *is a finer majorizing sequence for* $\{x_n\}$ *than* $\{t_n\}$*. More precisely, we have the following estimates (if* $L_0 < L$*):*

$$\| x_{n+1} - x_n \| \leq \alpha_{n+1} - \alpha_n < t_{n+1} - t_n < v_{n+1} - v_n, \quad (n \geq 1),$$

$$\| x_n - x^\star \| \leq \alpha^\star - \alpha_n < t^\star - t_n < v^\star - v_n, \quad (n \geq 0),$$

$$\alpha_n < t_n < v_n, \quad (n \geq 2),$$

and

$$\alpha^\star \leq t^\star \leq r,$$

where,

$$\alpha^\star{=}\lim_{n \longrightarrow \infty} \alpha_n.$$

Remark 9. *Our new technique of recurrent functions can be used on other Newton–type methods [1]–[9], so we can obtain the similar improvements as in the case of Newton's method above. These advantages are generalized, since we use the (needed, and more precise than the Lipschitz) center–Lipschitz condition for the computation of the upper bounds of the norms* $\| F'(x_n)^{-1}\ F'(x_0) \|$*. This modification leads to more precise majorizing sequences, which in turn motivate the introduction of the recurrent functions.*

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Ioannis K. Argyros
Cameron University,
Department of Mathematics Sciences,
Lawton, OK 73505, USA.
e-mail: iargyros@cameron.edu

Saïd Hilout
Poitiers University,
Laboratoire de Mathématiques et Applications,
Bd. Pierre et Marie Curie, Téléport 2, B.P. 30179,
86962 Futuroscope Chasseneuil Cedex, France.
e-mail: said.hilout@math.univ–poitiers.fr