Alternating Sign- & Magog ⬠'s

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Overview:

1) Definitions

2) Main results

3) History

4) Sketch of proof

Def: (G. 2023) Let $n \in \mathbb{N}$, $0 \leq l \leq n-2 < r \leq 2n-3$.

An (n,l,r)-*alternating sign pentagon (ASP)* is an array of the following form:

$$
\begin{array}{cccccc}
a_{1,l+1} & a_{1,l+2} & \cdots & \cdots & a_{1,r-1} & a_{1,r} \\
a_{2,l+1} & a_{2,l+2} & \cdots & \cdots & a_{2,r-1} & a_{2,r} \\
\vdots & \vdots & & & \vdots & \vdots \\
 & & & \cdots & a_{2n-r-2,r-1} & a_{2n-r-2,r} \\
a_{l+1,l+1} & a_{l+1,l+2} & \cdots & \cdots & a_{2n-(r+1),r-1} & \\
 & a_{l+2,l+2} & \cdots & \iddots & & \\
 & & \ddots & a_{n,n} & &
\end{array}
$$

with $a_{i,j} \in \{0, \pm 1\}$ s.t.

i) the non-zero entries alternate in sign along rows and columns,

ii) the row sums are all equal to 1,

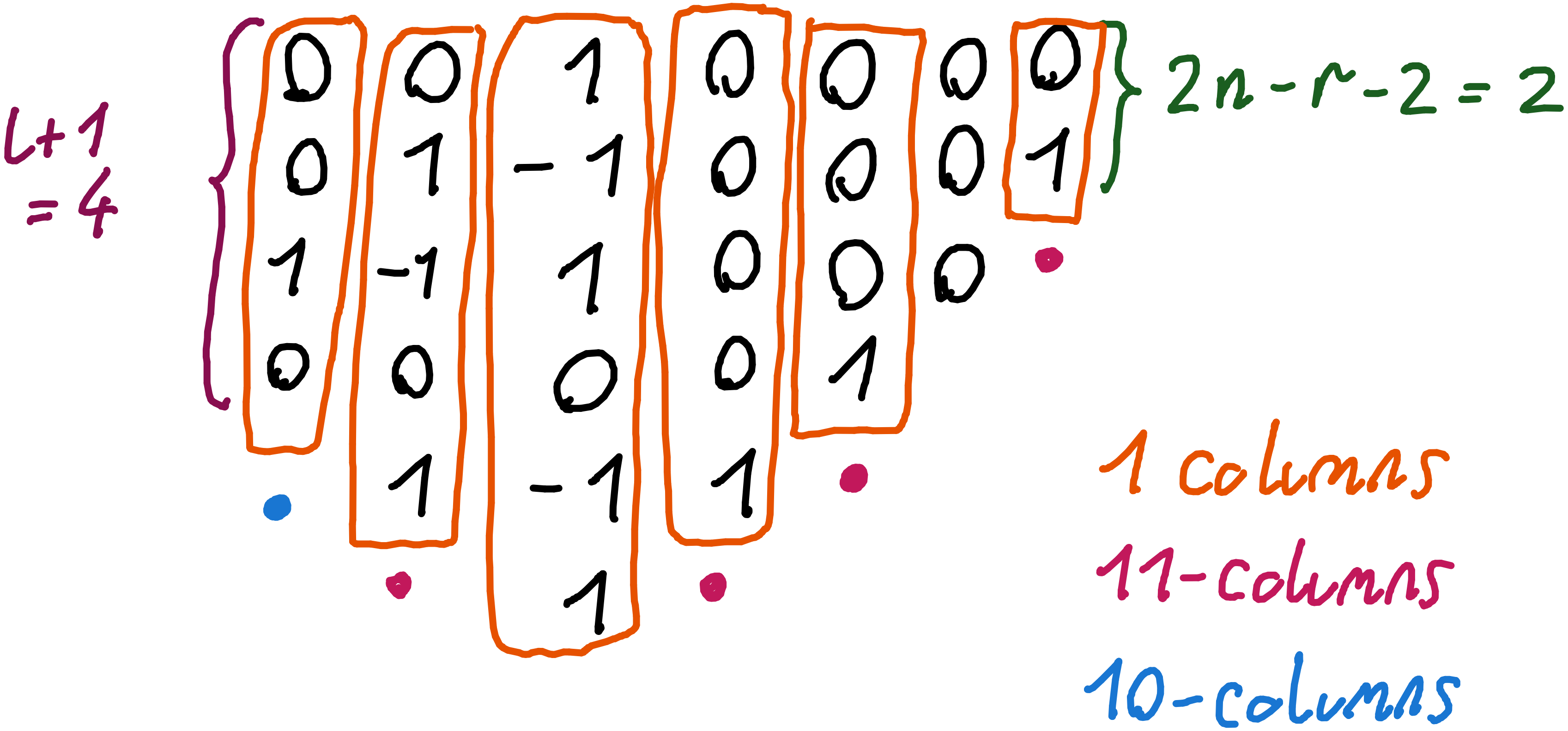
iii) the top most non-zero entry of each column is equal to 1 if it exists.

- A 1-column of an ASP is a column that sums up to 1.
- A 11-column is a 1-column with bottom entry 1.
- A 10-column is a 1-column with bottom entry 0.

We define for an (n,l,r)-ASP P the following statistic

$$\rho(P) := \#(\text{11-columns to the left of the central column of } P) + \#(\text{10-columns to the right of the central column of } P) + 1$$

Example: The following is a $(6,3,8)$-ASP with $\rho(P)=2$.



$\rho(P) :=$ #(11-columns to the left of the central column of P)

$+$ #(10-columns to the right of the central column of P) $+1 = 1 + 0 + 1 = 2$

Def: (G. 2023) Let $n \in \mathbb{N}$. An (n,k,l)-Magog pentagon is an array of positive integers consisting of the top k rows and the first l $\searrow$-diagonals, counted from top right to bottom left, of

$$\begin{array}{llll}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2\,n-1} \\
\vdots & \vdots & \iddots & \\
a_{n-1\,1} & a_{n-1\,2} & & \\
a_{n1} & & &
\end{array}$$

such that

i) entries along rows are weakly increasing,

ii) entries along columns are weakly decreasing,

iii) $\forall\, 1 \leq i \leq n:\ a_{1,i} \leq i$.

We define a statistic on an (n,k,l)-Magog pentagon M by

$$\tau(M) := n + \sum_{i=1}^{k} (a_{n-1,i} - a_{n,i}).$$

Example: A $(10, 4, 11)$-Magog pentagon with $\tau(M) = 5$ is displayed below

$$
\text{4 rows}\left\{
\begin{array}{cccccccccc}
1 & 2 & 2 & 4 & 5 & 6 & 7 & 7 & 8 & 9 \\
1 & 2 & 2 & 4 & 5 & 5 & 5 & 5 & 7 & \\
 & 2 & 2 & 4 & 4 & 4 & 4 & 5 & & \\
 & & 2 & 2 & 2 & 2 & 3 & & &
\end{array}
\right.
$$

1 2 3 4 5 6

$\downarrow$ - diagonals

11 10 9 8 7

$$\tau = 10 + (-1) + (-2) + (-1) + (-1) = 5$$

Thm (G. 2023) Let $n \in \mathbb{N}$, $1 \leq p \leq n$, $0 \leq \ell \leq n-2 < r \leq 2n-3$ such that $\ell + r < 2n-2$ and $r - \ell > n-3$ then the following sets are equinumerous:

- (n,ℓ,r)-ASPs P with $\varrho(P) = p$,
- $(n, 2n-3-r, 2n-3-\ell)$-ASPs T with $\varrho(T) = n+1-p$,
- $(n, r+2-n, r-\ell)$-Magog pentagons M with $\tau(M) = p$.

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Example: For $n=3, l=0, r=2$ we have:

$$\begin{array}{cccc} 1&0&0&0\\ &1&0&0\\ &&1& \end{array}\ \rho=3,\quad \begin{array}{cccc} 0&0&0&1\\ &1&0&0\\ &&1& \end{array}\ \rho=3,\quad \begin{array}{cccc} 0&0&1&0\\ &1&-1&1\\ &&1& \end{array}\ \rho=2,\quad \begin{array}{cccc} 1&0&0&0\\ &0&0&1\\ &&1& \end{array}\ \rho=2,\quad \begin{array}{cccc} 0&1&0&0\\ &0&0&1\\ &&1& \end{array}\ \rho=1$$

$$\begin{array}{cccc} 0&0&0&1\\ &0&0&1\\ &&1& \end{array}\ \rho=1,\quad \begin{array}{cccc} 1&0&0&0\\ &0&0&1\\ &&1& \end{array}\ \rho=1,\quad \begin{array}{cccc} 0&1&0&0\\ &1&-1&1\\ &&1& \end{array}\ \rho=2,\quad \begin{array}{cccc} 0&0&0&1\\ &1&0&0\\ &&1& \end{array}\ \rho=2,\quad \begin{array}{cccc} 0&0&1&0\\ &1&0&0\\ &&1& \end{array}\ \rho=3$$

$$\begin{array}{c} 1\ 1\\ \tau=3 \end{array},\quad \begin{array}{c} 2\ 2\\ \tau=3 \end{array},\quad \begin{array}{c} 2\ 3\\ \tau=2 \end{array},\quad \begin{array}{c} 1\ 2\\ \tau=2 \end{array},\quad \begin{array}{c} 1\ 3\\ \tau=1 \end{array}$$

Thm: (G. 2023) Let $n \in \mathbb{N}$, $1 \le p \le n$, $0 \le \ell \le n-2 < r \le 2n-3$ such that $\ell + r < 2n-2$ and $r - \ell > n-3$ then the generating function for the set of $(n, r+2-n, r-\ell)$-Magog pentagons w.r.t. τ is given by

$$t \cdot \mathrm{Pf}_{1 \le k_2 < k_1 \le n-1}\left[\sum_{\substack{e_1, e_2 = 1 \\ e_1 < e_2}}^{r+1} \det_{1 \le i,j \le 2}\left(t\binom{k_j - 1}{e_i - k_j} - t\binom{k_j - 1}{r - e_i - \ell + 2n - 1 - k_j} + \binom{k_j - 1}{e_i - 1 - k_j} - \binom{k_j - 1}{r - e_i - \ell + 2n - k_j}\right)\right]$$

if n is odd. If n is even we have to add an n-th column to the Pfaffian with entries

$$a_{j,n} := \sum_{e_1 = 1}^{r+1} t\binom{j-1}{e_1 - j} - t\binom{j-1}{r - e_1 - \ell - 2n + 1 - j} + \binom{j-1}{e_1 - 1 - j} - \binom{j-1}{r - e_1 - \ell + 2n - j}.$$

$n \times n$ - ASMs

Conj: RR 86

Proof: Zeilberger 96,
Kuperberg 97,
Fischer 2007

easy bijection

(n,n)-Gog trapezoids

$$\prod_{i=1}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

calc based proof:
Zeilberger 96
(no bijection yet!)

calc based
proof: Fischer
2018

(n,n)-Magog trapezoids

easy bijection

ASTs of order n

Proof: ABF 2016

TSSCPP's inside $2n \times 2n \times 2n$ box

Proof: Andrews 94

Arrow index: bijection, calculation, generalisation/refinement, conjecture

$n \times n$ - ASMs

(n,n)-Gog trapezoids

$n \times n$-ASMs M with $T_R(M) = n-1-k$, $T_L(M) = l-1$

(n,k,l)-Gog pentagons
BC 2016

BF2018

GOGAm pentagons
BC 2016

(n,l,r)-ASPs

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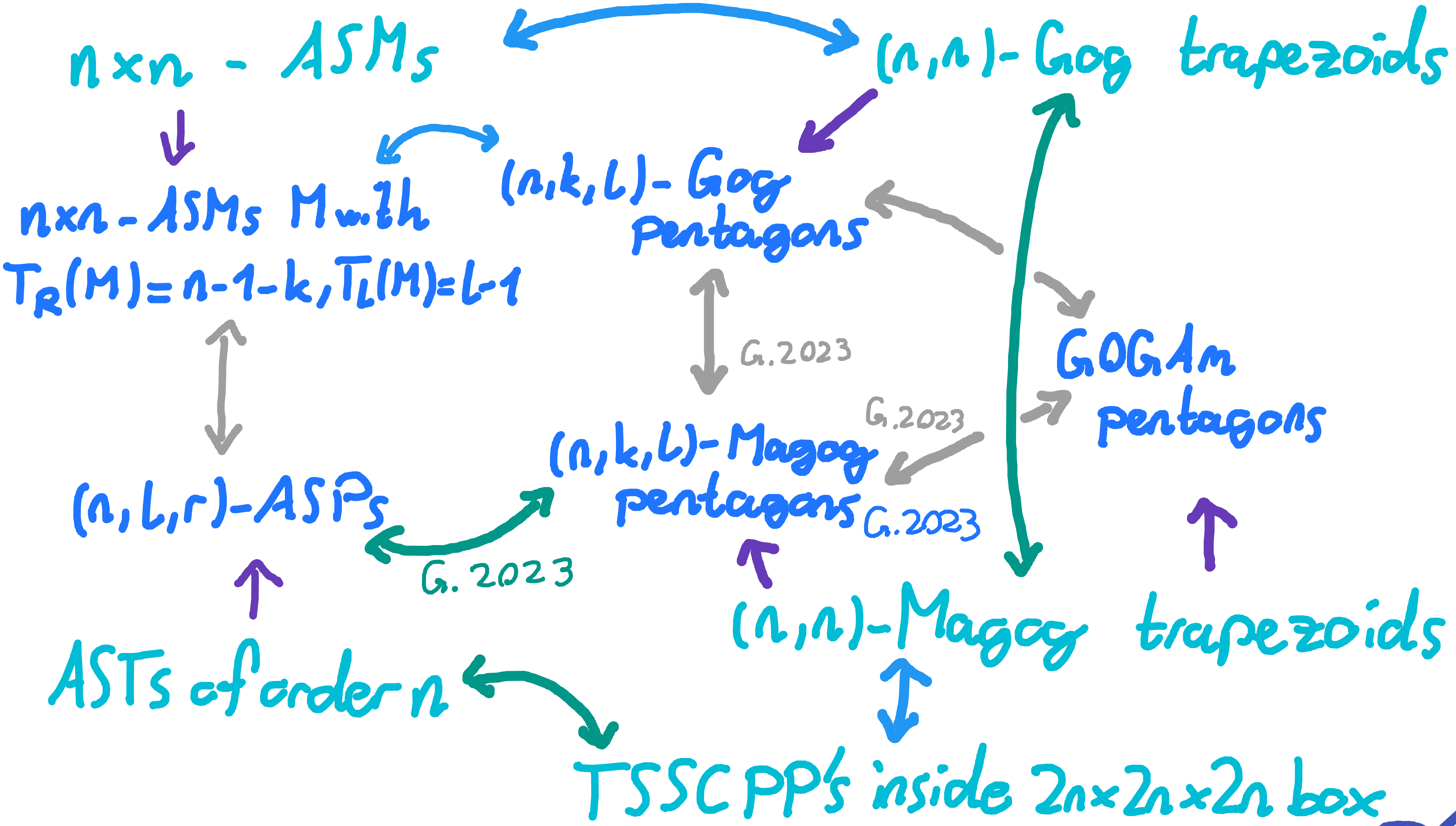

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Thm: (G. 2023) Let $n \in \mathbb{N}$, $1 \le p \le n$, $0 \le l \le n-2 < r \le 2n-3$ such that $l+r < 2n-2$ and $r-l > n-3$ then the generating function for the set of $(n, r+2-n, r-l)$-Magog pentagons w.r.t. τ is given by

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Thm: (Fischer 2018) Let $n \in \mathbb{N}$, $1 \le p \le n$. The number of ASTs T of order n with $\rho(T) = p$ and non-central 1 columns in positions $0 \le a_1 < \ldots < a_{n-1} \le 2n-3$ is given by the constant term w.r.t. $x_1, \ldots, x_{n-1}, t$ of:

$$t^{-p+1} \prod_{i=1}^{n-1} (t + x_i) x_i^{-a_i} \prod_{1 \le i < j \le n-1} (1 + x_i + x_i x_j)(x_j - x_i).$$

Cor: (G. 2023) Let $n \in \mathbb{N}$, $1 \le p \le n$, $0 \le l \le n-2 < r \le 2n-3$. The number of (n,k,l)-ASPs P with $\rho(P) = p$ is given by the constant term w.r.t. $x_1, \ldots, x_{n-1}, t$ of

$$t^{1-p} \sum_{l \le a_1 < \ldots < a_{n-1} \le r} \prod_{i=1}^{n-1} (t + x_i) x_i^{-a_i} \prod_{1 \le i < j \le n-1} (1 + x_i x_j)(x_j - x_i).$$

Goal: Evaluate constant term!

Define for a function $f(x_1,\dots,x_n)$:

$$\mathrm{Sym}_{x_1,\dots,x_n}\left[f(x_1,\dots,x_n)\right] := \sum_{\sigma\in S_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

$$\mathrm{ASym}_{x_1,\dots,x_n}\left[f(x_1,\dots,x_n)\right] := \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\, f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

Then show that previous CT is equal to CT w.r.t. $x_1,\dots,x_{n-1},t$ of

$$\frac{t^{1-p}}{(n-1)!}\prod_{i=1}^{n-1}(t+x_i)x_i^{-r}\prod_{1\le i<j\le n-1}(x_i-x_j)$$

$$\times\,\mathrm{ASym}_{x_1,\dots,x_{n-1}}\left[\prod_{1\le i<j\le n-1}(1+x_i+x_ix_j)\sum_{0\le k_1<k_2<\dots<k_{n-1}\le r-l}\ \prod_{i=1}^{n-1}x_i^{k_i}\right].$$

Thm: (Fischer 2022)

For $n \geq 1$ we have

$$\operatorname{ASym}_{x_1,\ldots,x_n}\left[\prod_{1\le i<j\le n}(1+wx_i+x_j+x_ix_j)\sum_{0\le k_1<\ldots<k_n\le m}\prod_{i=1}^{n}x_i^{k_i}\right]\cdot\prod_{1\le i<j\le n}(x_j-x_i)^{-1}$$

$$= \frac{\det_{1\le i,j\le n}\left(x_i^{j-1}(1+x_i)^{j-1}(1+wx_i)^{n-j}-x_i^{m+2n-j}(1+x_i^{-1})^{j-1}(1+wx_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)(x_j-x_i)}$$

and identity equivalent to "Littlewood identity" are some of the ingredients to show that $C_{\overline{r}}$ is given by

$$t^{1-p}\sum_{1\le e_1<\ldots<e_{n-1}\le r+1}\det_{1\le i,j\le n-1}\left[t\binom{j-1}{e_i-j}-t\binom{j-1}{r-e_i-l+2n-1-j}+\binom{j-1}{e_i-j-1}+\binom{j-1}{r-e_i-l+2n-j}\right]$$

Use tools from lattice path combinatorics to see that

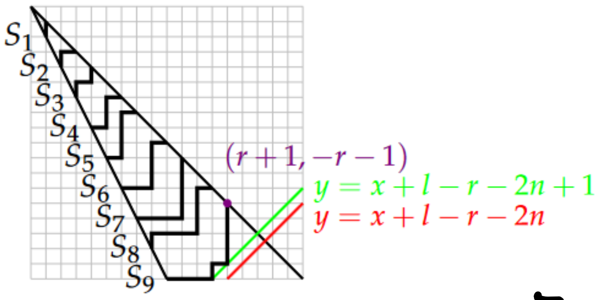
$$t^{1-p} \sum_{1 \le e_1 < \dots < e_{n-1} \le r+1} \det_{1 \le i,j \le n-1} \left[t\binom{j-1}{e_i - j} - t\binom{j-1}{r-e_i-l+2n-1-j} + \binom{j-1}{e_i-j-1} + \binom{j-1}{r-e_i-l+2n-j} \right]$$

is equal to GF of the weighted set $\mathcal{D}_n^{l,r}$ of all non-intersecting $n-1$ tuples of lattice paths consisting of north & east steps in the integer lattice with

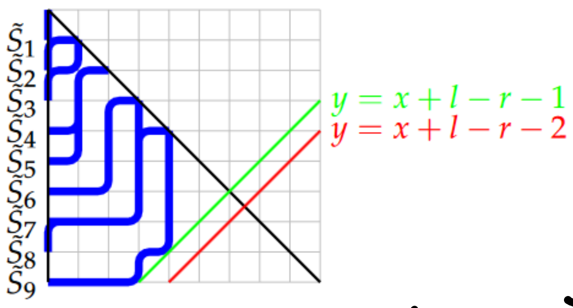
1) starting points $S_1 = (1,-2), \dots, S_{n-1} = (n-1, -2(n-1))$
2) end points in $E_1 = (1,-1), \dots, E_{r+1} = (r+1, -(r+1))$
3) the path starting at S_{n-1} staying weakly above $y = x + l - r - 2n + 1$

where a path is weighted by t if it ends with a north step.

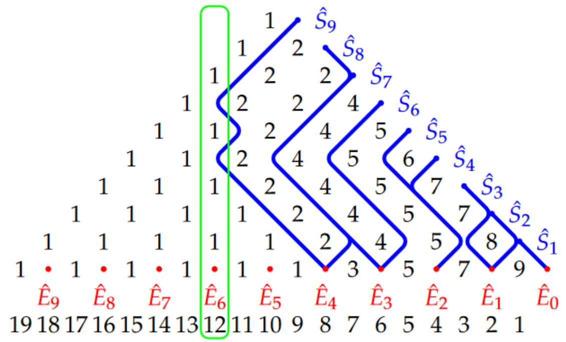
To obtain Magog pentagons do the following and carefully keep track of statistics and properties:



shift path at S_j by $(x,y)\to(x-j,y+j)$



rotate & fill in number j to the left of path at $\hat S_{n-j}$ starting with 1 up to n.



To obtain GF in terms of Pfaffian use:

Thm (Stembridge 1990) Let G be an acyclic, directed graph, $u_1,\dots,u_r\in V(G)$, $I\subseteq V(G)$ totally ordered s.t. for all $i<j\in[r]$, $v<v'$ in I any directed path from u_j to v in G intersects all directed paths from u_i to v' in G. Denote by $GF_0[u_1,..,u_r;I|G]=\sum_{(P_1,..,P_r)}\prod_{i=1}^{r} wt(P_i)$ where we sum over all r-tuples of non-intersecting lattice paths with starting points $u_1,..,u_r$ & end points in I in G. If r is even then

$$GF_0[u_1,..,u_r;I|G]=\mathrm{Pf}_{1\le i<j\le r}\left(GF_0[u_i,u_j;I|G]\right)$$

Thank you very much for your attention and the opportunity to give this talk!

Have a nice day!