Colored shuffle compatibility and ask zeta functions

Vassilis Dionyssis Moustakas

University of Pisa, Pisa (Italy)

91ème Séminaire Lotharingien de Combinatoire

Salobrẽna (Spain), March 19, 2024

Based on joint work with

Angela Carnevale

(University of Galway)

Tobias Rossmann

(University of Galway)

Outline

- Shuffle compatibility
- Colored shuffle compatibility
- Application: Hadamard products of ask zeta functions

Shuffle compatibility

Stanley's shuffling theorem

We let

- $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$

and for $\pi \in \mathfrak{S}_n$

- $\mathrm{Des}(\pi) := \{i \in [n-1] : \pi(i) > \pi(i+1)\}$

be the descent set of π.

For two disjoint permutations π and σ of length n and m, respectively, we let

- $\pi \sqcup\!\sqcup \sigma := \{\tau \in \mathfrak{S}_{n+m} : \pi, \sigma \text{ appear as subsequences of } \tau\}$

be the set of all shuffles of π and σ.

Theorem

For two disjoint permutations π and σ of length n and m, respectively, the multiset

$$\{\mathrm{Des}(\tau) : \tau \in \pi \sqcup\!\sqcup \sigma\}$$

depends only on $\mathrm{Des}(\pi), \mathrm{Des}(\sigma), n$ and m.

Quasisymmetric functions and P-partitions

We let

- $\mathbf{x} = (x_1, x_2, \dots)$ be an infinite sequence of commuting indeterminates,
- QSym be the $\mathbb{Q}$-algebra of quasisymmetric functions

and

$$F_{n,S}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

be the fundamental quasisymmetric function corresponding to $S \subseteq [n-1]$.

Theorem

If π and σ are two disjoint permutations of length n and m, respectively, then

$$F_{n,\mathrm{Des}(\pi)}(\mathbf{x}) F_{m,\mathrm{Des}(\sigma)}(\mathbf{x}) = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} F_{n+m,\mathrm{Des}(\tau)}(\mathbf{x}).$$

Quasisymmetric functions and P-partitions

Let P be a poset on $[n]$. A P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $i <_P j$ implies $f(i) \leq f(j)$
- $i <_P j$ and $i >_{\mathbb{Z}} j$ implies $f(i) < f(j)$.

Consider weight enumerator

$$F(P;\mathbf{x}) := \sum_{\substack{f:P\to\mathbb{Z}_{>0} \\ f = P\text{-partition}}} x_{f(1)}x_{f(2)}\cdots x_{f(n)} \in \mathrm{QSym}\,.$$

Theorem

- We have
$$F(P;\mathbf{x}) = \sum_{\pi\in\mathcal{L}(P)} F(\underline{\pi};\mathbf{x}) = \sum_{\pi\in\mathcal{L}(P)} F_{n,\mathrm{Des}(\pi)}(\mathbf{x}),$$
where $\mathcal{L}(P) \subseteq \mathfrak{S}_n$ is the set of all linear extensions of P.
- If P and Q are two posets on disjoint sets, then
$$F(P;\mathbf{x})F(Q;\mathbf{x}) = F(P+Q;\mathbf{x}).$$

Shuffle algebras

A permutation statistic stat is called shuffle compatible if for any two disjoint permutations π and σ, the multiset

$$\{\mathsf{stat}(\tau) : \tau \in \pi \sqcup\!\sqcup \sigma\}$$

depends only on $\mathsf{stat}(\pi)$, $\mathsf{stat}(\sigma)$ and the lengths of π and σ.
Such statistics define an equivalence relation $\sim_{\mathsf{stat}}$ on the set of all permutations by letting

$$\pi \sim_{\mathsf{stat}} \sigma \iff \pi \text{ and } \sigma \text{ have the same length, and } \mathsf{stat}(\pi) = \mathsf{stat}(\sigma).$$

The space $\mathcal{A}_{\mathsf{stat}}$ of equivalence classes of permutations with multiplication given by

$$[\pi]_{\mathsf{stat}}[\sigma]_{\mathsf{stat}} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} [\tau]_{\mathsf{stat}}$$

is called the shuffle algebra of stat.

Shuffle algebras

Theorem (Gessel–Zhuang, '18)

The descent set statistic Des is shuffle compatible and the corresponding shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ is isomorphic to QSym via the linear map

$$[\pi]_{\mathrm{Des}} \;\mapsto\; F_{n,\mathrm{Des}(\pi)}.$$

Hadamard product and shuffle algebras

The Hadamard product of two formal power series in t is defined by

$$\left(\sum_{n\geq 0} a_n t^n\right) * \left(\sum_{n\geq 0} b_n t^n\right) := \sum_{n\geq 0} a_n b_n t^n.$$

Hadamard product and shuffle algebras

The Hadamard product of two formal power series in t is defined by

$$\left(\sum_{n\geq 0} a_n t^n\right) * \left(\sum_{n\geq 0} b_n t^n\right) := \sum_{n\geq 0} a_n b_n t^n.$$

We let

- $\mathbb{Q}[\![t*]\!]$ be the space of formal power series in t with the Hadamard product

and for $\pi \in \mathfrak{S}_n$

- $\mathrm{des}(\pi) := |\,\mathrm{Des}(\pi)|$
- $\mathrm{comaj}(\pi) := \sum_{i\in \mathrm{Des}(\pi)}(n-i)$

be the descent number and the comajor index of π.

Theorem (Gessel–Zhuang, '18)

The shuffle algebra $\mathcal{A}_{(\mathrm{des},\mathrm{comaj})}$ is isomorphic to certain subalgebra of $\mathbb{Q}[q,z][\![t*]\!]$ via the linear map

$$[\pi]_{(\mathrm{des},\mathrm{comaj})} \mapsto \begin{cases} \frac{q^{\mathrm{comaj}(\pi)}\, t^{\mathrm{des}(\pi)+1}}{(1-t)(1-qt)\cdots(1-q^n t)} z^n, & \text{if } \pi \in \mathfrak{S}_n \\ \frac{1}{1-t}, & \text{if } \pi = \varnothing. \end{cases}$$

Ingredients for Shuffle-Compatibility

- descent set
- quasisymmetric functions
- P-partitions
- principal specialization

Colored shuffle compatibility

Colored descent set

We fix $r \in \mathbb{Z}_{>0}$, and think of

- $\mathbb{Z}_r = \{0 >' 1 >' \cdots >' r-1\}$ as a set of colors.

A *r-colored permutation* of length n is a pair (π, ϵ), where $\pi \in \mathfrak{S}_n$ and $\epsilon \in \mathbb{Z}_r^n$.
We let

- $\mathfrak{S}_{n,r}$ be the group of r-colored permutations of length n,

where the product is given by

$$(\pi, \epsilon)(\sigma, \delta) = (\pi\sigma, \sigma(\epsilon) + \delta).$$

Colored descent set

We fix $r \in \mathbb{Z}_{>0}$, and think of

- $\mathbb{Z}_r = \{0 >' 1 >' \cdots >' r-1\}$ as a set of colors.

A *r-colored permutation* of length n is a pair (π, ϵ), where $\pi \in \mathfrak{S}_n$ and $\epsilon \in \mathbb{Z}_r^n$. We let

- $\mathfrak{S}_{n,r}$ be the group of r-colored permutations of length n,

where the product is given by

$$(\pi, \epsilon)(\sigma, \delta) = (\pi\sigma, \sigma(\epsilon) + \delta).$$

Example

$$3^1\, 2^2 1^0\, 4^0\, 5^1\, 6^1 \;\in\; \mathfrak{S}_{6,3}$$

Colored descent set

Definition (Mantaci–Reutenauer, '95)

The colored descent set $\mathrm{sDes}(\pi, \epsilon)$ of $(\pi, \epsilon) \in \mathfrak{S}_{n,r}$ consists of pairs (i, ϵ_i) for $i \in [n-1]$ such that

- $\epsilon_i \neq \epsilon_{i+1}$, or
- $\epsilon_i = \epsilon_{i+1}$ and $i \in \mathrm{Des}(\pi)$

together with (n, ϵ_n).

Colored descent set

Definition (Mantaci–Reutenauer, '95)

The colored descent set $\mathrm{sDes}(\pi, \epsilon)$ of $(\pi, \epsilon) \in \mathfrak{S}_{n,r}$ consists of pairs (i, ϵ_i) for $i \in [n-1]$ such that

- $\epsilon_i \neq \epsilon_{i+1}$, or
- $\epsilon_i = \epsilon_{i+1}$ and $i \in \mathrm{Des}(\pi)$

together with (n, ϵ_n).

Example

$$\mathrm{sDes}(3^1\, 2^2 1^0\, 4^0\, 5^1\, 6^1) = \{(1,1), (2,2), (4,0), (6,1)\}$$

Colored quasisymmetric functions

For each color $j \in \mathbb{Z}_r$, we let

- $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots)$ be an infinite sequence of commuting indeterminates
- $\mathbf{X}^r = \left(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(r-1)}\right)$.

Colored quasisymmetric functions

Definition (Poirier, '98, Baumann–Hohlweg '08, Bergeron–Hohlweg, '06)

The fundamental colored quasisymmetric function corresponding to a colored permutation $u = (\pi, \epsilon) \in \mathfrak{S}_{n,r}$ is defined by

$$F_u := F_{\mathrm{sDes}(u)}(\mathbf{X}^r) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ \epsilon_{s_j} \le \epsilon_{s_{j+1}} \Rightarrow i_{s_j} < i_{s_j+1}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)},$$

where $\mathrm{sDes}(u) = \{(s_1, \epsilon_{s_1}) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} (s_k, \epsilon_{s_k})\}$.

Colored quasisymmetric functions

Definition (Poirier, '98, Baumann–Hohlweg '08, Bergeron–Hohlweg, '06)

The fundamental colored quasisymmetric function corresponding to a colored permutation $u = (\pi, \epsilon) \in \mathfrak{S}_{n,r}$ is defined by

$$F_u := F_{\mathrm{sDes}(u)}(\mathbf{X}^r) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ \epsilon_{s_j} \le \epsilon_{s_{j+1}} \Rightarrow i_{s_j} < i_{s_j+1}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)},$$

where $\mathrm{sDes}(u) = \{(s_1, \epsilon_{s_1}) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} (s_k, \epsilon_{s_k})\}$.

Example

For $u = 3^1\, 2^2 1^0\, 4^0\, 5^1\, 6^1$, we computed $\mathrm{sDes}(u) = \{(1,1), (2,2), (4,0), (6,1)\}$ and thus

$$F_u = \sum_{i_1 < i_2 \le i_3 \le i_4 < i_5 \le i_6} x_1^{(1)} x_2^{(2)} x_3^{(0)} x_4^{(0)} x_5^{(1)} x_6^{(1)}.$$

Colored quasisymmetric functions

Definition (Poirier, '98, Baumann–Hohlweg '08, Bergeron–Hohlweg, '06)

The fundamental colored quasisymmetric function corresponding to a colored permutation $u = (\pi, \epsilon) \in \mathfrak{S}_{n,r}$ is defined by

$$F_u := F_{\mathrm{sDes}(u)}(\mathbf{X}^r) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ \epsilon_{s_j} \le \epsilon_{s_{j+1}} \Rightarrow i_{s_j} < i_{s_{j+1}}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)},$$

where $\mathrm{sDes}(u) = \{(s_1, \epsilon_{s_1}) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} (s_k, \epsilon_{s_k})\}$.

Let

- $\mathrm{QSym}^{(r)}$ be the $\mathbb{Q}$-algebra of colored quasisymmetric functions, spanned by the fundamental colored quasisymmetric functions.

Colored quasisymmetric functions

Definition (Poirier, '98, Baumann–Hohlweg '08, Bergeron–Hohlweg, '06)

The fundamental colored quasisymmetric function corresponding to a colored permutation $u = (\pi, \epsilon) \in \mathfrak{S}_{n,r}$ is defined by

$$F_u := F_{\mathrm{sDes}(u)}(\mathbf{X}^r) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ \epsilon_{s_j} \le \epsilon_{s_{j+1}} \Rightarrow i_{s_j} < i_{s_{j+1}}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)},$$

where $\mathrm{sDes}(u) = \{(s_1, \epsilon_{s_1}) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} (s_k, \epsilon_{s_k})\}$.

Let

- $\mathrm{QSym}^{(r)}$ be the $\mathbb{Q}$-algebra of colored quasisymmetric functions, spanned by the fundamental colored quasisymmetric functions.

Question

How can we multiply two fundamental colored quasisymmetric functions?

Colored P-partitions

Let P be a poset on $[n] \times \mathbb{Z}_r$. A colored P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $u <_P v$ implies $f(u) \leq f(v)$
- $u <_p v$ and $u >_{\mathrm{rlex}} v$ implies $f(u) < f(v)$.

Consider the weight enumerator

$$F(P; \mathbf{X}^r) := \sum_{\substack{f: P \to \mathbb{Z}_{>0} \\ f = \text{colored } P\text{-partition}}} x_{f(1,\epsilon_1)}^{(\epsilon_1)} x_{f(2,\epsilon_2)}^{(\epsilon_2)} \cdots x_{f(n,\epsilon_n)}^{(\epsilon_n)} \in \mathrm{QSym}^{(r)} .$$

Colored P-partitions

Let P be a poset on $[n] \times \mathbb{Z}_r$. A colored P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $u <_P v$ implies $f(u) \leq f(v)$
- $u <_p v$ and $u >_{\mathrm{rlex}} v$ implies $f(u) < f(v)$.

Consider the weight enumerator

$$F(P; \mathbf{X}^r) := \sum_{\substack{f : P \to \mathbb{Z}_{>0} \\ f = \text{colored } P\text{-partition}}} x_{f(1,\epsilon_1)}^{(\epsilon_1)} x_{f(2,\epsilon_2)}^{(\epsilon_2)} \cdots x_{f(n,\epsilon_n)}^{(\epsilon_n)} \in \mathrm{QSym}^{(r)} .$$

Example

If $P = \underline{(\pi, \epsilon)}$ is an n-element (colored) chain represented by some $(\pi, \epsilon) \in \mathfrak{S}_{n,r}$, then

$$F(P; \mathbf{X}^r) = F_{(\pi,\epsilon)},$$

since $(\pi_i, \epsilon_i) >_{\mathrm{rlex}} (\pi_{i+1}, \epsilon_{i+1})$ translates to $\epsilon_i >' \epsilon_{i+1}$ or $\epsilon_i = \epsilon_{i+1}$ and $i \in \mathrm{Des}(\pi)$.

Colored P-partitions

Let P be a poset on $[n] \times \mathbb{Z}_r$. A colored P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $u <_P v$ implies $f(u) \leq f(v)$
- $u <_p v$ and $u >_{\mathrm{rlex}} v$ implies $f(u) < f(v)$.

Consider the weight enumerator

$$F(P;\mathbf{X}^r) := \sum_{\substack{f:P\to\mathbb{Z}_{>0} \\ f=\text{colored } P\text{-partition}}} x_{f(1,\epsilon_1)}^{(\epsilon_1)} x_{f(2,\epsilon_2)}^{(\epsilon_2)} \cdots x_{f(n,\epsilon_n)}^{(\epsilon_n)} \in \mathrm{QSym}^{(r)}.$$

Example

If $P = \underline{(\pi,\epsilon)}$ is an n-element (colored) chain represented by some $(\pi,\epsilon) \in \mathfrak{S}_{n,r}$, then

$$F(P;\mathbf{X}^r) = F_{(\pi,\epsilon)},$$

since $(\pi_i,\epsilon_i) >_{\mathrm{rlex}} (\pi_{i+1},\epsilon_{i+1})$ translates to $\epsilon_i < \epsilon_{i+1}$ or $\epsilon_i = \epsilon_{i+1}$ and $i \in \mathrm{Des}(\pi)$.

Colored P-partitions

Let P be a poset on $[n] \times \mathbb{Z}_r$. A colored P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $u <_P v$ implies $f(u) \leq f(v)$
- $u <_p v$ and $u >_{\mathrm{rlex}} v$ implies $f(u) < f(v)$.

Consider the weight enumerator

$$F(P; \mathbf{X}^r) := \sum_{\substack{f: P \to \mathbb{Z}_{>0} \\ f = \text{colored } P\text{-partition}}} x_{f(1,\epsilon_1)}^{(\epsilon_1)} x_{f(2,\epsilon_2)}^{(\epsilon_2)} \cdots x_{f(n,\epsilon_n)}^{(\epsilon_n)} \in \mathrm{QSym}^{(r)}.$$

Theorem (Hsiao–Petersen, '10)

- We have

$$F(P; \mathbf{X}^r) = \sum_{u \in \mathcal{L}(P)} F(\underline{u}; \mathbf{X}^{(r)}) = \sum_{u \in \mathcal{L}(P)} F_u$$

where $\mathcal{L}(P) \subseteq \mathfrak{S}_{n,r}$ is the set of all linear extensions of P.

- If P and Q are two (colored) posets on disjoint sets, then

$$F(P; \mathbf{X}^r) F(Q; \mathbf{X}^r) = F(P + Q; \mathbf{X}^r).$$

Colored P-partitions

Let P be a poset on $[n] \times \mathbb{Z}_r$. A colored P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that

- $u <_P v$ implies $f(u) \leq f(v)$
- $u <_p v$ and $u >_{\mathrm{rlex}} v$ implies $f(u) < f(v)$.

Consider the weight enumerator

$$F(P; \mathbf{X}^r) := \sum_{\substack{f : P \to \mathbb{Z}_{>0} \\ f = \text{colored } P\text{-partition}}} x_{f(1,\epsilon_1)}^{(\epsilon_1)} x_{f(2,\epsilon_2)}^{(\epsilon_2)} \cdots x_{f(n,\epsilon_n)}^{(\epsilon_n)} \in \mathrm{QSym}^{(r)} .$$

Corollary (Hsiao–Petersen, '10)

For two colored permutations u and v of length n and m, repsectively,

$$F_u F_v = \sum_{w \in u \sqcup\!\sqcup v} F_w .$$

In particular, the distribution of sDes on $u \sqcup\!\sqcup v$ depends only on $\mathrm{sDes}(u), \mathrm{sDes}(v)$, n and m.

Colored shuffle compatibility

A (**colored**) permutation statistic stat is called shuffle compatible if for any two disjoint r-**colored** permutations u and v, the multiset

$$\{\mathrm{stat}(w) : w \in u \sqcup\!\sqcup\, v\}$$

depends only on $\mathrm{stat}(u)$, $\mathrm{stat}(v)$ and the lengths of u and v.
Such statistic defines an equivalence relation $\sim_{\mathrm{stat}}$ on the set of all r-**colored** permutations by letting

$$u \sim_{\mathrm{stat}} v \iff u \text{ and } v \text{ have the same length, and } \mathrm{stat}(\pi) = \mathrm{stat}(\sigma).$$

The space $\mathcal{A}^{(r)}_{\mathrm{stat}}$ of equivalence classes of r-**colored** permutations with multiplication given by

$$[u]_{\mathrm{stat}}[v]_{\mathrm{stat}} = \sum_{w \in u \sqcup\!\sqcup v} [w]_{\mathrm{stat}}$$

is called the shuffle algebra of stat.

Colored shuffle compatibility

Theorem (M., '21)

The colored descent set statistic sDes is shuffle compatible and the corresponding shuffle algebra $\mathcal{A}^{(r)}_{\text{sDes}}$ is isomorphic to $\text{QSym}^{(r)}$ via the linear map

$$[u]_{\text{sDes}} \;\mapsto\; F_{\text{sDes}(u)}.$$

Hadamard product and (colored) shuffle algebras

For $(\pi, \epsilon) \in \mathfrak{S}_{n,r}$, let

$$\mathrm{des}(\pi, \epsilon) := |\{i \in [n-1] : \epsilon_i < \epsilon_{i+1}, \text{ or } \epsilon_i = \epsilon_{i+1} \text{ and } i \in \mathrm{Des}(\pi)\} \cup \{0 : \epsilon_1 > 0\}|$$

be the descent number of (π, ϵ) and let

- $\mathrm{comaj}(\pi, \epsilon) := \sum_{i \in \mathrm{Des}(\pi,\epsilon)} (n-i)$
- $\mathrm{col}_j(\pi, \epsilon) := |\{i \in [n] : \epsilon_i = j\}|$

be the comajor index and the number of j-colored entries of (π, ϵ), respectively.

Hadamard product and (colored) shuffle algebras

For $(\pi, \epsilon) \in \mathfrak{S}_{n,r}$, let

$$\mathrm{des}(\pi, \epsilon) := |\{i \in [n-1] : \epsilon_i < \epsilon_{i+1}, \text{ or } \epsilon_i = \epsilon_{i+1} \text{ and } i \in \mathrm{Des}(\pi)\} \cup \{0 : \epsilon_1 > 0\}|$$

be the descent number of (π, ϵ) and let

- $\mathrm{comaj}(\pi, \epsilon) := \sum_{i \in \mathrm{Des}(\pi, \epsilon)} (n - i)$
- $\mathrm{col}_j(\pi, \epsilon) := |\{i \in [n] : \epsilon_i = j\}|$

be the comajor index and the number of j-colored entries of (π, ϵ), respectively.

Example

For $u = 3^1\, 2^2\, 1^0\, 4^0\, 5^1\, 6^1 \in \mathfrak{S}_{6,3}$, we have

$$\begin{aligned} \mathrm{des}(u) &= 3 \\ \mathrm{comaj}(u) &= (6-0) + (6-1) + (6-4) = 13 \\ \mathrm{col}(u) &= (2, 3, 1). \end{aligned}$$

Hadamard product and (colored) shuffle algebras

Theorem (Carnevale, M., Rossmann, '23+)

The tuple $(\mathrm{des}, \mathrm{comaj}, \mathrm{col})$ is shuffle compatible and the shuffle algebra $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathrm{col})}$ is isomorphic to certain subalgebra of $\mathbb{Q}[q, z, p_0, p_1, \ldots, p_{r-1}][\![t*]\!]$ via the linear map

$$[u]_{(\mathrm{des},\mathrm{comaj},\mathrm{col})} \mapsto \begin{cases} \frac{p_0^{\mathrm{col}_0(u)} \cdots p_{r-1}^{\mathrm{col}_{r-1}(u)} q^{\mathrm{comaj}(u)} t^{\mathrm{des}(u)+1}}{(1-t)(1-qt)\cdots(1-q^n t)} z^n, & \text{if } u \in \mathfrak{S}_{n,r} \\ \frac{1}{1-t}, & \text{if } u = \varnothing. \end{cases}$$

Hadamard product and (colored) shuffle algebras

Theorem (Carnevale, M., Rossmann, '23+)

The tuple $(\mathrm{des}, \mathrm{comaj}, \mathrm{col})$ is shuffle compatible and the shuffle algebra $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathrm{col})}$ is isomorphic to certain subalgebra of $\mathbb{Q}[q, z, p_0, p_1, \ldots, p_{r-1}][\![t*]\!]$ via the linear map

$$[u]_{(\mathrm{des},\mathrm{comaj},\mathrm{col})} \mapsto \begin{cases} \frac{p_0^{\mathrm{col}_0(u)} \cdots p_{r-1}^{\mathrm{col}_{r-1}(u)} q^{\mathrm{comaj}(u)} t^{\mathrm{des}(u)+1}}{(1-t)(1-qt)\cdots(1-q^n t)} z^n, & \text{if } u \in \mathfrak{S}_{n,r} \\ \frac{1}{1-t}, & \text{if } u = \varnothing. \end{cases}$$

The proof is based on specializations of colored quasisymmetric functions (M., '21):

$$\begin{aligned} &x_1^{(0)} = p_0, \quad x_2^{(0)} = p_0 q, \quad x_3^{(0)} = p_0 q^2, \quad \ldots, \quad x_m^{(0)} = p_0 q^{m-1}, \quad x_{m+1}^{(0)} = \cdots = 0 \\ &x_1^{(j)} = 0, \quad x_2^{(j)} = p_j q, \quad x_3^{(j)} = p_j q^2, \quad \ldots, \quad x_m^{(j)} = p_j q^{m-1}, \quad x_{m+1}^{(j)} = \cdots = 0 \end{aligned}$$

for all $1 \leq j \leq r-1$.

Application: Hadamard products of ask zeta functions

Zeta functions of algebraic structures

- Let G be an algebraic structure (group, ring, module, variety, ...) and consider the associated zeta function $\zeta_G(s)$.

Zeta functions of algebraic structures

- Let G be an algebraic structure (group, ring, module, variety, ...) and consider the associated zeta function $\zeta_G(s)$.
- It is often the case that $\zeta_G(s)$ admits an Euler factorization

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$$

into local factors.

Zeta functions of algebraic structures

- Let G be an algebraic structure (group, ring, module, variety, ...) and consider the associated zeta function $\zeta_G(s)$.
- It is often the case that $\zeta_G(s)$ admits an Euler factorization

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$$

  into local factors.
- Results from p-adic integration imply that these local factors are rational functions in p^{-s}, i.e. of the form

$$\zeta_{G,p}(s) = W_p(p^{-s}),$$

  for some $W_p(t) \in \mathbb{Q}(t)$.

Zeta functions of algebraic structures

- Let G be an algebraic structure (group, ring, module, variety, ...) and consider the associated zeta function $\zeta_G(s)$.
- It is often the case that $\zeta_G(s)$ admits an Euler factorization

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$$

  into local factors.
- Results from p-adic integration imply that these local factors are rational functions in p^{-s}, i.e. of the form

$$\zeta_{G,p}(s) = W_p(p^{-s}),$$

  for some $W_p(t) \in \mathbb{Q}(t)$.
- In many cases of interest, local factors exhibit the following uniformity phenomenon: There exists a single bivariate rational function $W(q,t)$ such that

$$\zeta_{G,p}(s) = W(p, p^{-s})$$

  for all primes p (modulo a finite number of exceptions).

Zeta functions of algebraic structures

Problem

Understand, combinatorially if possible, $W(q, t)$.

Ask zeta functions

Definition

Let

- $\mathfrak{D}$ be a (compact) discrete valuation ring (think of integers mod p)
- $\mathfrak{m}$ be the (unique) maximal ideal of $\mathfrak{D}$
- q be the size of the residue field $\mathfrak{D}/\mathfrak{m}$.

The ask zeta function corresponding to $M \subset \mathrm{Mat}_{d \times e}(\mathfrak{D})$ is the formal power series

$$\mathsf{Z}_M^{\mathrm{ask}}(t) := \sum_{k \geq 0} a_k(M) t^k$$

where

$$a_k(M) := \frac{1}{|M_k|} \sum_{A \in M_k} |\ker(A)|$$

denotes the average size of the kernels within the reduction M_k of M modulo $\mathfrak{m}^k$.

Ask zeta functions

Examples

- For $M = \mathrm{Mat}_{d\times e}(\mathfrak{D})$,

$$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{Mat}_{d\times e}(\mathfrak{D})}(t) = \frac{1 - q^{-e}t}{(1-t)(1-q^{d-e}t)}.$$

Ask zeta functions

Examples

- For $M = \mathrm{Mat}_{d\times e}(\mathfrak{D})$,

$$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{Mat}_{d\times e}(\mathfrak{D})}(t) = \frac{1 - q^{-e}t}{(1-t)(1-q^{d-e}t)}.$$

- If $M = \mathfrak{so}_d(\mathfrak{D}) = \{A \in \mathfrak{gl}_d(\mathfrak{D}) : A + A^t = 0\}$, then

$$\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{D})}(t) = \frac{1 - q^{1-d}t}{(1-t)(1-qt)}.$$

Ask zeta functions

For a simple graph $G = ([n], E)$, let

- M_G be the set of $(n \times n)$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$, when $\{i, j\} \notin E$
- $\mathsf{Z}_G^{\mathrm{ask}}(t) := \mathsf{Z}_{M_G}^{\mathrm{ask}}(t)$ be the corresponding ask zeta function.

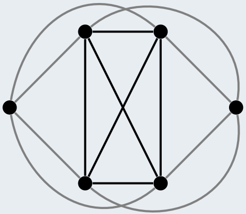
Ask zeta functions

For a simple graph $G = ([n], E)$, let

- M_G be the set of $(n \times n)$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$, when $\{i, j\} \notin E$
- $\mathsf{Z}^{\mathrm{ask}}_G(t) := \mathsf{Z}^{\mathrm{ask}}_{M_G}(t)$ be the corresponding ask zeta function.

Example

Let $G = K_{n+2} \vee \overline{K_n}$ is the join of the complete graph on $n + 2$ vertices and the edgeless graph on n vertices. For example, for $n = 2$



Ask zeta functions

For a simple graph $G = ([n], E)$, let

- M_G be the set of $(n \times n)$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$, when $\{i, j\} \notin E$
- $\mathsf{Z}_G^{\mathrm{ask}}(t) := \mathsf{Z}_{M_G}^{\mathrm{ask}}(t)$ be the corresponding ask zeta function.

Example

Let $G = K_{n+2} \vee \overline{K_n}$ is the join of the complete graph on $n+2$ vertices and the edgeless graph on n vertices. Then

$$\mathsf{Z}_G^{\mathrm{ask}}(t) = \frac{(1 - q^{-n-1}t)(1 - q^{-n-2}t)}{(1-t)(1-qt)(1-q^2t)}.$$

Ask zeta functions

For a simple graph $G = ([n], E)$, let

- M_G be the set of $(n \times n)$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$, when $\{i, j\} \notin E$
- $\mathsf{Z}_G^{\mathrm{ask}}(t) := \mathsf{Z}_{M_G}^{\mathrm{ask}}(t)$ be the corresponding ask zeta function.

Example

Let $G = K_{n+2} \vee \overline{K_n}$ is the join of the complete graph on $n+2$ vertices and the edgeless graph on n vertices. Then

$$\mathsf{Z}_G^{\mathrm{ask}}(t) = \frac{(1 - q^{-n-1}t)(1 - q^{-n-2}t)}{(1-t)(1-qt)(1-q^2t)}.$$

Claim

We can understand these ask zeta functions (and many more!) in a combinatorial way, through colored permutation statistics and shuffle-compatibility!

A reformulation in terms of colored configurations

Let $\mathfrak{S}$ be the set of all colored permutations with

- symbols taken from $\Sigma := \mathbb{Z}_{>0}$, and
- colors taken from $\Gamma := \mathbb{N}$.

A reformulation in terms of colored configurations

Let $\mathfrak{S}$ be the set of all colored permutations with

- symbols taken from $\Sigma := \mathbb{Z}_{>0}$, and
- colors taken from $\Gamma := \mathbb{N}$.

A colored configuration is an element of $\mathbb{Z}\mathfrak{S}$ of the form

$$f = \sum_{u \in \mathfrak{S}} f_u u,$$

where all but finitely many f_u are zero. The support of f is defined by

$$\mathrm{supp}(f) := \{u \in \mathfrak{S} : f_u \neq 0\}.$$

A reformulation in terms of colored configurations

Let $\mathfrak{S}$ be the set of all colored permutations with

- symbols taken from $\Sigma := \mathbb{Z}_{>0}$, and
- colors taken from $\Gamma := \mathbb{N}$.

A colored configuration is an element of $\mathbb{Z}\mathfrak{S}$ of the form

$$f = \sum_{u \in \mathfrak{S}} f_u u,$$

where all but finitely many f_u are zero. The support of f is defined by

$$\mathrm{supp}(f) := \{u \in \mathfrak{S} : f_u \neq 0\}.$$

A labelled colored configuration is a pair (f, α), where $f \in \mathbb{Z}\mathfrak{S}$ and $\alpha : \Gamma \to \{\pm q^k : k \in \mathbb{Z}\}$ is a map such that (possibly)

- $\alpha(c) \neq 1$ if c appears as the nonzero color of some colored permutation in f.

A reformulation in terms of colored configurations

Let $\mathfrak{S}$ be the set of all colored permutations with

- symbols taken from $\Sigma := \mathbb{Z}_{>0}$, and
- colors taken from $\Gamma := \mathbb{N}$.

A colored configuration is an element of $\mathbb{Z}\mathfrak{S}$ of the form

$$f = \sum_{u \in \mathfrak{S}} f_u u,$$

where all but finitely many f_u are zero. The support of f is defined by

$$\mathrm{supp}(f) := \{u \in \mathfrak{S} : f_u \neq 0\}.$$

A labelled colored configuration is a pair (f, α), where $f \in \mathbb{Z}\mathfrak{S}$ and $\alpha : \Gamma \to \{\pm q^k : k \in \mathbb{Z}\}$ is a map such that (possibly)

- $\alpha(c) \neq 1$ if c appears as the nonzero color of some colored permutation in f.

An example of a labelled colored configuration is the pair $(f = 1^0 + 1^1, \alpha)$ with

$$\alpha(1) = \pm q^k,\ \alpha(0) = \alpha(2) = \cdots = 1.$$

A reformulation in terms of colored configurations

Definition

For an integer $\epsilon \in \mathbb{Z}$ and a labelled colored configuration (f, α), we define

$$W^{\epsilon}_{f,\alpha}(q,t) := \sum_{u \in \mathrm{supp}(f)} f_u \, \frac{\alpha(u) q^{\epsilon\,\mathsf{comaj}(u)} t^{\mathsf{des}(u)}}{(1-t)(1-q^{\epsilon}t)\cdots(1-q^{\epsilon|u|}t)} \in \mathbb{Q}(q)[\![t]\!],$$

where $\alpha(u)$ denotes the product of the values of α at every color of u and $|u|$ denotes the length of u.

A reformulation in terms of colored configurations

Definition

For an integer $\epsilon \in \mathbb{Z}$ and a labelled colored configuration (f, α), we define

$$W^{\epsilon}_{f,\alpha}(q,t) := \sum_{u \in \mathrm{supp}(f)} f_u \frac{\alpha(u) q^{\epsilon\,\mathsf{comaj}(u)} t^{\mathsf{des}(u)}}{(1-t)(1-q^{\epsilon}t)\cdots(1-q^{\epsilon|u|}t)} \in \mathbb{Q}(q)[\![t]\!],$$

where $\alpha(u)$ denotes the product of the values of α at every color of u and $|u|$ denotes the length of u.

Example

For the labelled colored configuration $(f = 1^0 + 1^1, \alpha)$ with $\alpha(1) \neq 1$, we have

$$W^{\epsilon}_{f,\alpha}(q,t) = \frac{1 + \alpha(1) q^{\epsilon} t}{(1-t)(1-q^{\epsilon}t)}.$$

A reformulation in terms of colored configurations

Definition

For an integer $\epsilon \in \mathbb{Z}$ and a labelled colored configuration (f, α), we define

$$W^{\epsilon}_{f,\alpha}(q,t) := \sum_{u \in \mathrm{supp}(f)} f_u \frac{\alpha(u) q^{\epsilon\,\mathrm{comaj}(u)} t^{\mathrm{des}(u)}}{(1-t)(1-q^{\epsilon}t)\cdots(1-q^{\epsilon|u|}t)} \in \mathbb{Q}(q)[\![t]\!],$$

where $\alpha(u)$ denotes the product of the values of α at every color of u and $|u|$ denotes the length of u.

Osbervation

We have $\mathsf{Z}^{\mathrm{ask}}_{M}(t) = W^{\epsilon}_{f,\alpha}(q,t)$ for the following:

- $M = \mathrm{Mat}_{d\times e}(\mathfrak{D})$, and $f = 1^0 + 1^1, \alpha(1) = -q^{-d}$ and $\epsilon = d - e$.
- $M = \mathfrak{so}_d(\mathfrak{D})$, and $f = 1^0 + 1^1, \alpha(1) = -q^{-d}$, and $\epsilon = 1$.
- $M = M_G$, where $G = K_{n+2} \vee \overline{K_n}$ and

$$f = 1^0 2^0 + 1^0 2^2 + 1^1 2^0 + 1^1 2^2,\ \alpha(1) = \alpha(2) = -q^{-n-3},\ \epsilon = 1.$$

Hadamard products of ask zeta functions

Ask zeta functions satisfy the following property:

$$\mathsf{Z}^{\mathrm{ask}}_{M_1 \oplus M_2}(t) = \mathsf{Z}^{\mathrm{ask}}_{M_1}(t) * \mathsf{Z}^{\mathrm{ask}}_{M_1}(t).$$

Hadamard products of ask zeta functions

Ask zeta functions satisfy the following property:

$$\mathsf{Z}^{\mathrm{ask}}_{M_1 \oplus M_2}(t) = \mathsf{Z}^{\mathrm{ask}}_{M_1}(t) * \mathsf{Z}^{\mathrm{ask}}_{M_1}(t).$$

Theorem (Carnevale, M., Rossmann, '23+)

If (f, α) and (g, β) are two strongly disjoint labelled colored configurations, then

$$W^{\epsilon}_{f,\alpha}(q,t) * W^{\epsilon}_{g,\beta}(q,t) = W^{\epsilon}_{f \sqcup g, \alpha\beta}(q,t),$$

for all $\epsilon \in \mathbb{Z}$. In particular, for a fixed $\epsilon \in \mathbb{Z}$, the set

$$\{W^{\epsilon}_{f,\alpha}(q,t) : (f,\alpha) \text{ is a labelled colored configuration}\}$$

is closed under taking Hadamard products.

Hadamard products of ask zeta functions

Example

If $\epsilon = d_1 - e_1 = d_2 - e_2$, then

$$\begin{aligned}
\mathsf{Z}^{\mathrm{ask}}_{\mathrm{Mat}_{d_1\times e_1}(\mathfrak{O})\oplus\mathrm{Mat}_{d_2\times e_2}(\mathfrak{O})}(t) &= \mathsf{Z}^{\mathrm{ask}}_{\mathrm{Mat}_{d_1\times e_1}(\mathfrak{O})}(t) * \mathsf{Z}^{\mathrm{ask}}_{\mathrm{Mat}_{d_2\times e_2}(\mathfrak{O})}(t) \\
&= W^{\epsilon}_{1^0+1^1,1\mapsto -q^{-d_1}}(q,t) * W^{\epsilon}_{1^0+1^1,1\mapsto -q^{-d_2}}(q,t) \\
&= W^{\epsilon}_{1^0+1^1,1\mapsto -q^{-d_1}}(q,t) * W^{\epsilon}_{2^0+2^2,2\mapsto -q^{-d_2}}(q,t) \\
&= W^{\epsilon}_{(1^0+1^1)\sqcup\!\sqcup(2^0+2^2),\substack{1\mapsto -q^{-d_1}\\ 2\mapsto -q^{-d_2}}}(q,t).
\end{aligned}$$

We compute

$$(1^0+1^1)\sqcup\!\sqcup(2^0+2^2) = 1^0 2^0 + 2^0 1^0 + 1^0 2^2 + 2^2 1^0 + 1^1 2^0 + 2^0 1^1 + 1^1 2^2 + 2^2 1^1,$$

and therefore the ask zeta function above is equal to

$$\frac{1 + (1 - q^{-d_1} - q^{-d_2})q^{\epsilon}t + (-q^{-d_1} - q^{-d_2} + q^{-d_1-d_2})q^{2\epsilon}t + q^{-d_1-d_2}q^{3\epsilon}t^2}{(1-t)(1-q^{\epsilon}t)(1-q^{2\epsilon}t)}.$$

Thank you for your attention!

¡Gracias por su atención!

Ευχαριστώ για την προσοχή σας!