Total positivity and an inequality by Athanasiadis and Tzanaki

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Joint work with Volkmar Welker

f and h polynomials

Let Δ be a simplicial complex of dimension $d-1$ and $\sigma \in \Delta$ a face of Δ.

- The f-vector of Δ is the vector $f^{\Delta} = (f_{-1}, \ldots, f_{d-1})$ where $f_i = \#\{\sigma \in \Delta \mid \dim(\sigma) = i\}$.
- The f-polynomial of Δ is $f^{\Delta}(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i}$.
- The h-polynomial of Δ is $h^{\Delta}(x) = f^{\Delta}(x-1) = \sum_{i=0}^{d} h_i x^{d-i}$.
- The h-vector of Δ is $h^{\Delta} = (h_0, \ldots, h_d)$.

AT-inequality

Let $h^\Delta = (h_0^\Delta, \ldots, h_d^\Delta)$ be the h-vector a simplicial complex Δ of dimension $d-1$. Athanasiadis and Tzanaki study the inequalities

$$\frac{h_0^\Delta}{h_d^\Delta} \le \frac{h_1^\Delta}{h_{d-1}^\Delta} \le \cdots \le \frac{h_{d-1}^\Delta}{h_1^\Delta} \overset{(*)}{\le} \frac{h_d^\Delta}{h_0^\Delta}. \tag{1}$$

under the assumption all terms are defined.

- (1)holds for any Gorenstein* complex by the Dehn-Summerville equations $h_i = h_{d-i}$.

AT-inequality

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under the assumption all terms are defined.

Question (Athanasiadis, Tzanaki, 2021)

(1) may hold for all 2-Cohen-Macaulay simplicial complexes.

C. Athanasiadis, E. Tzanaki, Symmetric decompositions, triangulations and real-rootedness, Mathematika 2021.

Face uniform subdivision

A geometric realization $|\Delta|$ in some real vector space in which each face $\sigma \in \Delta$ is represented by a geometric simplex $|\sigma|$ of dimension $\dim(\sigma)$ such that $|\sigma| \cap |\tau| = |\sigma \cap \tau|$ for all $\sigma, \tau \in \Delta$.

A face uniform subdivision (or triangulation) of Δ is a simplicial complex $\Delta_{\mathscr{F}}$ with geometric realizations $|\Delta| = |\Delta_{\mathscr{F}}|$, such that

- each $|\sigma|$ for $\sigma \in \Delta$ is a union of $|\sigma'|$ for $\sigma' \in \Delta_{\mathscr{F}}$ and
- there are numbers f_{ij}, $0 \le i \le j \le \dim(\Delta)$ such that for any $\sigma \in \Delta$ we have $f_{ij} = \#\{\tau \in \Delta_{\mathscr{F}} \ : \ |\tau| \subseteq |\sigma|, \dim(\tau) = i\}$.

The transformation matrix of the subdivision

Proposition (Athanasiadis, 2022)

Let $\mathscr{F}$ be a face uniform triangulation in dimension $d-1$. Then there is a matrix $H_{\mathscr{F}} = (h_{ij})_{0\le i,j\le d}$ such that for any simplicial complex Δ of dimension $d-1$ we have

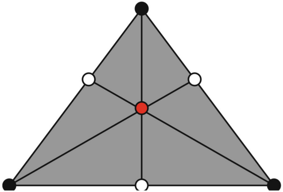
$$h^{\Delta_{\mathscr{F}}} = H_{\mathscr{F}}\, h^{\Delta}.$$

Moreover, we have $h_{ij} = h_{d-i,d-j}$ for $0 \le i, j \le d$.

C. Athanasiadis, Face numbers of uniform triangulations of simplicial complexes, Int. Math. Res. Not. 2022.

Example 1

- The barycentric subdivision of Δ: the simplicial complex of all chains in the poset of nonempty faces of Δ.



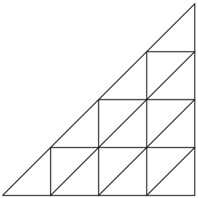
The barycentric subdivision of a triangle.

$$H_{\mathscr{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 4 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

♠ $h_{i,j} = \sharp\{\omega \in S_d : \operatorname{des}(\omega) = i, \ \omega(d) = j\}$.

Example 2

- The r^{th}-edgewise subdivision of Δ: the triangulation of a simplicial complex Δ by which every k-dimensional face of Δ is subdivided into r^k simplices of dimension k.



The 4^{th}-edgewise subdivision of a triangle.

$H_{\mathscr{F}}$ is the Amazing matrix.

♠ $$h_{i,j} = \sum_{k\geq 0} (-1)^k \binom{d+1}{k} \binom{d-1-i+(j+1-k)r}{d}.$$

The total positivity of matrix

An infinite matrix is called totally positive of order r (TP$_r$) if its minors of all orders $\leq r$ are nonnegative.

The matrix is called TP if its minors of all orders are nonnegative.

$$[a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Toeplitz matrix

Tp => Rz => Lc => U

$$[a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hankel matrix

Tp <=> SM => Lcx

The barycentric subdivision preserve the AT-inequality

Proposition

Let $\mathscr{F}$ be a face uniform subdivision such that $H_{\mathscr{F}}$ is TP_2.
Then for any simplicial complex Δ satisfying (1) *we have that $\Delta_{\mathscr{F}}$ satisfies* (1).

Theorem

Let $\mathscr{F}$ be the barycentric subdivision. Then $H_{\mathscr{F}}$ is TP_2.

Corollary

Let $\mathscr{F}$ be the barycentric subdivision. If Δ satisfies (1), *then so does $\Delta_{\mathscr{F}}$.*

The r^{th}-edgewise subdivision preserve the AT-inequality

Theorem (Diaconis, Fulman, 2009)

Let $\mathscr{F}$ be the r^{th}-edgewise subdivision. Then $H_{\mathscr{F}}$ is TP_2.

Corollary

Let $\mathscr{F}$ be the r^{th}-edgewise subdivision. If Δ satisfies (1)*, then so does $\Delta_{\mathscr{F}}$.*

P. Diaconis, J. Fulman, Carries, shuffling, and an amazing matrix, Am. Math. Mon. 2009.

Conjecture

Theorem (Mao, Wang, 2022)

Let $\mathscr{F}$ be the r^{th}-edgewise subdivision. Then $H_{\mathscr{F}}$ is TP.

Conjecture

Let $\mathscr{F}$ be the barycentric subdivision. Then $H_{\mathscr{F}}$ is TP.

J. Mao, Y. Wang, Proof of a conjecture on the total positivity of amazing matrices, Adv. in Appl. Math. 2022.

The inverse of $H_{\mathscr{F}}$

♣ The unsigned inverse of a TP matrix is still TP.

Let $\mathscr{F}$ be the barycentric subdivision.

Theorem

Let $P_j(x)$ be the generating polynomial of the j column of $H_{\mathscr{F}}^{-1}$, where $0 \le j \le d$. Then

$$P_j(x) = \frac{1}{d!} \prod_{k=1}^{d-j-1} (-kx + k + 1) \cdot \prod_{k=0}^{j-1} ((k+1)x - k).$$

Example of $H_{\mathscr{F}}$ is not TP_2

Let $\mathscr{F}$ be the subdivision of d-dimensional simplicial complexes which replaces each d-simplex by a cone over its boundary. The f_{ij} here take following form

$$f_{ij} = \begin{cases} 0 & \text{for } 0 \leq j < i < d \\ 1 & \text{for } j = i < d \\ \binom{d+1}{i} & \text{for } 0 \leq j < d = i \end{cases}$$

Recall

$$f_{ij} = \#\{\tau \in \Delta_{\mathscr{F}} \;:\; |\tau| \subseteq |\sigma|, \dim(\tau) = i\}.$$

Example of $H_{\mathscr{F}}$ is not TP_2

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Then $H_{\mathscr{F}}$ takes the following form:

$$H_{\mathscr{F}} = \begin{pmatrix} 1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

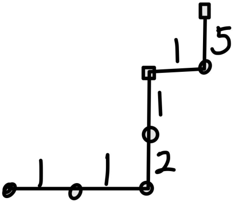
Labeled path

Let $P(d)$ be the set of d-tuples $\big((a_1,u_1),\ldots,(a_d,u_d)\big)$ in $\big(\{E,N\}\times\mathbb{N}\big)^d$, satisfying:

(L1) if $a_1 = E$ then $u_1 = 1$,

(L2) if $a_i = a_{i+1} = N$ are both vertical, or $a_i = a_{i+1} = E$ then $u_i \geq u_{i+1}$,

(L3) if $a_i \neq a_{i+1}$ then $u_i + u_{i+1} \leq i+1$.



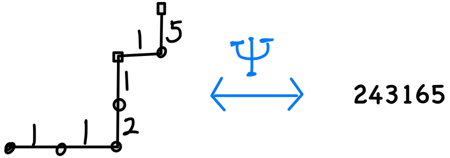
$\Psi : S_d \to P(d)$

For $\sigma = \sigma_1 \cdots \sigma_d \in S_d$, define $\Psi(\sigma) = \big((a_1, u_1), \ldots, (a_d, u_d)\big)$ where:

- $(a_1, u_1) = (E, 1)$
- for $2 \le i \le d$ we obtain (a_i, u_i) as follows.
  Let $\tau = \tau_1 \cdots \tau_i \in S_i$ such that for $1 \le \ell < j \le i$ we have

$$\tau_\ell < \tau_j \Leftrightarrow \sigma_\ell < \sigma_j.$$

  - If the position $i-1$ in σ or equivalently τ_i is a descent, let the $a_i = N$ and set $u_i = \tau_i$.
  - If the position $i-1$ in σ or equivalently τ_i is an ascent, let the $a_i = E$ and set $u_i = i+1-\tau_i$.



Bijection $\Psi : S_d \to P(d)$

Theorem (Bóna, Ehrenborg, 2000)

The map $\Psi : S_d \to P(d)$ is a bijection.

M. Bóna, R. Ehrenborg, A combinatorial proof of the log-concavity of the numbers of permutations with k runs, J. Combin. Theory Ser. A 2000.

- Let $P(d,i,j)$ be the set of labeled paths in $P(d)$ with i steps N and

$$u_d = \begin{cases} d-j & \text{if} \quad a_d = N \\ j+1 & \text{if} \quad a_d = E \end{cases}$$

- Let $A(d,i,j) = \sharp\{\sigma \in S_d : \mathrm{des}(\sigma) = i, \sigma(d) = d-j\}$.

Corollary

$\Psi : A(d,i,j) \to P(d,i,j)$ *is a bijection.*

TP_2

Proposition

For $d \geq 1$ and $0 \leq i, j \leq d-1$ there is an injection

$$\Phi : P(d,i,j+1) \times P(d,i+1,j) \to P(d,i,j) \times P(d,i+1,j+1).$$

Theorem

Let $\mathscr{F}$ be the barycentric subdivision. Then $H_{\mathscr{F}}$ is TP_2.

Thank you for your attention!