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CONTROLLABILITY RESULTS FOR SEMILINEAR FUNCTIONAL AND NEUTRAL FUNCTIONAL EVOLUTION EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper sufficient conditions are given ensuring the controllability of mild solutions defined on a bounded interval for two classes of first order semilinear functional and neutral functional differential equations involving evolution operators when the delay is infinite using the nonlinear alternative of Leray-Schauder type.

1 Introduction

Controllability of mild solutions defined on a bounded interval $J := [0, T]$ is considered, in this paper, for two classes of first order partial and neutral functional differential evolution equations with infinite delay in a real Banach space $(E, |\cdot|)$.

Firstly, in Section 3, we study the partial functional differential evolution equation with infinite delay of the form

$$y'(t) = A(t)y(t) + Cu(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J \tag{1.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.2}$$

where $f : J \times \mathcal{B} \to E$ and $\phi \in \mathcal{B}$ are given functions, the control function $u(.)$ is given in $L^2(J; E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{0 \le t \le T}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t \le T$. To study the system (1.1) – (1.2), we assume that the histories $y_t : (-\infty, 0] \to E$, $y_t(\theta) = y(t + \theta)$ belong to some abstract *phase space* $\mathcal{B}$, to be specified later. We consider in Section 4, the neutral functional differential evolution equation with infinite delay of the form

$$\frac{d}{dt}[y(t) - g(t, y_t)] = A(t)y(t) + Cu(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J \tag{1.3}$$

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$$y_0 = \phi \in \mathcal{B}, \tag{1.4}$$

where $A(\cdot)$, f, u, C and ϕ are as in problem (1.1)-(1.2) and $g : J \times \mathcal{B} \to E$ is a given function. Finally in Section 5, an example is given to demonstrate the results.

Partial functional and neutral functional differential equations arise in many areas of applied mathematics, we refer the reader to the book by Hale and Verduyn Lunel [31], Kolmanovskii and Myshkis [37] and Wu [49].

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space B plays an important role in the study of both qualitative and quantitative theory, a usual choice is a normed space satisfying suitable axioms, which was introduced by Hale and Kato [30], see also Kappel and Schappacher [36] and Schumacher [47]. For detailed discussion on this topic, we refer the reader to the book by Hino *et al.* [35], and the paper by Corduneanu and Lakshmikantham [20].

Controllability problem of linear and nonlinear systems represented by ODEs in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators (see [19, 43, 44]). More details and results can be found in the monographs [18, 21, 42, 51]. Triggiani [48] established sufficient conditions for controllability of linear and nonlinear systems in Banach space. Exact controllability of abstract semilinear equations has been studied by Lasiecka and Triggiani [39]. Quinn and Carmichael [46] have shown that the controllability problem can be converted into a fixed point problem. By means of a fixed point theorem Kwun *et al* [38] considered the controllability and observability of a class of delay Volterra systems. Fu in [25, 26] studies the controllability on a bounded interval of a class of neutral functional differential equations. Fu and Ezzinbi [27] considered the existence of mild and classical solutions for a class of neutral partial functional differential equations with nonlocal conditions, Balachandran and Dauer have considered various classes of first and second order semilinear ordinary, functional and neutral functional differential equations on Banach spaces in [10]. By means of fixed point arguments, Benchohra *et al.* have studied many classes of functional differential equations and inclusions and proposed some controllability results in [6, 12, 13, 14, 15, 16, 17]. See also the works by Gatsori [28] and Li *et al.* [40, 41, 42]. Adimy *et al* [1, 2, 3] studied partial functional and neutral functional differential equations with infinite delay. Belmekki *et al* [11] studied partial perturbed functional and neutral functional differential equations with infinite delay. Ezzinbi [23] studied the existence of mild solutions for functional partial differential equations with infinite delay. Henriquez [32] and Hernandez [33, 34] considered the existence and regularity of solutions to functional and neutral functional differential equations with unbounded delay.

Recently Baghli and Benchohra considered in [7] a class of partial functional evolution equation and in [8] a class of neutral functional evolution equations on a semiinfinite real interval and with a bounded delay. Extension of these works is given in [9] when the delay is infinite.

In this paper, we investigate the controllability of mild solutions of the previous evolution problems studied in [7, 8, 9] for the functional differential evolution problem (1.1)-(1.2) and the neutral case (1.3)-(1.4) on the finite interval J. Sufficient conditions are establish here to get the controllability of mild solutions which are fixed points of appropriate corresponding operators using the nonlinear alternative of Leray-Schauder type (see [29]).

2 Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J;E)$ be the Banach space of continuous functions with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

and $B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see Yosida [50] for instance).

Let $L^1(J;E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)| \ dt.$$

Consider the following space

$$B_T = \{y : (-\infty, T] \to E : y|_J \in C(J;E), \quad y_0 \in \mathcal{B}\},$$

where $y|_J$ is the restriction of y to J.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [30] and follow the terminology used in [35]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, T] \to E$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $y(t)$ with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

Denote $K_T = \sup\{K(t) : t \in J\}$ and $M_T = \sup\{M(t) : t \in J\}$.

(A_2) For the function $y(.)$ in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Remark 1.

1. *Condition (ii) in (A_1) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.*
2. *Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.*
3. *From the equivalence of (ii), we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$. This implies necessarily that $\phi(0) = \psi(0)$.*

Hereafter are some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [35].

Example 2. *Let the spaces*

BC *the space of bounded continuous functions defined from* $(-\infty, 0]$ *to* E;

BUC *the space of bounded uniformly continuous funct. defined from* $(-\infty, 0]$ *to* E;

$$C^\infty := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\right\};$$

$$C^0 := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\right\},$$ *endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 3. *Let g be a positive continuous function on $(-\infty, 0]$. We define :*

$$C_g := \left\{\phi \in C((-\infty, 0]; E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\};$$

$$C_g^0 := \left\{ \phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup \left\{ \frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0 \right\}.$$

We consider the following condition on the function g.

(g_1) *For all $a > 0$,* $\sup_{0 \leq t \leq a} \sup \left\{ \frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \leq -t \right\} < \infty.$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). They satisfy conditions (A_1) and (A_2) if g_1 holds.

Example 4. *For any real constant γ, we define the functional space C_γ by*

$$C_\gamma := \left\{ \phi \in C((-\infty, 0]; E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E \right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \leq 0\}.$$

Then in the space C_γ the axioms (A_1)-(A_3) are satisfied.

Definition 5. *A function $f : J \times \mathcal{B} \to E$ is said to be an L^1-Carathéodory function if it satisfies :*

(i) for each $t \in J$ the function $f(t, .) : \mathcal{B} \to E$ is continuous ;

(ii) for each $y \in \mathcal{B}$ the function $f(., y) : J \to E$ is measurable ;

(iii) for every positive integer k there exists $h_k \in L^1(J; \mathbb{R}^+)$ such that

$$|f(t, y)| \leq h_k(t) \quad \text{for all } \|y\|_{\mathcal{B}} \leq k \quad \text{and almost each } t \in J.$$

In what follows, we assume that $\{A(t)\}_{\,t\geq 0}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 6. *A family of bounded linear operators $\{U(t, s)\}_{(t,s)\in\Delta} : U(t, s) : E \to E$ for $(t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leq s \leq t \leq T\}$ is called an evolution system if the following properties are satisfied :*

1. $U(t, t) = I$ where I is the identity operator in E,

2. *$U(t,s)\,U(s,\tau)=U(t,\tau)$ for $0\le\tau\le s\le t\le T$,*

3. *$U(t,s)\in B(E)$ the space of bounded linear operators on E, where for every $(t,s)\in\Delta$ and for each $y\in E$, the mapping $(t,s)\to U(t,s)\,y$ is continuous.*

More details on evolution systems and their properties could be found on the books of Ahmed [4], Engel and Nagel [22] and Pazy [45].

Our results will be based on the following well known nonlinear alternative of Leray-Schauder type.

Theorem 7. *(Nonlinear Alternative of Leray-Schauder Type, [29]). Let X be a Banach space, Y a closed, convex subset of E, $\mathcal{U}$ an open subset of Y and $0\in X$. Suppose that $N:\overline{\mathcal{U}}\to Y$ is a continuous, compact map. Then either,*

$(C1)$ *N has a fixed point in $\overline{\mathcal{U}}$; or*

$(C2)$ *There exists $\lambda\in(0,1)$ and $x\in\partial\mathcal{U}$ (the boundary of $\mathcal{U}$ in Y) with $x=\lambda\,N(x)$.*

3 Semilinear Evolution Equations

Before stating and proving the main result, we give first the definition of mild solution of problem (1.1)-(1.2).

Definition 8. *We say that the function $y(\cdot):\mathbb{R}\to E$ is a mild solution of (1.1)-(1.2) if $y(t)=\phi(t)$ for all $t\in(-\infty,0]$ and y satisfies the following integral equation*

$$y(t)=U(t,0)\phi(0)+\int_0^t U(t,s)Cu(s)ds+\int_0^t U(t,s)f(s,y_s)ds \quad \text{for each } t\in J. \tag{3.1}$$

Definition 9. *The problem (1.1)-(1.2) is said to be controllable on the interval J if for every initial function $\phi\in\mathcal{B}$ and $y_1\in E$ there exists a control $u\in L^2(J;E)$ such that the mild solution $y(\cdot)$ of (1.1)-(1.2) satisfies $y(T)=y_1$.*

We will need to introduce the following hypotheses which are assumed hereafter:

$(H1)$ $U(t,s)$ is compact for $t-s>0$ and there exists a constant $\widehat{M}\ge 1$ such that :

$$\|U(t,s)\|_{B(E)}\le\widehat{M}\quad\text{for every } (t,s)\in\Delta.$$

$(H2)$ There exists a function $p\in L^1(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\to(0,\infty)$ and such that :

$$|f(t,u)|\le p(t)\;\psi(\|u\|_{\mathcal{B}})\text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

$(H3)$ The linear operator $W : L^2(J; E) \to E$ is defined by

$$Wu = \int_0^T U(T,s)Cu(s)ds,$$

has an induced invertible operator $\tilde{W}^{-1}$ which takes values in $L^2(J;E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that :

$$\|C\| \leq \widetilde{M} \quad \text{and} \quad \|\tilde{W}^{-1}\| \leq \widetilde{M}_1.$$

Remark 10. *For the construction of W and $\tilde{W}^{-1}$ see the paper by Carmichael and Quinn [46].*

Theorem 11. *Suppose that hypotheses $(H1)$-$(H3)$ are satisfied and moreover there exists a constant $M_\star > 0$ such that*

$$\frac{M_\star}{\beta + K_T\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1T + 1\right)\ \psi(M_\star)\ \|p\|_{L^1}} > 1, \tag{3.2}$$

with

$$\beta = \beta(\phi, y_1) = \left(K_T\widehat{M}H\left[1 + \widehat{M}\widetilde{M}\widetilde{M}_1T\right] + M_T\right)\|\phi\|_{\mathcal{B}} + K_T\widehat{M}\widetilde{M}\widetilde{M}_1T|y_1|.$$

Then the problem (1.1)-(1.2) is controllable on $(-\infty, T]$.

Proof. Transform the problem (1.1)-(1.2) into a fixed-point problem. Consider the operator $N : B_T \to B_T$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ \phi(0) + \int_0^t U(t,s)\ C\ u_y(s)\ ds & \\ + \int_0^t U(t,s)\ f(s,y_s)\ ds, & \text{if } t \in J. \end{cases} \tag{3.3}$$

Using assumption $(H3)$, for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \tilde{W}^{-1}\left[y_1 - U(T,0)\ \phi(0) - \int_0^T U(T,s)\ f(s,y_s)\ ds\right](t).$$

Noting that, we have

$$|u_y(t)| \leq \|\tilde{W}^{-1}\|\left[|y_1| + \|U(t,0)\|_{B(E)}|\phi(0)| + \int_0^T \|U(T,\tau)\|_{B(E)}|f(\tau,y_\tau)|d\tau\right].$$

From $(H2)$, we get

$$\begin{aligned} |u_y(t)| &\leq \widetilde{M}_1 \left[|y_1| + \widehat{M} H \|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T |f(\tau, y_\tau)| d\tau \right] \\ &\leq \widetilde{M}_1 \left[|y_1| + \widehat{M} H \|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(\tau)\ \psi(\|y_\tau\|_{\mathcal{B}})\ d\tau \right]. \end{aligned} \tag{3.4}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1.1)-(1.2).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_T$, set

$$y(t) = z(t) + x(t). \tag{3.5}$$

It is obvious that y satisfies (3.1) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s)\ C\ u_z(s)\ ds + \int_0^t U(t,s)\ f(s, z_s + x_s)\ ds \quad \text{for } t \in J.$$

Let

$$B_T^0 = \{ z \in B_T : z_0 = 0 \}.$$

For any $z \in B_T^0$ we have

$$\|z\|_T = \sup\{\ |z(t)|\ :\ t \in J\ \} + \|z_0\|_{\mathcal{B}} = \sup\{\ |z(t)|\ :\ t \in J\ \}.$$

Thus $(B_T^0, \|\cdot\|_T)$ is a Banach space.

Define the operator $F : B_T^0 \to B_T^0$ by :

$$F(z)(t) = \int_0^t U(t,s)\ C\ u_z(s)\ ds + \int_0^t U(t,s)\ f(s, z_s + x_s)\ ds \quad \text{for } t \in J. \tag{3.6}$$

Obviously the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point. The proof will be given in several steps.

Let us first show that the operator F is continuous and compact.

Step 1 : F is continuous.

Let $(z_n)_n$ be a sequence in B_T^0 such that $z_n \to z$ in B_T^0. Then using (3.4), we get

$$\begin{aligned} |F(z_n)(t) - F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \, \|C\| \, |u_{z_n}(s) - u_z(s)| \, ds \\ &+ \int_0^t \|U(t,s)\|_{B(E)} \, |f(s, z_{ns} + x_s) - f(s, z_s + x_s)| \, ds \\ &\leq \widehat{M}\widetilde{M} \int_0^t \widetilde{M}_1 \widehat{M} \int_0^T |f(\tau, z_{n\tau} + x_\tau) - f(\tau, z_\tau + x_\tau)| \, d\tau \, ds \\ &+ \widehat{M} \int_0^t |f(s, z_{ns} + x_s) - f(s, z_s + x_s)| \, ds \\ &\leq \widehat{M}^2 \widetilde{M}\widetilde{M}_1 T \int_0^T |f(s, z_{ns} + x_s) - f(s, z_s + x_s)| \, ds \\ &+ \widehat{M} \int_0^T |f(s, z_{ns} + x_s) - f(s, z_s + x_s)| \, ds. \end{aligned}$$

Since f is L^1-Carathéodory, we obtain by the Lebesgue dominated convergence theorem

$$|F(z_n)(t) - F(z)(t)| \to 0 \quad \text{as} \quad n \to +\infty.$$

Thus F is continuous.

Step 2 : F maps bounded sets of B_T^0 into bounded sets. For any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B_T^0 : \|z\|_T \leq d\}$ one has $\|F(z)\|_T \leq \ell$.

Let $z \in B_d$. By (3.4), we have for each $t \in J$

$$\begin{aligned} |F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \, \|C\| \, |u_z(s)| \, ds + \int_0^t \|U(t,s)\|_{B(E)} \, |f(s, z_s + x_s)| \, ds \\ &\leq \widehat{M}\widetilde{M} \int_0^t |u_z(s)| \, ds + \widehat{M} \int_0^t |f(s, z_s + x_s)| \, ds \\ &\leq \widehat{M}\widetilde{M} \int_0^t \widetilde{M}_1 \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(\tau) \, \psi(\|z_\tau + x_\tau\|_{\mathcal{B}}) \, d\tau \right] \, ds \\ &+ \widehat{M} \int_0^t |f(s, z_s + x_s)| \, ds \\ &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(s) \, \psi(\|z_s + x_s\|_{\mathcal{B}}) \, ds \right] \\ &+ \widehat{M} \int_0^t p(s) \, \psi(\|z_s + x_s\|_{\mathcal{B}}) \, ds \\ &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} \right] \\ &+ \widehat{M} \left(\widehat{M}\widetilde{M}\widetilde{M}_1 T + 1 \right) \int_0^T p(s) \, \psi(\|z_s + x_s)\|_{\mathcal{B}}) \, ds. \end{aligned}$$

Using the assumption (A_1), we get

$$\begin{aligned}
\|z_s + x_s\|_{\mathcal{B}} &\le \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\
&\le K(s)|z(s)| + M(s)\|z_0\|_{\mathcal{B}} + K(s)|x(s)| + M(s)\|x_0\|_{\mathcal{B}} \\
&\le K_T|z(s)| + K_T\|U(s,0)\|_{B(E)}|\phi(0)| + M_T\|\phi\|_{\mathcal{B}} \\
&\le K_T|z(s)| + K_T\widehat{M}H\|\phi\|_{\mathcal{B}} + M_T\|\phi\|_{\mathcal{B}} \\
&\le K_T|z(s)| + (K_T\widehat{M}H + M_T)\|\phi\|_{\mathcal{B}}.
\end{aligned}$$

Set $\alpha := (K_T\widehat{M}H + M_T)\|\phi\|_{\mathcal{B}}$ and $\delta := K_T d + \alpha$. Then,

$$\|z_s + x_s\|_{\mathcal{B}} \le K_T|z(s)| + \alpha \le \delta. \tag{3.7}$$

Using the nondecreasing character of ψ, we get for each $t \in J$

$$|F(z)(t)| \le \widehat{M}\widetilde{M}\widetilde{M}_1 T\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 T + 1\right)\ \psi(\delta)\ \|p\|_{L^1} := \ell.$$

Thus there exists a positive number ℓ such that

$$\|F(z)\|_T \le \ell.$$

Hence $F(B_d) \subset B_d$.

Step 3 : F maps bounded sets into equicontinuous sets of B_T^0. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned}
|F(z)(\tau_2) - F(z)(\tau_1)| \le\ & \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)}\ \|C\|\ |u_z(s)|\ ds \\
&+ \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)}\ |f(s, z_s + x_s)|\ ds \\
&+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)}\ \|C\|\ |u_z(s)|\ ds \\
&+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)}\ |f(s, z_s + x_s)|\ ds.
\end{aligned}$$

By the inequalities (3.4) and (3.7) and using the nondecreasing character of ψ, we get

$$|u_z(t)| \le \widetilde{M}_1\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M}\ \psi(\delta)\ \|p\|_{L^1}\right] := \omega. \tag{3.8}$$

Then

$$\begin{aligned} |F(z)(\tau_2)-F(z)(\tau_1)| &\leq \|C\|_{B(E)}\ \omega \int_0^{\tau_1} \|U(\tau_2,s)-U(\tau_1,s)\|_{B(E)}\ ds \\ &+ \psi(\delta)\int_0^{\tau_1} \|U(\tau_2,s)-U(\tau_1,s)\|_{B(E)}\ p(s)\ ds \\ &+ \|C\|_{B(E)}\ \omega \int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)}\ ds \\ &+ \psi(\delta)\int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)}\ p(s)\ ds. \end{aligned}$$

Noting that $|F(z)(\tau_2)-F(z)(\tau_1)|$ tends to zero as $\tau_2-\tau_1\to 0$ independently of $z\in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2-\tau_1\to 0$. Since $U(t,s)$ is a strongly continuous operator and the compactness of $U(t,s)$ for $t>s$ implies the continuity in the uniform operator topology (see [5, 45]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.
Let $t\in J$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $z\in B_d$ we define

$$\begin{aligned} F_\epsilon(z)(t) &= U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)\ C\ u_z(s)\ ds \\ &+ U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)\ f(s,z_s+x_s)\ ds. \end{aligned}$$

Since $U(t,s)$ is a compact operator, the set $Z_\epsilon(t)=\{F_\epsilon(z)(t):\ z\in B_d\}$ is precompact in E for every ϵ sufficiently small, $0<\epsilon<t$. Moreover using (3.8), we have

$$\begin{aligned} |F(z)(t)-F_\epsilon(z)(t)| &\leq \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ \|C\|\ |u_z(s)|\ ds \\ &+ \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ |f(s,z_s+x_s)|\ ds \\ &\leq \|C\|_{B(E)}\ \omega \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ ds \\ &+ \psi(\delta)\int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ p(s)\ ds. \end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{F(z)(t): z\in B_d\}$. Hence the set $\{F(z)(t): z\in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4 : For applying Theorem 7, we must check $(C2)$: i.e. it remains to show that the set

$$\mathcal{E} = \left\{ z \in B_T^0 : \ z = \lambda \, F(z) \ \text{ for some } 0 < \lambda < 1 \right\}$$

is bounded.

Let $z \in \mathcal{E}$. By (3.4), we have for each $t \in J$

$$\begin{aligned}
|z(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \, \|C\| \, |u_z(s)| \, ds + \int_0^t \|U(t,s)\|_{B(E)} \, |f(s, z_s + x_s)| \, ds \\
&\leq \widehat{M}\widetilde{M} \int_0^t \widetilde{M}_1 \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(\tau) \, \psi(\|z_\tau + x_\tau\|_{\mathcal{B}}) \, d\tau \right] ds \\
&+ \widehat{M} \int_0^t p(s) \, \psi(\|z_s + x_s\|_{\mathcal{B}}) \, ds \\
&\leq \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(s) \, \psi(\|z_s + x_s\|_{\mathcal{B}}) \, ds \right] \\
&+ \widehat{M} \int_0^t p(s) \, \psi(\|z_s + x_s\|_{\mathcal{B}}) \, ds.
\end{aligned}$$

Using the first inequality in (3.7) and the nondecreasing character of ψ, we get

$$\begin{aligned}
|z(t)| &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^T p(s) \, \psi(K_T|z(s)| + \alpha) \, ds \right] \\
&+ \widehat{M} \int_0^t p(s) \, \psi(K_T|z(s)| + \alpha) \, ds.
\end{aligned}$$

Then

$$\begin{aligned}
K_T|z(t)| + \alpha &\leq \alpha + K_T \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} \right. \\
&+ \left. \widehat{M} \int_0^T p(s) \, \psi(K_T|z(s)| + \alpha) \, ds \right] \\
&+ K_T \widehat{M} \int_0^t p(s) \, \psi(K_T|z(s)| + \alpha) \, ds.
\end{aligned}$$

Set $\beta := \alpha + K_T \widehat{M}\widetilde{M}\widetilde{M}_1 T \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} \right]$, thus

$$\begin{aligned}
K_T|z(t)| + \alpha &\leq \beta + K_T \widehat{M}^2 \widetilde{M}\widetilde{M}_1 T \int_0^T p(s) \, \psi(K_T|z(s)| + \alpha) \, ds \\
&+ K_T \widehat{M} \int_0^t p(s) \, \psi(K_T|z(s)| + \alpha) \, ds.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup \left\{ \, K_T|z(s)| + \alpha \ : \ 0 \leq s \leq t \, \right\}, \quad 0 \leq t \leq T.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_T|z(t^\star)| + \alpha$. If $t^\star \in J$, by the previous inequality, we have for $t \in J$

$$\mu(t) \leq \beta + K_T\widehat{M}^2\widetilde{M}\widetilde{M}_1T \int_0^T p(s)\ \psi(\mu(s))\ ds + K_T\widehat{M} \int_0^t p(s)\ \psi(\mu(s))\ ds.$$

Then, we have

$$\mu(t) \leq \beta + K_T\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1T + 1\right) \int_0^T p(s)\ \psi(\mu(s))\ ds.$$

Consequently,

$$\frac{\|z\|_T}{\beta + K_T\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1T + 1\right)\ \psi(\|z\|_T)\ \|p\|_{L^1}} \leq 1.$$

Then by (3.2), there exists a constant $M_\star$ such that $\|z\|_T \neq M_\star$. Set

$$\mathcal{U} = \{\ z \in B_T^0\ :\ \|z\|_T\ \leq\ M_\star + 1\ \}.$$

Clearly, $\mathcal{U}$ is a closed subset of B_T^0. From the choice of $\mathcal{U}$ there is no $z \in \partial\mathcal{U}$ such that $z = \lambda\ F(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 7 does not hold. As a consequence of the nonlinear alternative of Leray-Schauder type ([29]), we deduce that $(C1)$ holds : i.e. the operator F has a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, T]$ is a fixed point of the operator N, which is a mild solution of the problem (1.1)-(1.2). Thus the evolution system (1.1)-(1.2) is controllable on $(-\infty, T]$.

4 Semilinear Neutral Evolution Equations

In this section, we give controllability result for the neutral functional differential evolution problem with infinite delay (1.3)-(1.4). Firstly we define the mild solution.

Definition 12. *We say that the function $y(\cdot) : (-\infty, T] \to E$ is a mild solution of (1.3)-(1.4) if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$ and y satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t,y_t) + \int_0^t U(t,s)A(s)g(s,y_s)ds \\ &+ \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s,y_s)\ ds \quad \text{for each } t \in J. \end{aligned} \tag{4.1}$$

Definition 13. *The neutral evolution problem (1.3)-(1.4) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $y_1 \in E$ there exists a control $u \in L^2(J;E)$ such that the mild solution $y(\cdot)$ of (1.3)-(1.4) satisfies $y(T) = y_1$.*

We consider the hypotheses (H1)-(H3) and we will need the following assumptions :

$(H4)$ There exists a constant $\overline{M}_0 > 0$ such that :

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all} \quad t \in J.$$

$(H5)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_T}$ such that :

$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H6)$ There exists a constant $L_\star > 0$ such that :

$$|A(t)\ g(s,\phi) - A(t)\ g(\overline{s},\overline{\phi})| \leq L_\star\ (|s-\overline{s}| + \|\phi - \overline{\phi}\|_{\mathcal{B}})$$

for all $0 \leq t, s, \overline{s} \leq T$ and $\phi, \overline{\phi} \in \mathcal{B}$.

$(H7)$ The function g is completely continuous and for any bounded set $Q \subseteq B_T$ the set $\{t \to g(t,x_t) : x \in Q\}$ is equicontinuous in $C(J;E)$.

Theorem 14. *Suppose that hypotheses (H1)-(H7) are satisfied and moreover there exists a constant $M^{\star\star} > 0$ with*

$$\frac{M^{\star\star}}{\widetilde{\beta} + \widehat{M} K_T \dfrac{\widehat{M}\widetilde{M}\widetilde{M}_1 T + 1}{1 - \overline{M}_0 L K_T}\ [M^{\star\star} + \psi(M^{\star\star})]\ \|\zeta\|_{L^1}} > 1, \tag{4.2}$$

where $\zeta(t) = \max(L; p(t))$ and

$$\begin{aligned}
\widetilde{\beta} &= \widetilde{\beta}(\phi, y_1) = (K_T \widehat{M} H + M_T)\|\phi\|_{\mathcal{B}} + \frac{K_T}{1 - \overline{M}_0 L K_T} \times \\
&\quad \times \Big\{ \Big[\Big(\widehat{M} + 1\Big) \overline{M}_0 L + \widehat{M} L T \Big] \Big(1 + \widehat{M}\widetilde{M}\widetilde{M}_1 T\Big) \\
&\quad + \widehat{M} \Big[\overline{M}_0 L \Big(1 + \widehat{M}\widetilde{M}\widetilde{M}_1 T\Big) + \widetilde{M}\widetilde{M}_1 T \Big(\widehat{M} H + \overline{M}_0 L M_T\Big) \Big] \|\phi\|_{\mathcal{B}} \\
&\quad + \overline{M}_0 L (K_T \widehat{M} H + M_T)\|\phi\|_{\mathcal{B}} + \widehat{M}\widetilde{M}\widetilde{M}_1 T \left(1 + \overline{M}_0 L K_T\right) |y_1| \Big\}
\end{aligned}$$

then the neutral evolution problem (1.3)-(1.4) is controllable on $(-\infty, T]$.

Proof. Consider the operator $\widetilde{N} : B_T \to B_T$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t,y_t) & \\ + \displaystyle\int_0^t U(t,s) A(s) g(s,y_s) ds & \\ + \displaystyle\int_0^t U(t,s) C u_y(s) ds + \int_0^t U(t,s) f(s,y_s) ds, & \text{if } t \in J. \end{cases} \tag{4.3}$$

Using assumption $(H3)$, for arbitrary function $y(\cdot)$, we define the control

$$\begin{aligned} u_y(t) &= \tilde{W}^{-1}\left[y_1 - U(T,0)\left(\phi(0) - g(0,\phi)\right) - g(T,y_T)\right. \\ &- \left.\int_0^T U(T,s)A(s)g(s,y_s)ds - \int_0^T U(T,s)f(s,y_s)ds\right](t). \end{aligned}$$

Noting that

$$\begin{aligned} |u_y(t)| &\leq \|\tilde{W}^{-1}\| \left[\ |y_1| + \|U(t,0)\|_{B(E)}\left(|\phi(0)| + \|A^{-1}(0)\||A(0)g(0,\phi)|\right)\right. \\ &\quad + \|A^{-1}(T)\||A(T)g(T,y_T)| + \int_0^T \|U(T,\tau)\|_{B(E)}|A(\tau)g(\tau,y_\tau)|d\tau \\ &\quad \left. + \int_0^T \|U(T,\tau)\|_{B(E)}|f(\tau,y_\tau)|d\tau\right] \\ &\leq \widetilde{M}_1\left[\ |y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}}+1) + \overline{M}_0 L(\|y_T\|_{\mathcal{B}}+1)\right] \\ &+ \widetilde{M}_1\widehat{M}L\int_0^T(\|y_\tau\|_{\mathcal{B}}+1)d\tau + \widetilde{M}_1\widehat{M}\int_0^T |f(\tau,y_\tau)|d\tau. \end{aligned}$$

From $(H2)$, we get

$$\begin{aligned} |u_y(t)| &\leq \widetilde{M}_1\left[\ |y_1| + \widehat{M}\left(H + \overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M}LT\right] \\ &+ \widetilde{M}_1\overline{M}_0 L\|y_T\|_{\mathcal{B}} + \widetilde{M}_1\widehat{M}L\int_0^T \|y_\tau\|_{\mathcal{B}}\, d\tau + \widetilde{M}_1\widehat{M}\int_0^T |f(\tau,y_\tau)|\, d\tau \\ &\leq \widetilde{M}_1\left[|y_1| + \widehat{M}\left(H + \overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M}LT\right] \\ &+ \widetilde{M}_1\overline{M}_0 L\|y_T\|_{\mathcal{B}} + \widetilde{M}_1\widehat{M}L\int_0^T \|y_\tau\|_{\mathcal{B}}d\tau + \widetilde{M}_1\widehat{M}\int_0^T p(\tau)\psi(\|y_\tau\|_{\mathcal{B}})d\tau. \end{aligned} \tag{4.4}$$

We shall show that using this control the operator $\widetilde{N}$ has a fixed point $y(\cdot)$, which is a mild solution of the neutral evolution system (1.3)-(1.4).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_T$, set

$$y(t) = z(t) + x(t). \tag{4.5}$$

It is obvious that y satisfies (4.1) if and only if z satisfies $z_0 = 0$ and for $t \in J$, we get

$$\begin{aligned} z(t) &= g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s,z_s+x_s)ds \\ &+ \int_0^t U(t,s)Cu_z(s)ds + \int_0^t U(t,s)f(s,z_s+x_s)ds. \end{aligned}$$

Define the operator $\widetilde{F} : B_T^0 \to B_T^0$ by :

$$\begin{aligned}\widetilde{F}(z)(t) &= g(t, z_t + x_t) - U(t,0)\ g(0,\phi) + \int_0^t U(t,s)\ A(s)\ g(s, z_s + x_s)\ ds \\ &+ \int_0^t U(t,s)\ C\ u_z(s)\ ds + \int_0^t U(t,s)\ f(s, z_s + x_s)\ ds.\end{aligned} \tag{4.6}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F}$ has one, so it turns to prove that $\widetilde{F}$ has a fixed point. We can show as in Section 3 that the operator $\widetilde{F}$ is continuous and compact. To apply Theorem 7, we must check $(C2)$, i.e., it remains to show that the set

$$\mathcal{E} = \left\{ z \in B_0^T : \ z = \lambda\ \widetilde{F}(z) \ \text{ for some } 0 < \lambda < 1 \right\}$$

is bounded.

Let $z \in \mathcal{E}$. By $(H1)$ to $(H5)$ and (4.4), we have for each $t \in J$

$$\begin{aligned}|z(t)| &\le \|A^{-1}(t)\|\ |A(t)g(t, z_t + x_t)| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0)g(0,\phi)| \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s, z_s + x_s)|\ ds + \int_0^t \|U(t,s)\|_{B(E)}\ \|C\|\ |u_z(s)|\ ds \\
&+ \int_0^t \|U(t,s)\|_{B(E)}\ |f(s, z_s + x_s)|\ ds \\
&\le \overline{M}_0 L\left(\|z_t + x_t\|_{\mathcal{B}} + 1\right) + \widehat{M}\overline{M}_0 L\left(\|\phi\|_{\mathcal{B}} + 1\right) + \widehat{M} L \int_0^t \left(\|z_s + x_s\|_{\mathcal{B}} + 1\right)\ ds \\
&+ \widehat{M}\widetilde{M}\int_0^t \widetilde{M}_1 \Big[|y_1| + \widehat{M}\left(H + \overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M} + 1\right)\overline{M}_0 L + \widehat{M} L T \\
&+ \overline{M}_0 L \|z_T + x_T\|_{\mathcal{B}} + \widehat{M} L \int_0^T \|z_\tau + x_\tau\|_{\mathcal{B}} d\tau + \widehat{M}\int_0^T p(\tau)\psi(\|z_\tau + x_\tau\|_{\mathcal{B}}) d\tau \Big]\ ds \\
&+ \widehat{M}\int_0^t p(s)\ \psi(\|z_s + x_s\|_{\mathcal{B}})\ ds \\
&\le \left[\left(\widehat{M} + 1\right)\overline{M}_0 L + \widehat{M} L T\right]\left(1 + \widehat{M}\widetilde{M}\widetilde{M}_1 T\right) + \widehat{M}\widetilde{M}\widetilde{M}_1 T|y_1| \\
&+ \widehat{M}\left[\overline{M}_0 L\left(1 + \widehat{M}\widetilde{M}\widetilde{M}_1 T\right) + \widehat{M}\widetilde{M}\widetilde{M}_1 T H\right]\|\phi\|_{\mathcal{B}} + \widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 L T\|z_T + x_T\|_{\mathcal{B}} \\
&+ \overline{M}_0 L\|z_t + x_t\|_{\mathcal{B}} + \widehat{M} L\int_0^t \|z_s + x_s\|_{\mathcal{B}}\ ds + \widehat{M}^2\widetilde{M}\widetilde{M}_1 L T\int_0^T \|z_s + x_s\|_{\mathcal{B}}\ ds \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 T\int_0^T p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})\ ds + \widehat{M}\int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})\ ds.\end{aligned}$$

Noting that we have $\|z_T + x_T\|_{\mathcal{B}} \le K_T|y_1| + M_T\|\phi\|_{\mathcal{B}}$ and using the first inequality $\|z_t + x_t\|_{\mathcal{B}} \le K_T|z(t)| + \alpha$ in (3.7), then by the nondecreasing character of ψ, we obtain

$$
\begin{aligned}
|z(t)| \;\leq\; & \left[\left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}LT\right]\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)\\
+\; & \widehat{M}\left[\overline{M}_0 L\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)+\widetilde{M}\widetilde{M}_1 T\left(\widehat{M}H+\overline{M}_0 L M_T\right)\right]\|\phi\|_{\mathcal{B}}\\
+\; & \widehat{M}\widetilde{M}\widetilde{M}_1 T\left(1+\overline{M}_0 L K_T\right)|y_1|+\overline{M}_0 L\left(K_T|z(t)|+\alpha\right)\\
+\; & \widehat{M}L\int_0^t\left(K_T|z(s)|+\alpha\right)\,ds+\widehat{M}^2\widetilde{M}\widetilde{M}_1 LT\int_0^T\left(K_T|z(s)|+\alpha\right)\,ds\\
+\; & \widehat{M}^2\widetilde{M}\widetilde{M}_1 T\int_0^T p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds+\widehat{M}\int_0^t p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left(1-\overline{M}_0 L K_T\right)|z(t)| \;\leq\; & \left[\left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}LT\right]\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)+\overline{M}_0 L\alpha\\
+\; & \widehat{M}\left[\overline{M}_0 L\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)+\widetilde{M}\widetilde{M}_1 T\left(\widehat{M}H+\overline{M}_0 L M_T\right)\right]\|\phi\|_{\mathcal{B}}\\
+\; & \widehat{M}\widetilde{M}\widetilde{M}_1 T\left(1+\overline{M}_0 L K_T\right)|y_1|+\widehat{M}L\int_0^t\left(K_T|z(s)|+\alpha\right)\,ds\\
+\; & \widehat{M}^2\widetilde{M}\widetilde{M}_1 LT\int_0^T\left(K_T|z(s)|+\alpha\right)\,ds\\
+\; & \widehat{M}^2\widetilde{M}\widetilde{M}_1 T\int_0^T p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds\\
+\; & \widehat{M}\int_0^t p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds.
\end{aligned}
$$

Set

$$
\begin{aligned}
\widetilde{\beta} \;:=\; & \alpha+\frac{K_T}{1-\overline{M}_0 L K_T}\Big\{\left[\left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}LT\right]\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)+\overline{M}_0 L\alpha\\
& +\widehat{M}\left[\overline{M}_0 L\left(1+\widehat{M}\widetilde{M}\widetilde{M}_1 T\right)+\widetilde{M}\widetilde{M}_1 T\left(\widehat{M}H+\overline{M}_0 L M_T\right)\right]\|\phi\|_{\mathcal{B}}\\
& +\widehat{M}\widetilde{M}\widetilde{M}_1 T\left(1+\overline{M}_0 L K_T\right)|y_1|\Big\}
\end{aligned}
$$

thus

$$
\begin{aligned}
& K_T|z(t)|+\alpha\leq\widetilde{\beta}+\frac{\widehat{M}K_T}{1-\overline{M}_0 L K_T}\times\\
& \quad\times\Bigg[L\int_0^t\left(K_T|z(s)|+\alpha\right)\,ds+\widehat{M}\widetilde{M}\widetilde{M}_1 LT\int_0^T\left(K_T|z(s)|+\alpha\right)\,ds\\
& \quad+\widehat{M}\widetilde{M}\widetilde{M}_1 T\int_0^T p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds+\int_0^t p(s)\psi\left(K_T|z(s)|+\alpha\right)\,ds\Bigg].
\end{aligned}
$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ K_T|z(s)| + \alpha \ : \ 0 \le s \le t \ \}, \quad 0 \le t \le T.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_T|z(t^\star)| + \alpha$. If $t \in J$, by the previous inequality, we have for $t \in J$

$$\begin{aligned} \mu(t) \ &\le \ \widetilde{\beta} + \frac{\widehat{M}K_T}{1-\overline{M}_0LK_T}\left[L\int_0^t \mu(s)\ ds + \widehat{M}\widetilde{M}\widetilde{M}_1LT\int_0^T \mu(s)\ ds\right. \\ &+ \ \left.\widehat{M}\widetilde{M}\widetilde{M}_1T\int_0^T p(s)\psi(\mu(s))\ ds + \int_0^t p(s)\psi(\mu(s))\ ds\right]. \end{aligned}$$

Then, we have

$$\mu(t) \le \widetilde{\beta} + \widehat{M}K_T\frac{\widehat{M}\widetilde{M}\widetilde{M}_1T+1}{1-\overline{M}_0LK_T}\left[L\int_0^T \mu(s)\ ds + \int_0^T p(s)\psi(\mu(s))\ ds\right].$$

Set $\zeta(t) := \max(L; p(t))$ for $t \in J$

$$\mu(t) \le \widetilde{\beta} + \widehat{M}K_T\frac{\widehat{M}\widetilde{M}\widetilde{M}_1T+1}{1-\overline{M}_0LK_T}\ \int_0^T \zeta(s)\ \left[\mu(s) + \psi(\mu(s))\right]\ ds.$$

Consequently,

$$\frac{\|z\|_T}{\widetilde{\beta} + \widehat{M}K_T\dfrac{\widehat{M}\widetilde{M}\widetilde{M}_1T+1}{1-\overline{M}_0LK_T}\ \left[\|z\|_T + \psi(\|z\|_T)\right]\ \|\zeta\|_{L^1}} \ \le \ 1.$$

Then by (4.2), there exists a constant $M^{\star\star}$ such that $\|z\|_T \neq M^{\star\star}$. Set

$$\widetilde{\mathcal{U}} = \{ \ z \in B_T^0 \ : \ \|z\|_T \ \le \ M^{\star\star} + 1 \ \}.$$

Clearly, $\widetilde{\mathcal{U}}$ is a closed subset of B_T^0. From the choice of $\widetilde{\mathcal{U}}$ there is no $z \in \partial\widetilde{\mathcal{U}}$ such that $z = \lambda\ \widetilde{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 7 does not hold. As a consequence of the nonlinear alternative of Leray-Schauder type ([29]), we deduce that $(C1)$holds : i.e. the operator $\widetilde{F}$ has a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, T]$ is a fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem (1.3)-(1.4). Thus the evolution system (1.3)-(1.4) is controllable on $(-\infty, T]$.

5 An Example

To illustrate the previous results, we consider in this section the following model

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,\xi) - \displaystyle\int_{-\infty}^{0} T(\theta)w(t,v(t+\theta,\xi))d\theta\right] = a(t,\xi)\dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + \\ +d(\xi)u(t) + \displaystyle\int_{-\infty}^{0} P(\theta)r(t,v(t+\theta,\xi))d\theta \qquad t\in[0,T], \xi\in[0,\pi] \\ v(t,0) = \ v(t,\pi) \ = 0 \qquad t\in[0,T] \\ v(\theta,\xi) = \ v_0(\theta,\xi) \qquad -\infty < \theta \leq 0, \xi\in[0,\pi] \end{cases} \tag{5.1}$$

where $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t ; $T,P : (-\infty,0] \to \mathbb{R}$; $w,r : [0,T]\times\mathbb{R} \to \mathbb{R}$; $v_0 : (-\infty,0]\times[0,\pi] \to \mathbb{R}$ and $d:[0,\pi]\to\mathbb{R}$ are continuous functions. $u(\cdot):[0,\pi]\to\mathbb{R}$ is a given control.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain

$$D(A) = \{\ w\in E \ : \ w,\ w' \text{ are absolutely continuous},\ w''\in E,\ w(0)=w(\pi)=0\ \}$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption $(H1)$ and $(H4)$ (see [24]).

For the phase space $\mathcal{B}$, we choose the well known space $BUC(\mathbb{R}^-,E)$: the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\| = \sup_{\theta\leq 0}|\varphi(\theta)| \quad \text{for} \quad \varphi\in\mathcal{B}.$$

If we put for $\varphi\in BUC(\mathbb{R}^-,E)$ and $\xi\in[0,\pi]$

$$y(t)(\xi) = v(t,\xi),\ t\in[0,T],\ \xi\in[0,\pi],$$
$$\phi(\theta)(\xi) = v_0(\theta,\xi),\ -\infty<\theta\leq 0,\ \xi\in[0,\pi],$$
$$g(t,\varphi)(\xi) = \int_{-\infty}^{0} T(\theta)w(t,\varphi(\theta)(\xi))d\theta,\ -\infty<\theta\leq 0,\ \xi\in[0,\pi],$$

and

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta)r(t,\varphi(\theta)(\xi))d\theta,\ -\infty<\theta\leq 0,\ \xi\in[0,\pi]$$

Finally let $C\in B(\mathbb{R};E)$ be defined as

$$Cu(t)(\xi) = d(\xi)u(t),\ t\in[0,T],\ \xi\in[0,\pi].$$

Then, problem (5.1) takes the abstract neutral evolution form (1.3)-(1.4). In order to show the controllability of mild solutions of system (5.1), we suppose the following assumptions :

- w is Lipschitz with respect to its second argument. Let $lip(w)$ denotes the Lipschitz constant of w.
- There exist a continuous function $p \in L^1(J, \mathbb{R}^+)$ and a nondecreasing continuous function $\psi : [0, \infty) \to [0, \infty)$ such that

$$|r(t, u)| \leq p(t)\psi(|u|), \text{ for } t \in J, \text{ and } u \in \mathbb{R}.$$

- T and P are integrable on $(-\infty, 0]$.

By the dominated convergence theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. Moreover the mapping g is Lipschitz continuous in its second argument, in fact, we have

$$|g(t, \varphi_1) - g(t, \varphi_2)| \leq \overline{M}_0 L_\star lip(w) \int_{-\infty}^0 |T(\theta)|\, d\theta\, |\varphi_1 - \varphi_2|, \text{ for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

On the other hand, we have for $\varphi \in \mathcal{B}$ and $\xi \in [0, \pi]$

$$|f(t, \varphi)(\xi)| \leq \int_{-\infty}^0 |p(t)P(\theta)|\, \psi(|(\varphi(\theta))(\xi)|) d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$|f(t, \varphi)| \leq p(t) \int_{-\infty}^0 |P(\theta)|\, d\theta \psi(|\varphi|), \text{ for } \varphi \in \mathcal{B}.$$

Proposition 15. *Under the above assumptions, if we assume that condition (4.2) in Theorem 14 is true, $\varphi \in \mathcal{B}$, then the problem (5.1) is controllable on $(-\infty, \pi]$.*

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