Dual Specht Modules for the Rook Monoid

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The Rook Monoid

Let n be a positive integer and $[n] = \{1, \ldots, n\}$.

The rook monoid, denoted R_n, is the set of all partial permutations of $[n]$ endowed with the usual composition of partial functions.

Example

Let $\sigma, \tau \in R_5$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & - & 1 & 3 & 5 \end{pmatrix} \in R_5,\ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & - & - \end{pmatrix} \in S_3 \subseteq R_5.$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & 2 & 1 & - & - \end{pmatrix} \in R_5.$$

Representations of the Rook Monoid

Let $\mathbb{F}$ be a field of characteristic zero and let S_r be the symmetric group on $[r]$, for $r = 0, 1, \ldots, n$ (with $S_0 \cong S_1$).

It is clear that $|R_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r\,!$ and $S_r \subseteq R_n$, for $r = 0, 1, \ldots, n$.

The irreducible representations of R_n were described in the 1950's by W. D. Munn who showed how they can be built from the irreducible representations of S_r, with $r = 0, 1, \ldots, n$.

- $\mathbb{F}R_n$ is (split) semisimple;
- the isomorphism classes of simple $\mathbb{F}R_n$-modules are indexed by the set

  $$\{\mu \,:\, \mu \vdash r, r = 0, 1, \cdots n\},$$

  where $\mu \vdash r$ means that μ is a partition of r.

Main tools

Our description of a full set of representatives of the isomorphism classes of simple $\mathbb{F}R_n$ is associated with the following results:

- Schur–Weyl dualities;
- general theory of the functor $f : \mathrm{mod}(A) \to \mathrm{mod}(eAe)$

Schur–Weyl dualities

Definition

Let A and B be $\mathbb{F}$-algebras and let M be an (A, B)-bimodule. If $\rho : A \to End_{\mathbb{F}}(M)$ and $\psi : B \to End_{\mathbb{F}}(M)$ are the corresponding representations of A and B on M, we say that M satisfies Schur–Weyl duality if the image of each action in $End_{\mathbb{F}}(M)$ is the centraliser for the other. Equivalently,

$$\rho(A) = End_B(M) \text{ and } \psi(B) = End_A(M).$$

Let $V(\cong \mathbb{F}^d)$ be a vector space with basis $\{e_1, \cdots, e_d\}$ and let $GL(V) \cong GL_d(\mathbb{F})$ and $O(V) \cong O_d(\mathbb{F})$ be identified. Classical examples of Schur-Weyl dualities are

$$GL_d(\mathbb{F}) \circlearrowleft \otimes^n V \circlearrowright S_n \qquad \text{(Schur, 1927)}$$

$$O_d(\mathbb{F}) \circlearrowleft \otimes^n V \circlearrowright \mathcal{B}_n(d) \qquad \text{(Brauer, 1937)}$$

$$W_d \circlearrowleft \otimes^n V \circlearrowright \mathcal{P}_n(d) \qquad \text{(Jones, Martin 1994)}$$

Schur-Weyl duality for the rook monoid

Let $V(\cong \mathbb{F}^d)$ be a vector space with basis $\{e_1, \cdots, e_d\}$ and let

$$U = V \oplus W$$

with $W = \mathbb{F}e_\infty$ such that $\{e_1, \cdots, e_d, e_\infty\}$ is an $\mathbb{F}$-basis of U.

In 2002, L. Solomon defined an action of R_n via "place permutations" on the n-th tensor power $\otimes^n U$. He then showed that $\otimes^n U$ satisfies Schur-Weyl duality as an $(\mathbb{F}GL_d(\mathbb{F}), \mathbb{F}R_n)$-bimodule.

Theorem (Solomon, 2002)

Let $GL_d(\mathbb{F})$ act on $\otimes^n U$ by fixing $W = \mathbb{F}e_\infty$ and let $\phi : \mathbb{F}R_n \to End_{\mathbb{F}}(\otimes^n U)$ be defined by the right action of R_n over $\otimes^n U$ given by "place permutations". If $d \geq n$, there is an isomorphism of $\mathbb{F}$-algebras

$$\mathbb{F}R_n \cong End_{\mathbb{F}GL_d(\mathbb{F})}(\otimes^n U).$$

Schur-Weyl duality for the rook monoid via Schur algebras

Let $d \geq n$, let $V(\cong \mathbb{F}^d)$ be a vector space with basis $\{e_1, \cdots, e_d\}$ and let

$$U = V \oplus W$$

with $W = \mathbb{F}e_\infty$ such that $\{e_1, \cdots, e_d, e_\infty\}$ is an $\mathbb{F}$-basis of U.

For every $X \subseteq [n]$, set

$$\Gamma_X(d) = \{\alpha : \alpha : X \to [d] \text{ is a map}\}.$$

Example

Let $d = 7$ and $n = 5$. If $X = \{1, 4, 5\} \subseteq [5]$, then

$$\alpha = (\alpha(1), \alpha(4), \alpha(5)) = (7, 2, 2) \in \Gamma_X(7).$$

Schur-Weyl duality for the rook monoid via Schur algebras

Let $d \geq n$. For $X \subseteq [n]$ and $\alpha \in \Gamma_X(d)$, define $e_\alpha^\otimes \in \otimes^n U$ by

$$e_\alpha^\otimes = e_{\beta(1)} \otimes \cdots \otimes e_{\beta(n)}$$

where $\beta : [n] \mapsto [d] \in \Gamma_{[n]}(d)$ and $\beta(i) = \alpha(i)$ if $i \in X$ and $e_{\beta(i)} = e_\infty$ if $i \notin X$.

Example

As before, let $d = 7$, $n = 5$, let $X = \{1, 4, 5\} \subseteq [5]$ and let $\alpha = (\alpha(1), \alpha(4), \alpha(5)) = (7, 2, 2) \in \Gamma_X(7)$. Then

$$e_\alpha^\otimes = e_7 \otimes e_\infty \otimes e_\infty \otimes e_2 \otimes e_2 \in \otimes^5 U$$

The set $\{e_\alpha^\otimes : \alpha \in \Gamma_X(d), X \subseteq [n]\}$ is an $\mathbb{F}$-basis of $\otimes^n U$.

Schur-Weyl duality for the rook monoid via Schur algebras

Let $d \geq n$ and let $c_{i,j} : GL_d(\mathbb{F}) \to \mathbb{F}$ be defined by $c_{i,j}(g) = g_{i,j}$, for $1 \leq i,j \leq d$ and $g \in GL_d(\mathbb{F})$. If $X = \{x_1, \cdots, x_r\} \subseteq [n]$ and $\alpha, \beta \in \Gamma_X(d)$, then

$$c_{\alpha,\beta}(g) = c_{\alpha(x_1),\beta(x_1)}(g) \cdots c_{\alpha(x_r),\beta(x_r)}(g),$$

for all $g \in GL_d(\mathbb{F})$.

$\mathcal{A} = \mathcal{A}_{[n]}(d) =< c_{\alpha,\beta} : \alpha, \beta \in \Gamma_X(d), X \subseteq [n] >$ is the $\mathbb{F}$-space generated be all the monomials $c_{\alpha,\beta} : GL_d(\mathbb{F}) \to \mathbb{F}$.

The extended Schur algebra $\mathcal{S} = \mathcal{S}_{\mathbb{F}}(d, [n])$ is the dual $\mathbb{F}$-space of $\mathcal{A}$

$$\mathcal{S} = \mathcal{A}^* = Hom_{\mathbb{F}}(\mathcal{A}; \mathbb{F}).$$

$\mathcal{S}$ is a finite-dimensional $\mathbb{F}$-algebra. In fact, $\dim_{\mathbb{F}}(\mathcal{S}) = \binom{d^2+n}{n}$.

Schur-Weyl duality for the rook monoid via Schur algebras

- The category of finte-dimensional $\mathbb{F}GL_d(\mathbb{F})$-modules whose coefficient functions lie in $\mathcal{A}$ is equivalent to that of $\mathcal{S}$-modules.

- The $\mathbb{F}$-space $\otimes^n U$ has the structure of a left $\mathcal{S}$-module. For any $\xi \in \mathcal{S}$, $X \subseteq [n]$ and $\beta \in \Gamma_X(m)$, we have

$$\xi . e_\beta^{\otimes} = \sum_{\alpha \in \Gamma_X(m)} \xi(c_{\alpha,\beta}) e_\alpha^{\otimes}$$

Theorem (André, L. M.)

Let $d \geq n$. The representation $\rho : \mathcal{S} \mapsto End_{\mathbb{F}}(\otimes^n U)$ afforded by the left action of $\mathcal{S}$ on $\otimes^n U$ induces an isomorphism of $\mathbb{F}$-algebras

$$\mathcal{S} \cong End_{\mathbb{F}R_n}(\otimes^n U).$$

General theory of the functor $f : \mathrm{mod}(A) \to \mathrm{mod}(eAe)$

Let A be an $\mathbb{F}$-algebra and let $e \neq 0$ be an idempotent in A. Then:

- eAe is an algebra over $\mathbb{F}$;
- if M is an A-module, then $e\,M$ is an eAe-module.

In 1980, J. A. Green shows the following result (which he attributes in part to T. Martins and M. Auslander).

Theorem

Let A be an $\mathbb{F}$-algebra and let $\mathrm{mod}(A)$ be the category of A-modules of finite dimension. If $\{V_\lambda : \lambda \in \Lambda\}$ is a full set of simple A-modules in $\mathrm{mod}(A)$, then:

- if $M \in \mathrm{mod}(A)$ is simple, then eM is either zero or simple in $\mathrm{mod}(eAe)$;
- if $\Lambda' = \{\lambda \in \Lambda : eV_\lambda \neq 0\}$, then $\{eV_\lambda : \lambda \in \Lambda'\}$ is a complete set of simple eAe-modules in $\mathrm{mod}(eAe)$

The algebra $\mathcal{S}(\zeta)$ and the $\mathbb{F}R_n$-module $\zeta \otimes^n U$

Let $d \geq n$ and let $X \subseteq [n]$ be a set of size r. Then:

- $\iota_X : [r] \to X \subseteq [n] \subseteq [d]$ is the only order-preserving element of R_n with domain $[r]$ and range X;
- $\xi_X = \xi_{\iota_X, \iota_X}$ is an idempotent of $\mathcal{S}$ and so is

$$\xi = \sum_{X \subseteq [n]} \xi_{\iota_X, \iota_X} \in \mathcal{S}.$$

Let $\mathcal{S}(\zeta)$ be the $\mathbb{F}$-algebra $\mathcal{S}(\zeta) = \zeta \mathcal{S} \zeta$.

Theorem (André, L. M.)

If $d \geq n$, there is an isomorphism of $\mathbb{F}$-algebras $\mathcal{S}(\zeta) \cong \mathbb{F}R_n$. Under this identification, the left $\mathbb{F}R_n$-module $\zeta \otimes^n U$ has as $\mathbb{F}$-basis the set

$$\{e_\alpha^{\otimes} : \alpha \in \Gamma_X(d), \alpha : X \to [n] \text{ is injective}\}$$

and thus $\dim(\zeta \otimes^n U) = \dim(\mathbb{F}R_n)$.

Carter-Lusztig Modules

Let $1 \leq r \leq n \leq d$ and let $\mu = (\mu_1, \ldots, \mu_d) \vdash r$. A μ-tableau is a map $\mathfrak{T} : [\mu] \to [d]$, where $[\mu]$ is the Young diagram of μ.

If $d = n = 10$ and $\mu = (4, 1) \vdash 5 = r$, then $\mathfrak{T}$ is a μ-tableau.

$$\mathfrak{T} : \quad \begin{array}{|c|c|c|c|}\hline 5 & 4 & 7 & 5 \\ \hline 1 \\ \cline{1-1}\end{array}$$

If $\mathfrak{T} : [\mu] \to [r] \subseteq [d]$ is bijective, then $\mathfrak{T}$ is said to be basic. Let $\mathfrak{T}_\mu : [\mu] \to [r] \subseteq [d]$ be an arbitrary but fixed basic μ-tableau. For instance,

$$\mathfrak{T}_\mu : \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

The μ- tableau $\mathfrak{T}_\mu$ is standard.

Carter-Lusztig Modules

Let $1 \leq r \leq n \leq d$ and let $\mu = (\mu_1, \ldots, \mu_d) \vdash r$. Every μ-tableau $\mathfrak{T} : [\mu] \to [d]$ is of the form $\mathfrak{T} = \alpha \circ \mathfrak{T}_\mu$ for a unique $\alpha \in \Gamma_{[r]}(d)$, where $\mathfrak{T}_\mu$ is the basic μ-tableau.

If $d = n = 10$ and $\mu = (4, 2) \vdash 6 = r$, then $\mathfrak{T} = \alpha \circ \mathfrak{T}_\mu$, where

$$\mathfrak{T} : \begin{array}{|c|c|c|c|} \hline 1 & 5 & 5 & 7 \\ \hline 2 & 7 \\ \cline{1-2} \end{array} \quad \text{and} \quad \alpha = (1, 5, 5, 7, 2, 7).$$

If $l_\mu \in \Gamma_{[r]}(d)$ be the unique weakly increasing map such that

$$|\{k \in [r] : l_\mu(k) = i\}| = \mu_i,$$

for all $1 \leq i \leq d$, we write $\mathfrak{T}^\mu = l_\mu \circ \mathfrak{T}_\mu$. For instance, if $d = n = 10$ and $\mu = (4, 2) \vdash 6 = r$, then $\mathfrak{T}^\mu$ is given by

$$\mathfrak{T}^\mu : \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

Carter-Lusztig modules

Let $1 \le r \le n \le d$ and let $\mu = (\mu_1, \ldots, \mu_d) \vdash r$, where $s = l(\mu)$. Then $e_\mu^{\otimes} = e_{l_\mu}^{\otimes}$ is the tensor

$$e_\mu^{\otimes} = \underbrace{e_1 \otimes \ldots \otimes e_1}_{\mu_1 \text{ times}} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{\mu_2 \text{ times}} \otimes \ldots \otimes \underbrace{e_s \otimes \ldots e_s}_{\mu_s \text{ times}} \otimes \underbrace{e_\infty \otimes \ldots \otimes e_\infty}_{n-r \text{ times}}.$$

For each $\mu \vdash r$ with $1 \le r \le n \le d$, the cyclic $\mathcal{S}$-submodule of $\otimes^n U$ spanned by

$$e_\mu^{\otimes} c_\mu = \sum_{\sigma \in C(\mathfrak{T}_\mu)} sgn(\sigma) e_{l_\mu \sigma}^{\otimes},$$

where $c_\mu = \sum_{\sigma \in C(\mathfrak{T}_\mu)} sgn(\sigma)\sigma$ and $C(\mathfrak{T}_\mu)$ is the column stabiliser of $\mathfrak{T}_\mu$, is referred to as the Carter-Lusztig module U_μ (associated with μ). If $\mu = (0)$, we agree that $U_\mu = \mathbb{F} e_\emptyset^{\otimes} = e_\infty \otimes \ldots \otimes e_\infty$.

Theorem (André, L. M.)

If $d \ge n$, the set $\{U_\mu : \vdash r, r = 0, 1, \ldots, n\}$ is a complete set of representatives of the isomorphism classes of simple modules for $\mathcal{S}$.

Dual Specht Modules for the Rook Monoid

Theorem (André, L. M.)

If $d \geq n$, the set $\{\zeta U_\mu : \mu \vdash r, 0 \leq r \leq n\}$ is a complete set of representatives of the isomorphism classes of simple left $\mathcal{S}(\zeta)$-modules and thus also a complete set of representatives of the isomorphism classes of simple left $\mathbb{F}R_n$-modules.

Let $0 \leq r \leq n \leq d$ and $\mu \vdash r$. The simple $\mathbb{F}R_n$-module ζU_μ, denoted by $\mathcal{L}_\mu$, can be thought of as an analogue of the dual Specht module associated with μ for S_r.

Dual Specht Modules for the Rook Monoid

Theorem (André, L. M.)

Let $1 \leq r \leq n$ and let $\mu \vdash r$. The set

$$\{\xi_{\alpha,l_\mu} e_\mu^{\otimes} c_\mu : \alpha \in \Gamma_{[r]}(n), \alpha : [r] \to [n] \text{ is injective and } \alpha \circ \mathfrak{T}_\mu \text{ is standard}\}$$

is an $\mathbb{F}$-basis of the simple left $\mathbb{F}R_n$-module $\zeta U_\mu \subseteq \zeta \otimes^n U$.

As a consequence, if $\mu \vdash r$, we have that

$$\dim(\mathcal{L}_\mu) = \dim(\zeta U_\mu) = \binom{n}{r} f^\mu,$$

where f^μ is the number of basic μ-tableaux with values in $[r]$ which are standard.

Dual Specht Modules for the Rook Monoid

Theorem (André, L. M.)

Let $1 \leq r \leq n$, $\mu \vdash r$ and let $C(\mathfrak{T}_\mu)$ be the column stabiliser of $\mathfrak{T}_\mu$ and $R(\mathfrak{T}_\mu)$ the column stabiliser of $\mathfrak{T}_\mu$. The simple $\mathbb{F}R_n$-module $\mathcal{L}_\mu$ is isomorphic to the left ideal $\widehat{\mathcal{L}}_\mu$ of $\mathbb{F}R_n$, where

$$\widehat{\mathcal{L}}_\mu = \mathbb{F}R_n \widehat{r}_\mu \widehat{c}_\mu,$$

with

$$\widehat{r}_\mu = \eta_r r_\mu, \;\; \widehat{c}_\mu = \eta_r c_\mu, \;\; \eta_r = \sum_{\substack{X \subseteq \mathbf{n}, \\ |X|=r}} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} \epsilon_Y \in \mathbb{F}R_n$$

and $r_\mu = \sum_{\sigma \in R(\mathfrak{T}_\mu)} \sigma$ and $c_\mu = \sum_{\sigma \in C(\mathfrak{T}_\mu)} sgn(\sigma)\sigma$.

From Tableaux to Partial Tableaux

In 2002, C. Grood exhibited a full set of simple $\mathbb{C}R_n$-modules which are analogues of Specht modules for the symmetric group S_n. Her work relies on the notion of μ_r^n-tableaux.

Let $1 \leq r \leq n$ and let $\mu \vdash r$. A μ_r^n-tableau is the Young diagram of μ filled with r distinct entries from the set $[n] = \{1, 2, \ldots, n\}$.

Example

For example, if $n = 12$ and $\mu = (6) \vdash 6 = r$, then

2	1	3	7	5	9

is a μ_{12}^6-tableau. Clearly, if $\alpha = (2, 1, 3, 7, 5, 9) \in \Gamma_{[6]}(12)$, this is precisely $\alpha \circ \mathfrak{T}_\mu$.

From Tableaux to Partial Tableaux

Theorem (André, L. M.)

Let $1 \leq r \leq n$ and $\mu \vdash r$. Let R^μ be the corresponding analogue of the Specht module for S_n in Grood's sense. Then

$$\mathbb{F}R_n \widehat{c}_\mu \widehat{r}_\mu \cong R^\mu.$$

Hence, R^μ is dual to the left ideal $\widehat{\mathcal{L}}_\mu = \mathbb{F}R_n \widehat{r}_\mu \widehat{c}_\mu$ of $\mathbb{F}R_n$.