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THE EFFICIENCY OF MODIFIED JACKKNIFE AND RIDGE TYPE REGRESSION ESTIMATORS: A COMPARISON

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Abstract. A common problem in multiple regression models is multicollinearity, which produces undesirable effects on the least squares estimator. To circumvent this problem, two well known estimation procedures are often suggested in the literature. They are Generalized Ridge Regression (GRR) estimation suggested by Hoerl and Kennard [8] and the Jackknifed Ridge Regression (JRR) estimation suggested by Singh et al. [13]. The GRR estimation leads to a reduction in the sampling variance, whereas, JRR leads to a reduction in the bias. In this paper, we propose a new estimator namely, Modified Jackknife Ridge Regression Estimator (MJR). It is based on the criterion that combines the ideas underlying both the GRR and JRR estimators. We have investigated standard properties of this new estimator. From a simulation study, we find that the new estimator often outperforms the LASSO, and it is superior to both GRR and JRR estimators, using the mean squared error criterion. The conditions under which the MJR estimator is better than the other two competing estimators have been investigated.

1 Introduction

One of the major consequences of multicollinearity on the ordinary least squares (OLS) method of estimation is that it produces large variances for the estimated regression coefficients. For improving the precision of the OLS estimator, two standard procedures are (i) the Generalized Ridge Regression (GRR) and (ii) the Jackknifed Ridge Regression (JRR). The method of ridge regression is one of the most widely used "ad hoc" solution to the problem of multicollinearity (Hoerl and Kennard, [8]). The principle result concerning the ridge estimator is that, it is superior to the OLS estimator in terms of sampling variance even though biased. Hoerl and Kennard [8] and later Vinod [15] examined the performance of the ridge estimator using the MSE

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criterion and showed that there always exists a ridge estimator having smaller MSE than the OLS estimator (see also Vinod and Ullah [16] in this context). Hinkley [7] proposed a Jackknife method for multiple linear regression which later extended to Jackknifed ridge regression by Singh et al. [13]. Gruber [5] compared the Hoerl and Kennard ridge regression estimator with the Jackknifed ridge regression estimator and observed that, although the use of the Jackknife procedure reduces the bias considerably, the estimators may have large variances and MSE than the ordinary ridge regression estimators in certain situations. More recently it has been shown that effect Batah et al. [2] and improving precision for Jackknifing ridge type estimation Batah and Gore [1]. In this paper, we have suggested a new estimator for the regression parameter by modifying the GRR estimator in the line of JRR estimator. Some important properties of this estimator are studied. Further, we have established the MSE superiority of the proposed estimator over both the GRR and the JRR estimators. We have also derived conditions for MSE superiority of this estimator. The paper is organized as follows: The model as well as the GRR and the JRR estimators are described in Section 2. The proposed new estimator is introduced in Section 3. The performance of this estimator vis-a-vis the GRR and the JRR estimators are studied in Section 4. Section 5 considers a small simulation study to justify the superiority of the suggested estimator. The paper ends with some conclusions in Section 6.

2 The Model, GRR and JRR Estimators

Consider a multiple linear regression model

$$Y = X\beta + \epsilon, \tag{2.1}$$

where Y is an $(n \times 1)$ vector of observations on the dependent variable, X is an $(n \times p)$ matrix of observations on p non-stochastic independent variables, β is a $(p \times 1)$ vector of parameters associated with the p regressors and ϵ is an $(n \times 1)$ vector of disturbances having mean zero and variance-covariance matrix $\sigma^2 I_n$. We assume that two or more regressors in X are closely linearly related, so that the model suffers from the problem of multicollinearity. Let Λ and T be the matrices of eigenvalues and eigenvectors of $X'X$, then $T'X'XT = \Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$, λ_i being the i-th eigenvalue of $X'X$ and $T'T = TT' = I$. The orthogonal version of the model (2.1) is

$$Y = Z\gamma + \epsilon, \tag{2.2}$$

where $Z = XT$ and $\gamma = T'\beta$, using singular value decomposition (see, Vinod and Ullah, [16, p. 5] of X. The OLS estimator of γ is given by

$$\begin{aligned}\hat{\gamma}_{(LS)} &= (Z'Z)^{-1}Z'Y \\ &= \Lambda^{-1}Z'Y.\end{aligned} \tag{2.3}$$

Since $\gamma = T'\beta$ and $T'T = I$, the OLS estimator of β is given by

$$\hat{\beta}_{(LS)} = T\hat{\gamma}_{(LS)}. \tag{2.4}$$

The GRR estimator of γ is

$$\hat{\gamma}_R(K) = (\Lambda + K)^{-1}Z'Y = (\Lambda + K)^{-1}\Lambda\hat{\gamma}_{(LS)} = (I - K(\Lambda + K)^{-1})\hat{\gamma}_{(LS)}.$$

That is,

$$\hat{\gamma}_R(K) = (I - KA^{-1})\hat{\gamma}_{(LS)}, \tag{2.5}$$

where $K = diag(k_1, k_2, \ldots, k_p)$, $k_i \geq 0$, $i = 1, 2, \ldots, p$, and $A = \Lambda + K$. Thus, the GRR estimator of β is

$$\hat{\beta}_R(K) = T\hat{\gamma}_R(K) = TA^{-1}Z'Y = (X'X + TKT')^{-1}X'Y. \tag{2.6}$$

When $k_1 = k_2 = \ldots = k_p = k$, $k > 0$ the ordinary ridge regression (ORR) estimator of γ can be written as

$$\hat{\gamma}_R(k) = (\Lambda + kI_p)^{-1}Z'Y = A_k^{-1}Z'Y, \tag{2.7}$$

where $A_k = \Lambda + kI_p$.

By jackknifing the GRR estimator (2.5), Singh et al. [13] proposed the JRR estimator:

$$\hat{\gamma}_J(K) = (I + KA^{-1})\hat{\gamma}_R(K) = (I - K^2A^{-2})\hat{\gamma}_{(LS)}. \tag{2.8}$$

When $k_1 = k_2 = \ldots = k_p = k$, $k > 0$ the ordinary Jackknifed ridge regression (OJR) estimator of γ can be written as

$$\hat{\gamma}_J(k) = (I - k^2A_k^{-2})\hat{\gamma}_{(LS)}. \tag{2.9}$$

The coordinatewise estimators in (2.3), (2.5), and (2.8) are all of the form (Gruber, [6])

$$\hat{\gamma}_i = [f(k_i)]\hat{\gamma}_{(iLS)}, \tag{2.10}$$

where $0 < f(k_i) \leq 1$ and $\hat{\gamma}_{(iLS)}$ are the individual components of $\hat{\gamma}_{(LS)}$. For (2.3)

$$f(k_i) = 1, \tag{2.11}$$

for(2.5),

$$f(k_i) = \frac{\lambda_i}{\lambda_i + k_i}, \tag{2.12}$$

and for(2.8),

$$f(k_i) = \frac{\lambda_i^2 + 2\lambda_i k_i}{(\lambda_i + k_i)^2}. \tag{2.13}$$

Obviously, the OLS estimator refers to the case where $k_i = 0$. It is easily seen from (2.5) and (2.8) that $\hat{\gamma}_R(K)$ and $\hat{\gamma}_J(K)$ are always biased estimators of γ, and the bias equals $(-KA^{-1}\gamma)$ and $(-K^2A^{-2}\gamma)$ respectively. From Gruber [5] [6, p. 203-302] it may be noted that:

Variance

$$\begin{aligned} Var(\hat{\gamma}_R(K)) &= E[(\hat{\gamma}_R(K) - E(\hat{\gamma}_R(K)))(\hat{\gamma}_R(K) - E(\hat{\gamma}_R(K)))'] \\ &= \sigma^2(I - KA^{-1})\Lambda^{-1}(I - KA^{-1})', \end{aligned} \tag{2.14}$$

where

$$MSE(\hat{\gamma}_{(LS)}) = Var(\hat{\gamma}_{(LS)}) = \sigma^2\Lambda^{-1}, \tag{2.15}$$

and

$$\begin{aligned} Var(\hat{\gamma}_J(K)) &= E[(\hat{\gamma}_J(K) - E(\hat{\gamma}_J(K)))(\hat{\gamma}_J(K) - E(\hat{\gamma}_J(K)))'] \\ &= \sigma^2(I - K^2A^{-2})\Lambda^{-1}(I - K^2A^{-2})', \end{aligned} \tag{2.16}$$

Mean Squared Error (MSE)

$$\begin{aligned} MSE(\hat{\gamma}_R(K)) &= Var(\hat{\gamma}_R(K)) + [Bias(\hat{\gamma}_R(K))][Bias(\hat{\gamma}_R(K))]' \\ &= \sigma^2(I - KA^{-1})\Lambda^{-1}(I - KA^{-1})' + KA^{-1}\gamma\gamma'A^{-1}K, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} MSE(\hat{\gamma}_J(K)) &= Var(\hat{\gamma}_J(K)) + [Bias(\hat{\gamma}_J(K))][Bias(\hat{\gamma}_J(K))]' \\ &= \sigma^2(I - K^2A^{-2})\Lambda^{-1}(I - K^2A^{-2})' + K^2A^{-2}\gamma\gamma'A^{-2}K^2. \end{aligned} \tag{2.18}$$

3 The Proposed Estimator

In this section, a new estimator of γ is proposed. The proposed estimator is designated as the Modified Jackknife Ridge Regression estimator (MJR) denoted by $\hat{\gamma}_{MJ}(K)$:

$$\hat{\gamma}_{MJ}(K) = [I - K^2A^{-2}]\hat{\gamma}_R(K) = [I - K^2A^{-2}][I - KA^{-1}]\hat{\gamma}_{(LS)}. \tag{3.1}$$

When $k_1 = k_2 = \ldots = k_p = k, \;\; k > 0$ the MJR estimator is called the Modified Ordinary Jackknife Ridge Regression estimator (MOJR) denoted by $\hat{\gamma}_{MJ}(k)$:

$$\hat{\gamma}_{MJ}(k) = [I - k^2A_k^{-2}][I - kA_k^{-1}]\hat{\gamma}_{(LS)}.$$

Obviously, $\hat{\gamma}_{iMJ}(K) = \hat{\gamma}_{(iLS)}$ when $k_i = 0$. It may be noted that the proposed estimator MJR in (3.1) is obtained as in the case of JRR estimator - but with GRR instead of OLS. Accordingly, from (2.10)

$$f(k_i) = [1 - \frac{k_i^2}{(\lambda_i + k_i)^2}][1 - \frac{k_i}{(\lambda_i + k_i)}] = \frac{\lambda_i^3 + 2\lambda_i^2 k_i}{(\lambda_i + k_i)^3}. \tag{3.2}$$

The MJR estimator is also the Bayes estimator, if each γ_i independently follow a normal prior with $E(\gamma_i) = 0$ and $Var(\gamma_i) = \frac{f(k_i)}{1-f(k_i)} \frac{\sigma^2}{\lambda_i}$ with $f(k_i)$ as given in (3.2).

The expressions for bias, variance and mean squared errors may be obtained as below:

Bias

$$Bias(\hat{\gamma}_{MJ}(K)) = E(\hat{\gamma}_{MJ}(K)) - \gamma = -K[I + KA^{-1} - KA^{-2}K]A^{-1}\gamma. \tag{3.3}$$

Variance

$$\begin{aligned} Var(\hat{\gamma}_{MJ}(K)) &= E[(\hat{\gamma}_{MJ}(K) - E(\hat{\gamma}_{MJ}(K)))(\hat{\gamma}_{MJ}(K) - E(\hat{\gamma}_{MJ}(K)))'] \\ &= \sigma^2 W \Lambda^{-1} W', \end{aligned} \tag{3.4}$$

where $W = (I - K^2A^{-2})(I - KA^{-1})$.

Mean Squared Error (MSE)

$$\begin{aligned} MSE(\hat{\gamma}_{MJ}(K)) &= Var(\hat{\gamma}_{MJ}(K)) + [Bias(\hat{\gamma}_{MJ}(K))][Bias(\hat{\gamma}_{MJ}(K))]' \\ &= \sigma^2 W \Lambda^{-1} W' + K\Phi A^{-1}\gamma\gamma' A^{-1}\Phi' K. \end{aligned} \tag{3.5}$$

where $\Phi = [I + KA^{-1} - KA^{-2}K]$.

4 The Performance of the MJR Estimator by MSE Criterion

We have already seen in the previous section that, the estimator MJR is biased and hence the appropriate criterion for gauging the performance of this estimator is MSE. Next, we compare the performance of the MJR estimator vis-a-vis the GRR and the JRR estimators by this criterion.

4.1 Comparison between the MJR and the GRR estimators

As regards the performance by the sampling variance, we have the following theorem.

Theorem 1. *Let K be a $(p \times p)$ symmetric positive definite matrix. Then the MJR estimator has smaller variance than the GRR estimator.*

Proof. From (2.14) and (3.4) it can be shown that

$$Var(\hat{\gamma}_R(K)) - Var(\hat{\gamma}_{MJ}(K)) = \sigma^2 H,$$

where

$$\begin{aligned} H &= (I - KA^{-1})\Lambda^{-1}(I - KA^{-1})'[I - (I - K^2A^{-2})(I - K^2A^{-2})] \\ &= (I - KA^{-1})\Lambda^{-1}(I - KA^{-1})'[(I - (I - K^2A^{-2}))(I + (I - K^2A^{-2}))] \\ &= (I - KA^{-1})\Lambda^{-1}(I - KA^{-1})'[K^2A^{-2}(I + (I - K^2A^{-2}))] \end{aligned} \tag{4.1}$$

and $A = \Lambda + K$. Since A^{-1} is positive definite, $(I - KA^{-1})$ and $(I - K^2A^{-2})$ are positive definite matrices. It can be concluded that the matrix $[I + (I - K^2A^{-2})]$ is positive definite so that multiplying by $[K^2A^{-2}[I + (I - K^2A^{-2})]]$ will result in a positive definite matrix. Thus, we conclude that H is positive definite whenever K $(p \times p)$ is a symmetric positive definite matrix. This completes the proof. □

Next we prove a necessary and sufficient condition for the MJR estimator to outperform the GRR estimator using the MSE criterion. The proof requires the following lemma from Groß[4, p.356].

Lemma 2. *Let A be a symmetric positive definite $p \times p$ matrix, γ an $p \times 1$ vector and α a positive number. Then $\alpha A - \gamma\gamma'$ is nonnegative definite if and only if $\gamma' A^{-1}\gamma \le \alpha$ is satisfied.*

Theorem 3. *Let K be a $(p \times p)$ symmetric positive definite matrix. Then the difference*

$$\Delta = MSE(\gamma, \hat{\gamma}_R(K)) - MSE(\gamma, \hat{\gamma}_{MJ}(K))$$

is a nonnegative definite matrix if and only if the inequality

$$\gamma'[L^{-1}(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)L^{-1}]^{-1}\gamma \le 1, \tag{4.2}$$

is satisfied with $L = K(I + KA^{-1} - KA^{-2}K)A^{-1}$. In addition, $\Delta \ne 0$ whenever $p > 1$.

Proof. From (2.17) and (3.5) we have

$$\begin{aligned} \Delta &= MSE(\gamma, \hat{\gamma}_R(K)) - MSE(\gamma, \hat{\gamma}_{MJ}(K)) \\ &= \sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K - K\Phi A^{-1}\gamma\gamma' A^{-1}\Phi' K, \end{aligned} \tag{4.3}$$

where $\Phi = (I + KA^{-1} - KA^{-2}K)$ is a positive definite matrix. We have seen that H is positive definite from Theorem 1. Therefore, the difference $\Delta = MSE(\gamma, \hat{\gamma}_R(K)) - MSE(\gamma, \hat{\gamma}_{MJ}(K))$ is a nonnegative definite if and only if $L^{-1}\Delta L^{-1}$ is nonnegative definite. The matrix $L^{-1}\Delta L^{-1}$ can be written as

$$L^{-1}\Delta L^{-1} = L^{-1}(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)L^{-1} - \gamma\gamma'. \tag{4.4}$$

Since the matrix $(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)$ is symmetric positive definite, using Lemma 2, we may conclude that $L^{-1}\Delta L^{-1}$ is nonnegative definite if and only if the inequality

$$\gamma'[L^{-1}(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)L^{-1}]^{-1}\gamma \leq 1, \tag{4.5}$$

is satisfied. Moreover, $\Delta = 0$ if and only if $L^{-1}\Delta L^{-1} = 0$, that is

$$L^{-1}(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)L^{-1} = \gamma\gamma'. \tag{4.6}$$

The rank of the left hand matrix is p, while the rank of the right hand matrix is either 0 or 1. Therefore $\Delta = 0$ cannot hold true whenever $p > 1$. This completes the proof. $\square$

For the special case $K = kI_p$, the inequality

$$\gamma'[L^{-1}(\sigma^2 H + KA^{-1}\gamma\gamma' A^{-1}K)L^{-1}]^{-1}\gamma \leq 1,$$

becomes

$$\gamma'[L_k^{-1}(\sigma^2 H_k + k^2 A_k^{-1}\gamma\gamma' A_k^{-1})L_k^{-1}]^{-1}\gamma \leq 1,$$

is satisfied, where

$$L_k = k(I + kA_k^{-1} - k^2 A_k^{-2})A_k^{-1},$$

and

$$H = (I - kA_k^{-1})\Lambda^{-1}(I - kA_k^{-1})'[k^2 A_k^{-2}(I + (I - k^2 A_k^{-2}))].$$

In addition, $\Delta \neq 0$ whenever $k > 0$ and $p > 1$.

4.2 Comparison between the MJR and the JRR estimators

Here we show that the MJR estimator outperform the JRR estimator in terms of the sampling variance.

Theorem 4. *Let K be a $(p \times p)$ symmetric positive definite matrix. Then the MJR estimator has smaller variance than the JRR estimator.*

Proof. From (2.16) and (3.4) it can be shown that $Var(\hat{\gamma}_J(K)) - Var(\hat{\gamma}_{MJ}(K)) = \sigma^2\Omega$, where

$$\Omega = (I - K^2A^{-2})\Lambda^{-1}(I - K^2A^{-2})'[KA^{-1}(I + (I - KA^{-1}))]. \tag{4.7}$$

Rewriting the arguments leads to the conclusion that Ω is positive definite whenever K is a $(p \times p)$ symmetric positive definite matrix and hence the proof.

In the following theorem, we have obtained a necessary and sufficient condition for the MJR estimator to outperform JRR estimator in terms of matrix mean square error (MSE). The proof of the theorem is similar to that of Theorem 3. $\square$

Theorem 5. *Let K be a $(p \times p)$ symmetric positive definite matrix. Then the difference*

$$\Delta = MSE(\gamma, \hat{\gamma}_J(K)) - MSE(\gamma, \hat{\gamma}_{MJ}(K))$$

is a nonnegative definite matrix if and only if the inequality

$$\gamma'[L^{-1}(\sigma^2\Omega + K^2A^{-2}\gamma\gamma'A^{-2}K^2)L^{-1}]^{-1}\gamma \leq 1, \quad (4.8)$$

is satisfied. In addition, $\Delta \neq 0$ whenever $p > 1$.

Remark 6. *It may be noted that (4.2) and (4.8) are complex functions of K, as well as they depend on unknown parameters β and σ^2. And hence, it is difficult to establish the existence of K from (4.2) and (4.8). However, it may be guaranteed that when $K = kI_p$, with $k > 0$, these conditions are trivially met. (Infact, it can be shown that when $k \to 0$, the condition fails and otherwise it holds good, including when $k \to \infty$).*

5 A Simulation Study

In this section, we present a Monte Carlo study to compare the mean of the relative mean squared error of four estimators viz., LASSO (Least Absolute Shrinkage and Selection Operator) suggested in Tibshirani [14], ORR, OJR and MOJR with OLS. All simulations were conducted using MATLAB code. Each β_j is rewritten as $\beta_i^+ - \beta_i^-$, where β_i^+ and β_i^- are nonnegative. We have used the quadratic programming module 'quadprog' in MATLAB to find the LASSO solution. The true model $Y = X\beta + \sigma\epsilon$, is considered with $\beta = (1, 0, 1)'$. Here ϵ follows a standard normal distribution $N(0, 1)$ and the explanatory variables are generated from

$$x_{ij} = (1 - \rho^2)^{\frac{1}{2}} w_{ij} + \rho w_{ip}, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, p, \quad (5.1)$$

where w_{ij} are independent standard normal random numbers and ρ^2 is the correlation between x_{ij} and $x_{ij'}$ for $j, j' < p$ and $j \neq j'$, $j, j' = 1, 2, \ldots, p$. When j or $j' = p$, the correlation will be ρ. We consider $\rho = 0.9, 0.99, 0.999$ and 0.9999. These variables are then standardized so that $X'X$ and $X'Y$ are in the correlation forms. We have simulated the data with sample sizes $n = 15, 30, 50$ and 100 and $p = 3$. The variances of the error terms are taken as $\sigma^2 = 25$ and 100. The optimal k is given by $k_{HKB} = \frac{p\sigma^2}{\beta'\beta}$ (see Hoerl et al., [9]). We replace β by $\hat{\beta}_J(k)$ and σ^2 by $\hat{\sigma}^2_{(J)}$, where $\hat{\sigma}^2_{(J)}$ is defined as

$$\hat{\sigma}^2_{(J)} = \frac{(Y - X\hat{\beta}_J(k))'(Y - X\hat{\beta}_J(k))}{n - p}.$$

Accordingly,

$$\hat{k} = \frac{p\hat{\sigma}^2_{(J)}}{\hat{\beta}_J(k)'\hat{\beta}_J(k)}.$$

Our simulation is patterned as in McDonald and Galarneau [11] and Leng et al. [10]. This pattern was also adopted by Wichern and Churchill [17], and Gibbons [3]. When computing the estimator, we first transform the original linear model to canonical form to get the estimator of γ. Then the estimator of γ is transformed back to the estimator of β. For each choice of ρ, σ^2 and n, the experiment is replicated 1000 times and obtained the average MSE:

$$MSE(\hat{\beta}_i) = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\beta}_{ij} - \beta_i)^2, \tag{5.2}$$

where $\hat{\beta}_{ij}$ denote the estimate of the i-th parameter in j-th replication and β_1, β_2 and β_3 are the true parameter values. Numerical results of the simulation are summarized in Table 1. It may be noted that the performance of MOJR is excellent in comparison with that of the other estimators for all combinations of correlation between regressors and variance of errors σ^2, except with LASSO estimator (cf. Table 1). With LASSO, the suggested MOJR estimator has a mixed performance pattern in terms of MSE.

Remark 7. *Extensive simulations were carried out to study the behavior of the MOJR in comparison with the other estimators when the correlations are low with smaller σ. We have observed that the performance of MOJR estimator is quite good in those cases. When σ is large, the MSE of MOJR estimator is still comparable with the other two estimators, even if they are on the higher side.*

Remark 8. *It may be noted that the R^2 value of the analysis is quite sensitive to the choice of σ^2. This observation was also made by Peele and Ryan [12]. In our simulation study, we have observed that the R^2 values were between* 0.55 - 0.90 *for all choices of σ^2.*

Remark 9. *Even though the MOJR estimator does not always perform better than the LASSO estimator, their MSE's are comparable. However, it may be noted that, it is possible to derive an exact analytical expression for the MSE of MOJR estimator, whereas, in the case of LASSO, it is simply not possible.*

6 Conclusion

In this article we have combined the criteria underlying the GRR and JRR estimators to obtain a new estimator for the regression coefficients of a linear regression model which suffers from the problem of multicollinearity. The proposed estimator is designated as the modified jackknife ridge regression estimator or MJR. The performance of this estimator as compared to that of the GRR and JRR estimators has been studied using the MSE criterion. The conditions have been derived for

n	ρ	σ	$\frac{LASSO}{OLS}$	$\frac{ORR}{OLS}$	$\frac{OJR}{OLS}$	$\frac{MOJR}{OLS}$
15	0.9	5	0.4128	0.4096	0.6140	0.3050
		10	0.1572	0.2283	0.3333	0.1542
	0.99	5	0.0206	0.0125	0.0566	0.0013
		10	0.0162	0.7501	0.9371	0.7154
	0.999	5	0.0070	0.0041	0.0053	0.0035
		10	0.0037	0.0102	0.0195	0.0024
	0.9999	5	0.0112	0.0053	0.0054	0.0054
		10	0.000027	0.00012	0.000092	0.00014
30	0.9	5	0.1139	0.6210	0.8995	0.5598
		10	0.0371	0.8690	0.9798	0.8538
	0.99	5	0.0525	0.1197	0.3424	0.0294
		10	0.0484	0.1069	0.3122	0.0237
	0.999	5	0.1079	0.0091	0.0126	0.0078
		10	0.0027	0.0037	0.0140	0.00009
	0.9999	5	0.00021	0.00038	0.00039	0.00038
		10	0.00020	0.00034	0.00049	0.00028
50	0.9	5	0.4802	0.3722	0.8426	0.2641
		10	0.3019	0.1034	0.1453	0.1030
	0.99	5	0.0313	0.1465	0.4505	0.0117
		10	0.0015	0.0197	0.0020	0.0192
	0.999	5	0.2744	3.1085	17.2592	0.8827
		10	0.0677	0.2824	0.3048	0.0544
	0.9999	5	0.0095	0.0364	0.0655	0.0073
		10	0.0024	0.0043	0.0075	0.0016
100	0.9	5	3.3409	0.6441	0.9926	0.6371
		10	0.5087	0.3704	0.9702	0.3437
	0.99	5	0.0991	0.5299	0.8851	0.4603
		10	0.0282	0.2171	0.6105	0.0973
	0.999	5	0.0586	0.5294	0.9132	0.4296
		10	0.0184	0.2781	0.7647	0.0967
	0.9999	5	0.0031	0.0054	0.0138	0.0012
		10	0.0018	0.0013	0.0034	0.000109

Table 1:
Ratio of MSE of estimators when $\hat{k} = \frac{p\hat{\sigma}^2_{(J)}}{\hat{\beta}_J(k)'\hat{\beta}_J(k)}$

the superiority of this estimator over the other two estimators in terms of MSE. We have established that this suggested estimator has a smaller mean square error value than the GRR and JRR estimators. Even though the MJR estimator does not always perform better than LASSO estimator, their MSE's are comparable.

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