Schröder trees, antipode formulas and non-commutative probability

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Séminaire Lotharingien de Combinatoire 91
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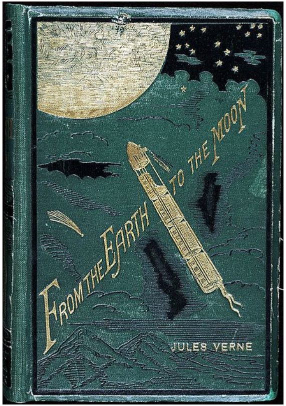


"From the Earth to the Moon: A Direct Route in 97 Hours, 20 Minutes",

by Jules Verne (1865)

Purpose of this talk: give a survey of some links between notions in *non-commutative probability*, algebra and combinatorics.

Adrián Celestino

Content

- Non-commutative probability
- Hopf algebras and the Ebrahimi-Fard-Patras construction
- Applications
- Future work

Part of the talk is based on joint work with Adrián Celestino: "*Schröder trees, antipode formulas and non-commutative probability*" (arXiv:2311.07824).

Non-commutative probability

Let G be a finite, simple, *rooted graph*, with set of vertices $\{v_1, v_2, \ldots, v_\ell\}$.

For every $n \geq 0$, consider

$$m_n(G) := \#\text{ closed walks of length } n \text{ starting at the root.}$$

Let v_1 be the root. If Adj(G) denotes the adjacency matrix of G, then

$$(\mathrm{Adj}(G)_{1,1})^n = m_n(G).$$

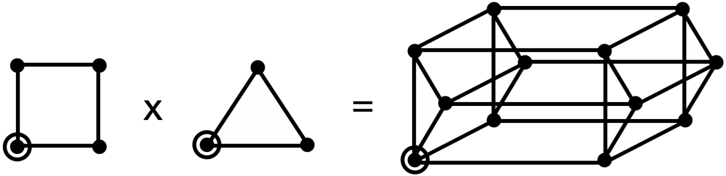
In the space of adjacency matrices, this defines a random variable $A := \mathrm{Adj}(G)$ for which

$$\mathbb{E}[A^n] := m_n(G).$$

Related to the study of "growing graphs", there are binary operations $G_1 * G_2$ on rooted graphs for which we can look at

$$\mathbb{E}[(\mathrm{Adj}(G_1 * G_2))^n].$$

Cartesian product ↔ (classical) independence



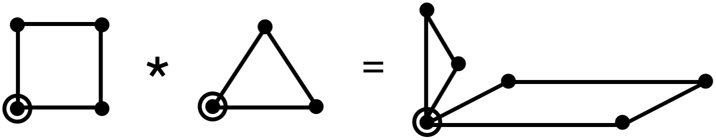
The adjacency matrix of the Cartesian product $G_1 \times G_2$ is

$$\mathrm{Adj}(G_1 \times G_2) = \mathrm{Adj}(G_1) \otimes I_2 + I_1 \otimes \mathrm{Adj}(G_2).$$

The random variables $\mathrm{Adj}(G_1) \otimes I_2$ and $I_1 \otimes \mathrm{Adj}(G_2)$ are independent, so

$$\begin{aligned}\mathbb{E}\Big[\mathrm{Adj}(G_1 \times G_2)^n\Big] &= \mathbb{E}\Big[(\mathrm{Adj}(G_1) \otimes I_2 + I_1 \otimes \mathrm{Adj}(G_2))^n\Big] \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\Big[(\mathrm{Adj}(G_1) \otimes I_2)^k\Big] \mathbb{E}\Big[(I_1 \otimes \mathrm{Adj}(G_2))^{n-k}\Big]\end{aligned}$$

Star product ↔ "Boolean" independence



The adjacency matrix of the Cartesian product $G_1 \star G_2$ is

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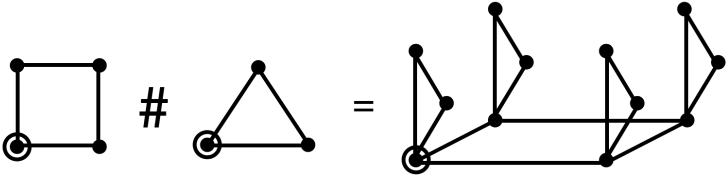
The random variables $\mathrm{Adj}(G_1) \otimes P_2$ and $P_1 \otimes \mathrm{Adj}(G_2)$ are not independent. Still, there is a combinatorial way to calculate the expectation of non-commutative monomials:

$$\mathbb{E}[\mathrm{GESSEL}] = \mathbb{E}[\mathrm{G}]\,\mathbb{E}[\mathrm{E}]\,\mathbb{E}[\mathrm{S}^2]\,\mathbb{E}[\mathrm{E}]\,\mathbb{E}[\mathrm{L}] = \mathbb{E}[\mathrm{G}]\,\mathbb{E}[\mathrm{E}]^2\,\mathbb{E}[\mathrm{S}^2]\,\mathbb{E}[\mathrm{L}].$$

Comb product $\leftrightarrow$ "monotone" independence



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$$\mathbb{E}[\mathrm{GESIRASEL}] = \mathbb{E}[\mathrm{S}]\,\mathbb{E}[\mathrm{R}]\,\mathbb{E}[\mathrm{S}]\,\mathbb{E}[\mathrm{I}]\,\mathbb{E}[\mathrm{GEAEL}].$$

- The field of *Free Probability* was introduced by Dan-Virgil Voiculescu in the 1980s.
- Investigate the notion of “freeness" in analogy to the concept of “independence” from (classical) probability theory.
- A combinatorial theory of freeness was developed by Nica and Speicher in the 1990s.
- Voiculescu discovered freeness also asymptotically for many kinds of random matrices (1991).

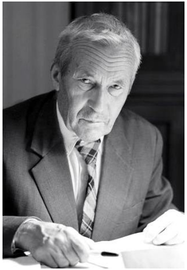
Dan Voiculescu , 2015

Commutative vs non-commutative

Voiculescu: "'Free probability is a probability theory adapted to dealing with variables which have the highest degree of noncommutativity. *Failure of commutativity may occur in many ways*."

- Quantum mechanics' commutation relation: $XY - YX = I$.
- Free product of groups.
- Independent random matrices tend to be asymptotically freely independent, under certain conditions.

Classical probability space



Andrey Kolmogorov

A **probability space** (Kolmogorov, 1930's) is given by the following data:

- a set Ω (sample space),
- a collection $\mathcal{F}$ (event space),
- $\mathbb{P} : \mathcal{F} \to [0,1]$ (probability function),

satisfying several axioms.

Expectation: for every bounded random variable $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega).$$

Intuition: replace $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ by a more general pair $(\mathcal{A}, \varphi)$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

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Examples: $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, $\left(\mathrm{Mat}_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\right)$, $(\mathrm{Mat}_n(\Omega), \varphi)$,

$$\varphi(a) := \int_\Omega \mathrm{tr}(a(\omega))\, d\mathbb{P}(\omega)$$

Non-commutative probability space

Random variable: $a \in \mathcal{A}$

Moments: $(\varphi(a), \varphi(a^2), \varphi(a^3), \dots) \longleftrightarrow \mu : \mathbb{C}[x] \to \mathbb{C}, \mu(t^i) := \varphi(a^i)$

Join distribution of $(a_1, \dots, a_k)$: if $1 \leq i_1, \dots, i_n \leq k$,

$$\mu : \mathbb{C}\langle t_1, \dots, t_k \rangle \to \mathbb{C} \quad , \quad \mu(t_{i_1} \cdots t_{i_n}) := \varphi(a_{i_1} \cdots a_{i_n})$$

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In a (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the notion of independence between two random variables $X, Y : \Omega \to \mathbb{C}$ implies

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n).$$

Free independence

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The family $\{\mathcal{A}_i\}_{i \in I}$ of algebras is **freely independent** if for every $n \in \mathbb{N}$ and for every choice of $(i_1, \ldots, i_n)$ of "different neighbouring indices" (i.e., $i_{j-1} \neq i_j \neq i_{j+1}$), we have

$$\varphi(a_1 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{A}_{i_j}$ and $\varphi(a_j) = 0$, for every $1 \leq j \leq n$.

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Sets of variables in $(\mathcal{A}, \varphi)$ are free if the algebras they generate are free.

It looks artificial...

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$$\begin{aligned} 0 &= \varphi((a - \varphi(a) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})) \\ &= \varphi(ab) - \varphi(a \cdot 1_{\mathcal{A}})\varphi(b) - \varphi(a)\varphi(1_{\mathcal{A}} \cdot b) + \varphi(a)\varphi(b)\varphi(1_{\mathcal{A}}) \\ &= \varphi(ab) - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b) \end{aligned}$$

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Therefore, $\varphi(ab) = \varphi(a)\varphi(b)$.

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Let $(\mathcal{A}, \varphi)$ be a n.c.p.s. and let $\{a_1, a_2\}, \{b\} \subseteq \mathcal{A}$ free n.c.r.v.

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What is $\varphi(a_1 b a_2)$?

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What is $\varphi(a_1 b a_2)$? From

$$\varphi\Big((a_1 - \varphi(a_1) \cdot 1_{\mathcal{A}})(b - \varphi(b) \cdot 1_{\mathcal{A}})(a_2 - \varphi(a_2) \cdot 1_{\mathcal{A}})\Big) = 0,$$

we obtains

$$\varphi(a_1 b a_2) = \varphi(a_1 a_2)\varphi(b).$$

Free independence

Free independence provides a rule to compute mixed moments.

If $\{a_1, a_2\}, \{b_1, b_2\} \subseteq \mathcal{A}$ free n.c.r.v, what is $\varphi(abab)$?

$$\begin{aligned}\varphi(a_1 b_1 a_2 b_2) =& \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) \\ & - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2).\end{aligned}$$

$$\Rightarrow \varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2.$$

Freeness from the free product

Voiculescu gave the definition of freeness in the context of von Neumann algebras of free products of groups.

$$F(G) := \{\alpha : G \to \mathbb{C} : |\{g \in G \mid \alpha(g) \neq 0\}| < \infty\},$$

$$(\alpha * \beta)(g) := \sum_{h \in G} \alpha(gh^{-1})\beta(h),$$

$$\varphi_G : F(G) \to \mathbb{C} \qquad , \qquad \alpha \mapsto \alpha(e).$$

$F(G)$ is linearly generated by $\{\delta_g : g \in G\}$, where

$$\delta_g(h) = \begin{cases} 1, & h = g \\ 0, & h \neq g \end{cases}$$

Freeness from the free product

Theorem

If $\{G_i\}_{i \in I}$ *subgroups of* G *are algebraically free, then* $\{F(G_i)\}_{i \in I} \subseteq F(G)$ *are freely independent in* $(F(G), \varphi_G)$*.*

Sketch of the proof:
Consider $(i_1, \ldots, i_n) \in I^n$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$, and $\alpha_k \in F(G_{i_k})$ such that $\alpha_k(e) = 0$, for $1 \leq k \leq n$.

$$\begin{aligned} \varphi(\alpha_1 * \cdots * \alpha_n) &= (\alpha_1 * \cdots * \alpha_n)(e) \\ &= \sum_{\substack{g_1, \ldots, g_n \in G \\ g_1 \cdots g_n = e}} \alpha_1(g_1) \cdots \alpha_n(g_n). \end{aligned}$$

Since $G_{i_1}, \ldots, G_{i_n}$ are algebraically free. there exists k such that $g_k = e$, leading to $\varphi(\alpha_1 * \cdots * \alpha_n)$.

Non-commutative independences

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$. Let $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that $i_j \neq i_{j+1}$.

The family $\{\mathcal{A}_i\}_{i \in I}$ is

- **freely independent** if
$$\varphi(a_1 \cdots a_n) = 0,$$
when $\varphi(a_j) = 0$, for all $1 \leq j \leq n$;
- **boolean independent** if
$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n);$$
- **monotone independent** if
$$\varphi(a_1 \cdots a_n) = \varphi(a_j)\varphi(a_1 \cdots a_{j-1} \cdot a_{j+1} \cdots a_n),$$

Other notions: *conditional monotone*, *cyclic monotone*, ...

Back to the examples (free case)

$$\begin{aligned}
\varphi(ab) &= \varphi(a)\varphi(b) \\
\varphi(a_1 b a_2) &= \varphi(a_1 a_2)\varphi(b) \\
\varphi(a_1 b_1 a_2 b_2) &= \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) \\
&\quad - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2) \\
\varphi(a_1 b_1 c b_2 a_2 d a_3) &= \varphi(a_1 a_2 a_3)\varphi(b_1 b_2)\varphi(c)\varphi(d).
\end{aligned}$$

"Non-crossing moments" factorize; "crossing moments" don't factorize.

Back to $(\mathcal{A}, \varphi)$

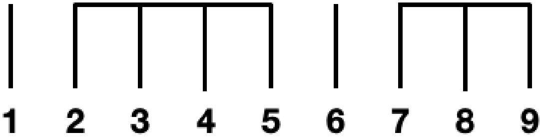
Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathcal{A}$.

Consider $\{f_n : \mathcal{A}^n \to \mathbb{C} \,|\, n \geq 0\}$ a family of multilinear functionals.

Let $\pi = \{B_1, \ldots, B_k\} \in NC(n)$. We define

$$f_\pi(a_1, \ldots, a_n) := \prod_{\substack{B \in \pi \\ B = \{b_1 < b_2 < \cdots < b_r\}}} f_{|B|}(b_1, b_2, \ldots, b_r).$$

Back to $(\mathcal{A}, \varphi)$



If $\pi = \{\{1\}, \{2,3,4,5\}, \{6\}, \{7,8,9\}\}$, then

$$f_\pi(a_1, \ldots, a_9) = f_1(a_1)\, f_4(a_2, a_3, a_4, a_5)\, f_1(a_6)\, f_3(a_7, a_8, a_9).$$

Moment to cumulant relations in $(\mathcal{A}, \varphi)$

Consider the multilinear functionals

$\{r_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Free cumulants) , $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Boolean cumulants) , $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Monotone cumulants)

defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} r_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_{\mathrm{Int}}(n)} b_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{\tau(\pi)!} h_\pi(a_1, \ldots, a_n).$$

Hopf algebras

Saj-nicole A. Joni and
Gian-Carlo Rota (1932-1999)

- Classical Hopf algebras: Borel, Cartier, Hopf (1940-1950).
- Motivation: algebraic topology, homological algebra, study of loop spaces, algebras of operations (Steenrod), homology of Eilenberg–MacLane spaces.

- Joni-Rota: "*A great many problems in combinatorics are concerned in assembling, or disassembling, large objects out of pieces of prescribed shape, as in the familiar board puzzles.*"

Hopf algebras

A **Hopf algebra** $(H, m, \iota, \Delta, \varepsilon, S)$ consists of

- an associative algebra (H, m, ι);
- a coassociative coalgebra (H, Δ, ε);
- compatibility between the product and the coproduct;
- the identity map $\mathrm{id} : H \to H$ is invertible in the **convolution algebra** $(\mathrm{End}(H), *)$, where

$$f * g := \Delta \circ (f \otimes g) \circ m.$$

The inverse of id, denoted by S, is called *the antipode of* H.
Finding an optimal formula for the antipode is not easy. It provides a rich information about hidden combinatorial structures on H.

Double tensor Hopf algebra

Double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$: non-commutative and non-cocommutative Hopf algebra, with graduation

$$\mathsf{T}(\mathsf{T}_+(V))_n := \bigoplus_{n_1+\cdots+n_k=n} V^{\otimes n_1} \otimes \cdots \otimes V^{\otimes n_k}.$$

Elements in $\mathsf{T}(\mathsf{T}_+(V))_n$ are written as (linear combinations of) words with bars

$$w_1|\cdots|w_k,$$

where $w_i \in V^{\otimes n_i}$ for some $n_1 + \cdots + n_k = n$. We call this elements **words on (non-empty) words**.

Double tensor Hopf algebra

Let V be a $\mathbb{K}$-vector space.

If $k \geq 0$, we write elementary tensors from $V^{\otimes k}$ as **words**, $u_1 u_2 \cdots u_k$, with $u_i \in V$. We called the $\mathbb{K}$-vector spaces

$$\mathsf{T}(V) := \bigoplus_{k \geq 0} V^{\otimes k} \quad , \quad \mathsf{T}_+(V) := \bigoplus_{k \geq 1} V^{\otimes k}$$

the **tensor module** and **reduced tensor module**, respectively, generated by V.

- *Product rule*: if $u \in T(T_+(V))_n$ and $v \in_m$, then

$$u|v := u_1|\cdots|u_r|v_1|\cdots|v_s \in T(T_+(V))_{n+m}.$$

- *Coproduct rule*: given a word $u = u_1 \cdots u_n \in V^{\otimes n}$ and $A = \{a_1, \dots, a_k\} \subset \mathbb{N}$, we write $u_A := u_{a_1} \cdots u_{a_k}$.
Consider the map $\Delta : T_+(V) \to T(V) \otimes T(T_+(V))$ given by

$$\begin{aligned}\Delta(u) : &= \sum_{A \subseteq [n]} u_A \otimes u_{K(A,[n])} \\ &= \sum_{A \subseteq [n]} u_A \otimes u_{K_1}|\cdots|u_{K_r}.\end{aligned}$$

Finally, we extend the map Δ multiplicatively to all of $T(T_+(V))$, by setting

$$\Delta(w_1|\cdots|w_k) := \Delta(w_1)\cdots\Delta(w_k).$$

For example, we have

$$\Delta(abc) = 1 \otimes abc + a \otimes bc + \mathbf{b} \otimes \mathbf{a}|\mathbf{c} + c \otimes ab + ab \otimes c + ac \otimes b + bc \otimes a + 1 \otimes abc;$$

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Algebraic approach to cumulants (Ebrahimi-Fard, Patras)

- $(\mathcal{A}, \varphi)$ non-commutative probability space.
- $H = T(T_+(\mathcal{A}))$ *words on non-empty words on* $\mathcal{A}$.
- The coproduct Δ in H is *codendriform*: $\Delta = \Delta_< + \Delta_>$.
- The space $(\mathrm{Hom}_{\mathrm{lin}}(H, \mathbb{K}), <, >)$ is a dendriform algebra, with $* =< + >$.
- The linear form φ is extended to $T_+(\mathcal{A})$ by defining to all words $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$

$$\varphi(a_1 a_2 \cdots a_n) := \varphi(a_1 \cdot_{\mathcal{A}} a_2 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n).$$

This is the **multivariate moment** of u.

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  This is the **multivariate moment** of u.
  The map φ is then extended multiplicatively to a map $\Phi : T(T_+(\mathcal{A})) \to \mathbb{K}$ with $\Phi(\mathbf{1}) := 1$ and

$$\Phi(u_1 | \cdots | u_k) := \varphi(u_1) \cdots \varphi(u_k).$$

Cumulants as infinitesimal characters

Proposition (Ebrahimi-Fard, Patras -2015)

Let $\rho, \kappa, \beta \in \mathfrak{g}(\mathcal{A})$ *the infinitesimal characters solving*

$$\Phi = \exp_*(\rho),$$

$$\Phi = \epsilon + \kappa \prec \Phi$$

and

$$\Phi = \epsilon + \Phi \succ \beta.$$

Then, ρ, κ, β *correspond to the* ***monotone cumulants, free cumulants and boolean cumulants****, respectively.*

For any word $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$, we have

$$h_n(a_1, \ldots, a_n) = \rho(u), r_n(a_1, \ldots, a_n) = \kappa(u), b_n(a_1, \ldots, a_n) = \beta(u).$$

Characters

The set of group-like elements $\mathrm{G}(V) \subset \mathcal{L}_V$ forms a group with respect to the convolution $*$. The inverse of an element $\Phi \in \mathrm{G}(V)$ is

$$\Phi^{-1} = \Phi \circ S.$$

The set $\mathfrak{g}(V) \subset \mathcal{L}_V$ of **infinitesimal characters** forms a Lie algebra with Lie bracket defined by the commutator in $\mathcal{L}_V$.

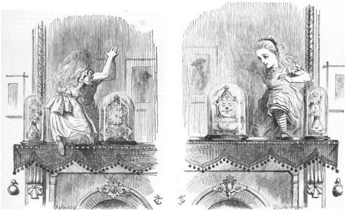
Inversion formulas

Proposition (Ebrahimi-Fard, Patras (2018))

The free cumulant κ *and boolean cumulant* β *satisfy the relations*

$$\kappa = (\Phi - \epsilon) \prec \Phi^{-1} \ \textit{and}\ \beta = \Phi^{-1} \succ (\Phi - \epsilon).$$

"We can look at κ and β through the inversion formula $\Phi^{-1} = \Phi \circ S$."



Antipode formula for the double tensor algebra

The Takeuchi's formula for the antipode

$$S(w) = \sum_{k\geq 0} (-1)^k \,|^{(k-1)} \circ (\mathrm{id} - \iota\varepsilon)^{\otimes k} \circ \Delta^{(k-1)}(w),$$

where $|^{-1} := \iota$ and $\Delta^{(-1)} := \varepsilon$, may contains several cancellations ($S(a|bcd)$ contains 75 terms, which reduces to 11 after cancellation).

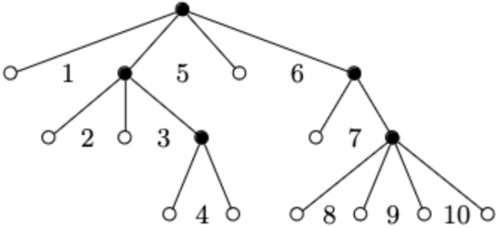
The following result helps to efficiently determines the antipode of $\mathsf{T}(\mathsf{T}_+(V))$.

Theorem (Celestino - V.)

Let $w = u_1 u_2 \cdots u_n \in V^{\otimes n}$. The action of the antipode over u is given by the following cancellation-free and grouping-free formula:

$$S(w) = \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)} w_t,$$

where $\mathrm{Sch}(n)$ is the set of Schroder trees with $n+1$ leaves.



$w_t = 156|23|4|7|89\underline{10}$

Proposition (Josuat-Vergès, Menous, Novelli,Thibon /Arizmendi, Celestino /Celestino - V.)

Let $(\mathcal{A}, \varphi)$ *be a non-commutative probability space and* $\{k_n\}_{n\geq 1}$ *be its free cumulants. Then, for any* $a_1, \ldots, a_n \in (\mathcal{A}$ *we have:*

$$k_n(a_1, \ldots, a_n) = \sum_{t \in PSch(n)} (-1)^{i(t)-1} \varphi_{\pi(t)}(a_1, \ldots, a_n).$$

If $\{b_n\}_{n\geq 1}$ *are the Boolean cumulants, then*

$$b_n(a_1, \ldots, a_n) = \sum_{t \in BSch(n)} (-1)^{i(t)-1} \varphi_{\pi(t)}(a_1, \ldots, a_n).$$

Species

André Joyal, Alain Connes, Olivia Caramello and Laurent Lafforgue, IHES (2015)

The theory of *combinatorial species* was introduced by André Joyal in 1980. Species can be seen as a *categorification* of generating functions. It provides a categorical foundation for enumerative combinatorics.

Species

A **set-species** is a functor

$$\mathsf{p} : \mathsf{set}^{\times} \longrightarrow \mathsf{set}.$$

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The **Cauchy product** of two species p and q is given by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

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The category of species is symmetric monoidal.
We can speak of monoids, comonoids, ..., in species.

$$\mathsf{h}[S] \otimes \mathsf{h}[T] \xrightarrow{\mu_{S,T}} \mathsf{h}[I] \qquad \mathsf{h}[I] \xrightarrow{\Delta_{S,T}} \mathsf{h}[S] \otimes \mathsf{h}[T].$$

Examples of species

- Species E of **sets**:

$$E[I] := \mathbb{K}\{*_I\}.$$

- Species E_n of n-**sets**:

$$E_n[I] := \begin{cases} \mathbb{K}\{*_I\}, & \text{if } |I| = n; \\ (0), & \text{if } |I| \neq n. \end{cases}$$

- Species $X := E_1$ of sets of one element.
- Species Π of **partitions**.
- Species L of **linear orders**.
- Species G of **graphs**:

$$G[I] := \mathbb{K}\{\text{ finite graphs with vertices in } I\}.$$

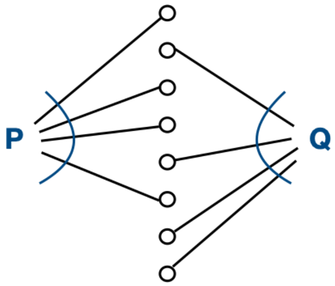
Operations on species

- **Sum of species**

$$(p + q)[I] := p[I] \oplus q[I].$$

- **Product of species** (Cauchy product)

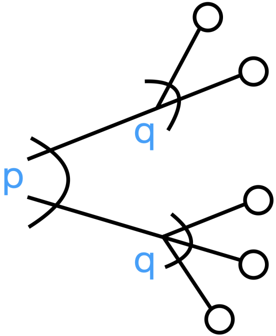
$$(p \cdot q)[I] := \bigoplus_{I = S \sqcup T} p[S] \otimes q[T].$$



Operations on species

- **Composition of species**

$$(p \circ q)[I] := \bigoplus_{\pi \in \Pi[I]} p[\pi] \otimes \bigotimes_{B \in \pi} q[B].$$



Generating function of a species

To every species p it is associated its **exponential generating function**:

$$\mathsf{p}(x) := \sum_{n \geq 0} \dim_{\mathbb{K}} \mathsf{p}[n] \frac{x^n}{n!}.$$

We have:

$$(\mathsf{p} + \mathsf{q})(x) = \mathsf{p}(x) + \mathsf{q}(x),$$

$$(\mathsf{p} \cdot \mathsf{q})(x) = \mathsf{p}(x) \cdot \mathsf{q}(x),$$

$$(\mathsf{p} \circ \mathsf{q})(x) = \mathsf{p}(x) \circ \mathsf{q}(x).$$

For the last identity, $\mathsf{q}[\emptyset] := (0)$.

Cumulants from Hopf monoids (Aguiar-Mahajan)

Let h be a species.
The n-th cumulant of h is

$$k_n(h) = \sum_{\pi \vdash I} \mu(\{I\}, \pi) \dim_{\mathbb{k}} h(\pi),$$

where $h(\pi) := \bigotimes_{B \in \pi} h[B]$.

Species	Moments	Cumulants	Distribution
L *linear orders*	$n!$	$(n-1)!$	Exponential of par. 1
E *sets*	1	$\delta_{n,1}$	Dirac measure $\delta = 1$
Π *partitions*	Bell_n	1	Poisson of par. 1
Σ *ordered partitions*	$\mathrm{OrdBell}_n$	$\sum_{k \geq 1} k^n / 2^k$	Geometric of par. 1

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it is not evident that the integers $k_n(h)$ are non-negative.

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From the formula

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it is not evident that the integers $k_n(h)$ are non-negative.

Proposition (Aguiar-Mahajan)

For any finite-dimensional cocommutative connected bimonoid h, *the dimension of its primitive part is*

$$\dim_{\Bbbk} \mathcal{P}(h)[I] = k_{|I|}(h).$$

Free and boolean cumulants of h

The free cumulants of h are the integers $c_n(h)$ defined by

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Question: are these integers non-negative? What conditions on h?

The cumulant-to-moment formulas come from different notions of "connected structures" of combinatorial objects.

Theorem (V. - 2024)

Let p *be a positive species.*

- *if* $\mathsf{h} = \mathsf{E} \circ \mathsf{p}$, *then,* $k_{|I|}(\mathsf{h}) = \dim_{\Bbbk} \mathsf{p}[I]$*;*
- *if* $\mathsf{h} = \mathsf{E} \circ_{\mathsf{NC}} \mathsf{p}$, *then,* $c_{|I|}(\mathsf{h}) = \dim_{\Bbbk} \mathsf{p}[I]$*;*
- *if* $\mathsf{h} = \mathsf{E} \diamond \mathsf{p}$, *then,* $b_{|I|}(\mathsf{h}) = \dim_{\Bbbk} \mathsf{p}[I]$*.*

Work in progress

- An algebraic model for several notions of non-commutative independences was presented by Ebrahimi-Fard and Patras. It involves infinitesimal characters on a certain Hopf algebra.
- Understanding this approach in terms of species and algebraic structures in the monoidal category of species (monoids, comonoids, lie monoids, bimonoids) might give a better insight of the combinatorics behind moment-to-cumulant formulae.
- Universality of $\mathrm{E} \circ_{\mathrm{NC}} \mathrm{p}$ (analogue to the *free and cofree* monoid in species).
- Operadic notion using non-crossing composition (rigid and classic species).
- What's next?

Geometrical notion of independence(s)?

Polytope	Hopf monoid	Independence
Permutahedron	Π	Classical
Associahedron	F	Monotone
Cyclohedron	C	Conditional monotone
$\vdots$	$\vdots$	$\vdots$

Joint work with Cesar Ceballos, Adrián Celestino and Franz Lehner (ANR-FWF International Cooperation Project PAGCAP - *Beyond Permutahedra and Associahedra: Geometry, Combinatorics, Algebra, and Probability*).

¡Gracias!

Save the date!

“Recent Perspectives on Non-crossing Partitions through Algebra, Combinatorics, and Probability”, Feb. 17, 2025 — Feb. 21, 2025.