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NON-SIMULTANEOUS BLOW-UP FOR A SEMILINEAR PARABOLIC SYSTEM WITH NONLINEAR MEMORY

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Abstract. In this note, we study the possibility of non-simultaneous blow-up for positive solutions to the following system,

$$u_t - \Delta u = u^{q_1} \int_0^t v^{p_1}(x,s)ds, \quad v_t - \Delta v = v^{q_2} \int_0^t u^{p_2}(x,s)ds.$$

Under appropriate hypotheses, we prove that u blows up while v fails to blow up if and only if $q_1 > 1$ and $p_2 < 2(q_1 - 1)$.

1 Introduction

In this note, we study blowing up solutions of the following parabolic system

$$\begin{aligned} u_t - \Delta u &= u^{q_1} \int_0^t v^{p_1}(x,s)ds, \qquad (x,t) \in \mathbb{R}^N \times (0,T), \\ v_t - \Delta v &= v^{q_2} \int_0^t u^{p_2}(x,s)ds, \qquad (x,t) \in \mathbb{R}^N \times (0,T), \end{aligned} \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where the parameters $p_i, q_i > 0$ $(i = 1,2)$ and the functions $u_0(x), v_0(x)$ are positive and bounded. As a physical motivation, the single equation

$$u_t - \Delta u = u^q \int_0^t u^p(x,s)ds \quad p,\ q > 0, \tag{1.3}$$

plays an important role in the theory of nuclear reactor kinetics (see [8] for physical motivation), which has been studied extensively (see [7], [9], [10], [15], [16], [17]).

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Recently, Du [2] studied the blow-up conditions for problem (1.1) with homogeneous Dirichlet boundary data in a bounded domain Ω with smooth boundary $\partial\Omega$. A simple modification of the proofs (similar to Remark 3), we can get that the solution (u, v) blows up in finite time T if and only if the parameters p_i, q_i verify any one of the following conditions,

$$(i)\ q_1 > 1, \quad (ii)\ q_2 > 1, \quad (iii)\ p_1 p_2 > (1 - q_1)(1 - q_2).$$

At that time T we have

$$\limsup_{t \uparrow T} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}) = \infty.$$

However, a priori there is no reason why both functions, u and v, should go to infinity simultaneously at the time T. Indeed, as we will show, for certain choices of the parameters p_i, q_i there are initial data for which one of the components of the system remains bounded while the other blows up. We denote this phenomenon as *non-simultaneous blow-up.*

The problem of studying non-simultaneous blow-up attracts a lot of authors and a large amount of papers have been appear (see for example [1], [12], [13], [14], [19]). However, it seems that there is no paper discussing the non-simultaneous blow-up conditions for equations with memory. In this paper we will characterize the range of parameters for which non-simultaneous blow-up occurs (for suitable initial data). Our results are stated as follows.

Theorem 1. *If $q_1 > 1$ and $p_2 < 2(q_1 - 1)$, then there exist initial data u_0, v_0 such that u blows up while v remains bounded.*

The condition $q_1 > 1$ guarantees blow-up for u, while the condition $p_2 < 2(q_1 - 1)$ implies that the coupling between u and v is weak enough to have non-simultaneous blow-up. Under a hypothesis on the blow-up rate for u we can prove the reciprocal.

Theorem 2. *If u blows up at time T at some point x_0 and v remains bounded up to time T, with*

$$u(x, t) \geq c(T - t)^{-1/(q_1 - 1)}, \quad |x - x_0| \leq K\sqrt{T - t}, \tag{1.4}$$

where c and K are positive constants, then $q_1 > 1$ and $p_2 < 2(q_1 - 1)$.

Remark 3. *Both theorems can be extended to the case of a bounded smooth domain Ω with homogeneous Dirichlet boundary data, provided that the blow-up set of u lies in a compact subset of Ω (see [4] for conditions that guarantee this fact). The only difference arises from an extra boundary term in the right hand side of (2.1) is the form*

$$\int_0^t \int_{\partial\Omega} \Gamma(x - y, t - \tau) \frac{\partial u}{\partial \nu} d\sigma(y) d\tau$$

where ν is the unit outer normal vector on $\partial\Omega$. As u remains bounded in a small neighborhood of $\partial\Omega$, this term is negative and bounded. Hence it does not play any role in the proofs. For simplicity we perform the proofs in $\mathbb{R}^N$ and leave the details in the case of a bounded domain to the readers.

Remark 4. *The hypothesis on the blow-up rate, (1.4), is known to hold for q_1 subcritical, i.e. $1 < q_1 < (N+2)/(N-2)_+$, both in $\mathbb{R}^N$ and in a bounded convex smooth domain (see [5], [11] and [18]). It also holds without any restriction on the exponent if $u_t \geq 0$ in the case of a bounded convex smooth domain (see [4]). However, it may fail for large q_1 in high dimensions (see [6]).*

2 Proofs Of The Theorems

Proof. This is the proof of Theorem 1.1. We choose u_0 and v_0 radial and decreasing with $|x|$. Thus u and v are radial and decreasing with $|x|$.

The idea is to fix v_0 and then choose u_0 large enough to guarantee that u blows up alone. Observe that every positive solution with u_0 large has finite time blow-up as $q_1 > 1$. Moreover, if u_0 is large with v_0 fixed then T becomes small where T is the blow-up time.

We will consider two cases, $q_2 > 1$ and $q_2 \leq 1$. In the first case, we fix $0 < v_0(x) < 1/4$. We want to use the representation formula obtained from the fundamental solution. Let $\Gamma(x,t)$ be the fundamental solution of the heat equation, then

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

As v is a solution of

$$v_t - \Delta v = v^{q_2} \int_0^t u^{p_2}(x,s)ds,$$

we have (see [3])

$$v(x,t) = \int_{\mathbb{R}^N} \Gamma(x-y,t)v_0(y)dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-\tau)v^{q_2}(y,\tau) \int_0^\tau u^{p_2}(y,s)dsdyd\tau. \tag{2.1}$$

We set $V(t) = \sup_{\mathbb{R}^N[0,t]} v$. As the initial data are radial and decreasing, u verifies

$$u(x,t) \leq C(T-t)^{-1/(q_1-1)}$$

with C independent of u_0 (see [5]).

(1) When $q_1 < 1 + p_2$, hence,

$$v^{q_2}(y,\tau) \int_0^\tau u^{p_2}(y,s)ds \leq Cv^{q_2}(y,\tau)(T-t)^{(q_1-1-p_2)/(q_1-1)}.$$

Since $V(t)$ is nondecreasing, from (2.1) we obtain

$$V(t) \le V(0) + CV^{q_2}(t)\int_0^t (T-t)^{(q_1-1-p_2)/(q_1-1)}d\tau.$$

We choose u_0 large enough in order to get T small. Since $\frac{1+p_2-q_1}{q_1-1} < 1$, the integral is smaller than $1/(2C)$ if T is small. Hence,

$$V(t) \le V(0) + \frac{1}{2}V^{q_2}(t). \tag{2.2}$$

We claim that $V(t) < 1$ for all $0 < t < T$. Suppose not and let $0 < t_0 < T$ be the first time such that $V(t_0) = 1$ (i.e $V(t) < 1$ for all $0 < t < t_0$). For $0 < t \le t_0$ we have $V^{q_2}(t) \le V(t)$. Therefore

$$\frac{1}{2}V(t_0) \le V(0) \le \frac{1}{4},$$

a contradiction with $V(t_0) = 1$.

Next, we assume $q_2 \le 1$. We choose $C \ge v_0 \ge 1$. Thus $V(t) \ge 1$. Arguing as before we obtain again (2.2). Now $V^{q_2}(t) \le V(t)$. Hence (2.2) produces

$$\frac{1}{2}V(t) \le V(0) \le C.$$

(2) When $q_1 = 1 + p_2$, we choose u_0 large enough such that $T < 1$, hence,

$$v^{q_2}(y,\tau)\int_0^\tau u^{p_2}(y,s)ds \le -Cv^{q_2}(y,\tau)\ln(T-\tau).$$

Since $V(t)$ is nondecreasing, from (2.1) we obtain,

$$\begin{aligned} V(t) &\le V(0) - CV^{q_2}(t)\int_0^t \ln(T-s)ds \\ &= V(0) + C(T-t)V^{q_2}(t)\ln(T-t) + C(t - T\ln T)V^{q_2}(t) \\ &\le V(0) + C(T - T\ln T)V^{q_2}(t) \le V(0) + \frac{1}{2}V^{q_2}(t), \end{aligned}$$

if we choose T small enough such that $T - T\ln T \le 1/(2C)$ (this inequality holds since $T - T\ln T$ is increasing in T for $0 < T < 1$ and $\lim_{T\downarrow 0} T - T\ln T = 0$). The following discussion is the same as (1).

(3) When $q_1 > 1 + p_2$, hence,

$$v^{q_2}(y,\tau)\int_0^\tau u^{p_2}(y,s)ds \le Cv^{q_2}(y,\tau)T^{(q_1-1-p_2)/(q_1-1)}.$$

Since $V(t)$ is nondecreasing, from (2.1) we obtain,

$$V(t) \le V(0) + CtV^{q_2}(t)T^{(q_1-1-p_2)/(q_1-1)}.$$

We choose u_0 large enough in order to get T small such that

$$V(t) \leq V(0) + \frac{1}{2}V^{q_2}(t).$$

The following discussion is the same as (1) and we get Theorem 1.1. □

Proof. This is the proof of Theorem 1.2. The function u is a subsolution of

$$u_t = \Delta u + Cu^{q_1}, \tag{2.3}$$

which has finite time blow-up. Solutions of (2.3) are global in time if $q_1 \leq 1$. To see this we can compare with a flat solution of (2.3), i.e. a solution of $u_t = Cu^{q_1}$ with initial data $u(x,0) = \|u_0\|_{L^\infty}$. Hence we must have $q_1 > 1$.

Next we prove that $p_2 < 2(q_1 - 1)$ (we only consider the case $q_1 \neq 1 + p_2$ since if $q_1 = 1 + p_2$, it is obvious that $p_2 < 2(q_1 - 1)$), let x_0 be the blow-up point for u. As $v(y,\tau)$ is bounded from below in $|y - x_0| \leq C\sqrt{T-t}$, from (2.1)we get

$$V(t) \geq v(x_0,t) \geq C\int_0^t \int_{\{|y-x_0|\leq C\sqrt{T-t}\}} \Gamma(x_0 - y, t-\tau)\int_0^\tau u^{p_2}(y,s)dsdyd\tau.$$

By assumption, there exists a constant c such that

$$u(x,t) \geq c(T-t)^{-1/(q_1-1)}, \qquad |x - x_0| \leq K\sqrt{T-t}.$$

With this bound for u we obtain

$$\begin{aligned} V(t) &\geq \frac{C(q_1-1)}{q_1-1-p_2}\int_0^t T^{\frac{q_1-1-p_2}{q_1-1}} - (T-\tau)^{\frac{q_1-1-p_2}{q_1-1}}\int_{\{|y-x_0|\leq C\sqrt{T-t}\}} \Gamma(x_0-y,t-\tau)dyd\tau \\ &\geq \frac{C(q_1-1)}{q_1-1-p_2}\int_0^t T^{\frac{q_1-1-p_2}{q_1-1}} - (T-\tau)^{\frac{q_1-1-p_2}{q_1-1}}\int_{\{|y-x_0|\leq C\sqrt{T-t}\}} e^{-|\omega|^2/4}d\omega d\tau \\ &\geq \frac{C(q_1-1)}{q_1-1-p_2}\int_0^t T^{\frac{q_1-1-p_2}{q_1-1}} - (T-\tau)^{\frac{q_1-1-p_2}{q_1-1}}d\tau. \end{aligned}$$

If $\frac{1+p_2-q_1}{q_1-1} \geq 1$, the last integral diverges as $t \uparrow T$. Hence, so does v. Therefore, if v remains bounded $p_2 < 2(q_1 - 1)$. The proof of Theorem 1.2 is complete. □

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