Séminaire Lotharingien de Combinatoire **88** (2023), Article B88b

ON FINITE ANALOGS OF SCHMIDT'S PROBLEM AND ITS VARIANTS

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To George and Peter with appreciation of their friendship

Abstract. We refine Schmidt's problem and a partition identity related to 2-color partitions which we will refer to as Andrews–Paule–Uncu theorem. We approach the problem using Boulet–Stanley weights and a formula on Rogers–Szegő polynomials by Berkovich and Warnaar and present various Schmidt's problem alike theorems and their refinements. We study many variants of Schmidt's problem. In particular, we consider partitions with a bound on the largest part and uncounted odd-indexed parts. Our new Schmidt-type results include the use of even-indexed parts' sums, alternating sum of parts, and hook lengths as well as the odd-indexed parts' sum which appears in the original Schmidt's problem. We also translate some of our Schmidt's problem alike relations to weighted partition counts with multiplicative weights in relation to Rogers–Ramanujan partitions.

1. Introduction

A *partition* π is a non-increasing finite sequence $\pi = (\lambda_1, \lambda_2, \dots)$ of positive integers. The elements λ_i that appear in the sequence π are called *parts* of π. The *total number of parts* is denoted by $\#(\pi)$. For positive integers i, we call λ_{2i-1} odd-indexed parts, and λ_{2i} even-indexed parts of π. The sum of all the parts of a partition π is called the *size* of this partition and is denoted by $|\pi|$. We say π is a partition of n if its size is n. The empty sequence $\emptyset$ is considered as the unique partition of zero.

Let $\mathcal{D}$ be the set of partitions into distinct parts, and let $\mathcal{P}$ be the set of all the (ordinary) partitions. The bivariate generating functions for these sets, where the exponent of x is counting the number of parts and the exponent of q is keeping track of the size of the partitions are known to have infinite product representations. Explicitly

$$\sum_{\pi\in\mathcal{D}} x^{\#(\pi)} q^{|\pi|} = (-xq;q)_\infty \quad \text{and} \quad \sum_{\pi\in\mathcal{P}} x^{\#(\pi)} q^{|\pi|} = \frac{1}{(xq;q)_\infty},$$

where

$$(a;q)_l := \prod_{i=0}^{l-1} (1-aq^i), \quad \text{for} \quad l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

Date: December 28, 2023.

2010 *Mathematics Subject Classification.* 05A15, 05A17, 05A19, 11B34, 11B75, 11P81.

Key words and phrases. Partitions, q-series, Schmidt-type theorems, Boulet–Stanley weights, Rogers–Szegő polynomials, Sylvester's bijection.

Research of the second author is partly supported by EPSRC grant number EP/T015713/1 and partly by FWF grant P-34501N.

the standard q-Pochhammer symbol [3].

In 1999, although the original problem was submitted in 1997, F. Schmidt [18] shared his curious observations on a connection between partitions into distinct parts and ordinary partitions. For a given partition $\pi = (\lambda_1, \lambda_2, \dots)$, let

$$\mathcal{O}(\pi) := \lambda_1 + \lambda_3 + \lambda_5 + \cdots ,$$

be the *sum of odd-indexed parts*. Then, Schmidt observed the following.

Theorem 1.1 (Schmidt)**.** *For any integer n, we have*

$$|\{\pi \in \mathcal{D} : \mathcal{O}(\pi) = n\}| = |\{\pi \in \mathcal{P} : |\pi| = n\}|.$$

This result has been proven and rediscovered by many over the years. For example, P. Mork [17] gave a solution to Schmidt's problem, the second author gave an independent proof in 2018 [19] without the knowledge of Schmidt's problem, and recently Andrews and Paule [5] gave another proof of the same theorem in the last installation of their MacMahon's partition analysis series. The Andrews–Paule paper is particularly important because it led many new researchers into this area and many new proofs of Schmidt's theorem and its variants started appearing in a short period of time. Some examples that give novel proofs for Schmidt's problem are Alladi [2], Li and Yee [16], and Bridges joint with the second author [10].

The original way the second author discovered and proved Schmidt's problem in 2016 uses Boulet–Stanley weights [11]. While working on her doctorate under the supervision of R. Stanley, Boulet wrote a 4-parameter generalization of the generating functions for the number of partitions where, given a Ferrers diagram, they decorated the odd-indexed parts with alternating variables a and b (starting with a) and decorated the even-indexed parts with alternating c and d (starting with c). In this setting, instead of q keeping track of the size, we have a four-variable function $\omega_\pi(a, b, c, d)$ where each variable counts the number of respective variables in the decorated Ferrers diagram. For example, the 4-decorated Ferrers diagram of the partition $\pi = (12, 10, 7, 5, 2)$ and the respective weight function are given next.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline a & b & a & b & a & b & a & b & a & b & a & b \\
\hline c & d & c & d & c & d & c & d & c & d \\
\cline{1-10} a & b & a & b & a & b & a \\
\cline{1-7} c & d & c & d & c \\
\cline{1-5} a & b \\
\cline{1-2}
\end{array} \quad , \quad \omega_\pi(a, b, c, d) = a^{11} b^{10} c^8 d^7.$$

Boulet showed that the four-variable generating functions for the ordinary partitions and for partitions into distinct parts also have product representations.

Theorem 1.2 (Boulet)**.** *For variables a, b, c, and d and $Q := abcd$, we have*

$$\Psi(a, b, c, d) := \sum_{\pi \in D} \omega_\pi(a, b, c, d) = \frac{(-a, -abc; Q)_\infty}{(ab; Q)_\infty}, \tag{1.1}$$

$$\Phi(a, b, c, d) := \sum_{\pi \in \mathcal{P}} \omega_\pi(a, b, c, d) = \frac{(-a, -abc; Q)_\infty}{(ab, ac, Q; Q)_\infty}. \tag{1.2}$$

It is easy to check that with the trivial choice $a = b = c = d = q$, $\omega_\pi(q, q, q, q) = q^{|\pi|}$, the 4-parameter generating functions become the generating function for the number of partitions and number of partitions into distinct parts, respectively. The choice $(a, b, c, d) = (q, q, 1, 1)$, solves Schmidt's problem and yields another similar theorem of the sort for ordinary partitions.

Theorem 1.3 (Theorems 1.3 and 6.3 [19]).

$$\Psi(q, q, 1, 1) = \sum_{\pi\in\mathcal{D}} q^{\mathcal{O}(\pi)} = \frac{1}{(q; q)_\infty}, \tag{1.3}$$

$$\Phi(q, q, 1, 1) = \sum_{\pi\in\mathcal{P}} q^{\mathcal{O}(\pi)} = \frac{1}{(q; q)_\infty^2}. \tag{1.4}$$

This was the approach of the second author [19]. In the same paper, he also showed that Alladi's weighted theorem on Rogers–Ramanujan type partitions [1] is equivalent to Schmidt's problem.

Another independent proof of the right-most equality of (1.4) recently appeared in the paper [5] of Andrews and Paule, where they also solved Schmidt's problem. The combinatorial version of (1.4) is given as follows.

Theorem 1.4 (Andrews–Paule, Theorem 2 [5]). *For any integer n,*

$$|\{\pi \in \mathcal{P} : \mathcal{O}(\pi) = n\}| = |\{(\pi_1, \pi_2) \in \mathcal{P} \times \mathcal{P} : |\pi_1| + |\pi_2| = n\}|.$$

The number of partitions where the odd-indexed parts sum to n is equal to the number of 2-color partitions of n.

This discovery created an influx of research interests towards Theorem 1.4 (or equivalently (1.4)). Some examples of these new works proving, dissecting and refining (1.4) are [2, 4, 10, 12, 15, 16]. In some of these texts, (1.4) is referred to as the Andrews–Paule theorem, however considering the second author's earlier discovery and proof of this result [19, Theorem 6.3], we will refer to this result as the Andrews–Paule–Uncu theorem.

In this paper, we study Schmidt-type results systematically following the footsteps of [19]. We focus on different substitutions of Stanley–Boulet weights in Theorem 1.2 and a finite analog of it due to Ishikawa and Zeng (see Theorem 2.1 in Section 2). These substitutions lead to the observation of generating function relations in q-series notation [3]. Then we present combinatorial interpretations of these results. We also provide combinatorial proofs of some results using the Sylvester bijection.

In particular, this study leads to two refinements of the Andrews–Paule–Uncu theorem. Let $\mathcal{P}_{\leq N}$ be the set of partitions with parts $\leq N$, and let

$$\gamma(\pi) := \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots$$

be the *alternating sum of parts* of a given partition $\pi = (\lambda_1, \lambda_2, \dots)$. Then, one refinement of the Andrews–Paule–Uncu theorem with two parameters is as follows.

Theorem 1.5. *Let $U_{N,j}(n)$ be the number of partitions $\pi \in \mathcal{P}_{\leq N}$ such that $\mathcal{O}(\pi) = n$ and $\gamma(\pi) = j$ and let $T_{N,j}(n)$ be the number of 2-color partitions (red and green) of n*

with exactly j red parts and the largest green part at most $N-j$. Then,

$$U_{N,j}(n) = T_{N,j}(n).$$

Letting N tend to infinity and summing over all possible γ values, we obtain Theorem 1.4. We remark that a combinatorial proof and a refinement (different from Theorem 1.5) of the Andrews–Paule–Uncu theorem was found by Ji [15].

We also give various results involving the *even-indexed parts' sum* statistics

$$\mathcal{E}(\pi) := \lambda_2 + \lambda_4 + \lambda_6 + \cdots,$$

where $\pi = (\lambda_1, \lambda_2, \lambda_3, \dots)$. One such relation is given below.

Theorem 1.6. *The number of partitions π into distinct parts where $\mathcal{E}(\pi) = n$ and $\gamma(\pi) = j$ is equal to the number of partitions of n into parts $\leq j$.*

An example of this theorem with $n = 5$ and $j = 4$ is given in the next table.

$\mathcal{E}(\pi) = 5$ & $\gamma(\pi) = 4$	$\lvert\pi\rvert = 5$ & parts ≤ 4
$(9,5),\ (8,5,1),\ (7,5,2)$	$(4,1),\ (3,2),\ (3,1,1),$
$(7,4,2,1),\ (6,5,3),\ (6,4,3,1).$	$(2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$

The organization of this paper is as follows. In Section 2, we will recall refinements of the generating functions proved by Boulet, compare them with a result by Berkovich and Warnaar on Rogers–Szegő polynomials to get some weighted partition identities involving the alternating sum of parts and the sum of odd-indexed parts statistics. In Section 3, we study the refinements of Schmidt's problem and the Andrews–Paule–Uncu theorem, first in generating function form (see Theorems 3.1 and 3.2) using the results of Section 2. The rest of the results of Section 3 are the one and two parameter refinements of Schmidt's problem and the Andrews–Paule–Uncu theorem, both of which focus on the odd-indexed parts sum of partitions. We also provide a proof of Theorem 1.5. Section 4 is reserved for some new substitutions of Boulet–Stanley weights into the generating function relations presented in Sections 1 and 2. This will lead to the discovery of new Schmidt-type results which involve weighting partitions with respect to the total of the even-indexed parts. Most of this section has generating function connections; some combinatorial results (see Theorems 4.2 and 4.3) are provided to showcase the connection of the found generating function relations to partition identities. Section 5 has new Schmidt-type results that focus on adding even-indexed parts rather than the odd-indexed parts, including a proof of Theorem 1.6. Similar to Section 4, we first provide the generating function connections and then present a combinatorial partition identity as their corollary, see Theorem 5.5. Finally, Section 6 has the weighted counts connections between the Schmidt-type statistics and multiplicative weights that were earlier studied by Alladi [1] and by the second author [19].

2. Refinements for Boulet's generating functions and Rogers–Szegő polynomials

In [7] the authors made an extensive study on Boulet's results by imposing bounds on the largest part and the number of parts of the partitions. It should be noted that

Ishikawa and Zeng [14] were the first ones to present four-variable generating functions for distinct and ordinary partitions by imposing a single bound on the largest part of partitions. In [7], the authors gave a different representation of the singly bounded generating functions and also gave two doubly bounded generating functions. Later, Fu and Zeng [13] worked on the questions and techniques discussed in [6, 7], and they presented doubly bounded generating functions with uniform bounds for the 4-decorated Ferrers diagrams of ordinary partitions and of partitions into distinct parts.

Let $\mathcal{P}_{\leq N}$ and $\mathcal{D}_{\leq N}$ be the sets of partitions from the sets $\mathcal{P}$ and $\mathcal{D}$, respectively, with the extra bound N on the largest part. Define the generating functions

$$\Psi_N(a,b,c,d) := \sum_{\pi\in\mathcal{D}_{\leq N}} \omega_\pi(a,b,c,d), \tag{2.1}$$

$$\Phi_N(a,b,c,d) := \sum_{\pi\in\mathcal{P}_{\leq N}} \omega_\pi(a,b,c,d), \tag{2.2}$$

which are finite analogs of Boulet's generating functions for the weighted count of 4-variable decorated Ferrers diagrams. In [14], Ishikawa and Zeng wrote explicit formulas for (2.1) and (2.2) using Pfaffians and also by studying equations the related generating functions satisfy using combinatorial considerations.

Theorem 2.1 (Ishikawa–Zeng). *For a non-zero integer N, variables a, b, c, and d, we have*

$$\Psi_{2N+\nu}(a,b,c,d) = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{N-i+\nu}(-c;Q)_i (ab)^i, \tag{2.3}$$

$$\Phi_{2N+\nu}(a,b,c,d) = \frac{1}{(ac;Q)_{N+\nu}(Q;Q)_N} \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{N-i+\nu}(-c;Q)_i (ab)^i, \tag{2.4}$$

where $\nu\in\{0,1\}$ and $Q = abcd$.

In Theorem 2.1 and throughout the rest of the paper we use the standard definition [3] of the *q-binomial coefficients*:

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_{n+m}}{(q;q)_n(q;q)_m}, & \text{if } \ n,m\geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

A combinatorially constructed companion of Theorem 2.1 is given in [6].

Another useful connection is the first author and Warnaar's theorem [8] on the Rogers–Szegő polynomials

$$H_N(z,q) := \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q z^k.$$

Theorem 2.2 (Berkovich–Warnaar). *Let N be a non-negative integer. The Rogers–Szegő polynomials can be expressed as*

$$H_{2N+\nu}(zq,q^2)=\sum_{k=0}^{N}\begin{bmatrix}N\\k\end{bmatrix}_{q^4}(-zq;q^4)_{N-k+\nu}(-q/z;q^4)_k(zq)^{2k}, \tag{2.5}$$

where $\nu=0$ or 1.

We see that the right-hand sides of (2.3) and (2.5) coincide for a particular choice of (a,b,c,d):

$$H_N(zq,q^2)=\Psi_N(zq,zq,q/z,q/z). \tag{2.6}$$

In [7], the authors showed that

$$\Psi_N(zq,zq,q/z,q/z)=\sum_{\pi\in\mathcal{D}_{\leq N}}q^{|\pi|}z^{\gamma(\pi)}, \tag{2.7}$$

where $\gamma(\pi)=\lambda_1-\lambda_2+\lambda_3-\lambda_4+\cdots$, as before. Moreover, in the same paper the authors, using (2.6), also showed that the coefficient of the term z^k in (2.7) is $q^k\left[\begin{smallmatrix}N\\k\end{smallmatrix}\right]_{q^2}$.

Next, by first shifting z to zq and then mapping $q^2\mapsto q$, we obtain the following theorem.

Theorem 2.3. *For a non-negative integer N, we have*

$$\Psi_N(qz,qz,1/z,1/z)=\sum_{k=0}^{N}q^kz^k\begin{bmatrix}N\\k\end{bmatrix}_q, \tag{2.8}$$

$$\Phi_N(qz,qz,1/z,1/z)=\frac{\Psi_N(qz,qz,1/z,1/z)}{(q;q)_N}=\sum_{k=0}^{N}\frac{q^kz^k}{(q;q)_k(q;q)_{N-k}}. \tag{2.9}$$

3. Another look at Schmidt-type results and their refinements

We start by noting that the left-hand sides in Theorem 2.3 can be related to generating functions of some partitions.

Theorem 3.1. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}}q^{\mathcal{O}(\pi)}z^{\gamma(\pi)}=\sum_{k=0}^{N}q^kz^k\begin{bmatrix}N\\k\end{bmatrix}_q, \tag{3.1}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}}q^{\mathcal{O}(\pi)}z^{\gamma(\pi)}=\sum_{k=0}^{N}\frac{q^kz^k}{(q;q)_k(q;q)_{N-k}}. \tag{3.2}$$

Note that comparison of coefficients of z^j on both sides of (3.2) implies Theorem 1.5.

We also point out that these generating functions have alternate representations by Theorem 2.1.

Theorem 3.2. *Let N be an integer and $\nu = 0$ or 1. Then we have*

(3.3)
$$\Psi_{2N+\nu}(q,q,1,1) = \sum_{\pi\in\mathcal{D}_{\leq 2N+\nu}} q^{\mathcal{O}(\pi)} = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_{q^2} (-q;q^2)_{N-i+\nu}(-1;q^2)_i q^{2i},$$

(3.4)
$$\Phi_{2N+\nu}(q,q,1,1) = \sum_{\pi\in\mathcal{P}_{\leq 2N+\nu}} q^{\mathcal{O}(\pi)} = \frac{1}{(q;q)_{2N+\nu}} \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_{q^2} (-q;q^2)_{N-i+\nu}(-1;q^2)_i q^{2i}.$$

The right-hand sides of (3.3) and (3.4) are sum representations for the bounded analogs of Schmidt's problem (1.3) and the Andrews–Paule–Uncu theorem (1.4), respectively.

Equation (3.1) with $z = 1$ can be interpreted as the following combinatorial theorem.

Theorem 3.3. *Let $S_N(n)$ be the number of partitions into distinct parts $\pi = (\lambda_1, \lambda_2, \dots)$ with $\mathcal{O}(\pi) = n$ and $\lambda_1 \leq N$. Let $\Gamma_N(n)$ be the number of partitions of n where the largest hook length is $\leq N$. Then,*

$$S_N(n) = \Gamma_N(n).$$

Recall that, given a Ferrers diagram, the *hook length* of a box of the diagram is defined as one plus the number of boxes directly to the right and directly below the chosen box. It is then easy to see that the largest hook length is the hook length of the top-left-most box in a Ferrers diagram. Thus, we could also define $\Gamma_N(n)$ as the number of partitions of n where the quantity "the number of parts plus the largest part minus 1" is less than or equal to N.

We exemplify Theorem 3.3 with $n = N = 4$ and list the partitions counted by $S_N(n)$ and $\Gamma_N(n)$ below.

$S_4(4) = 5$	$\Gamma_4(4) = 5$
$(4,3)$	(4)
$(4,2)$	$(3,1)$
$(4,1)$	$(2,2)$
(4)	$(2,1,1)$
$(3,2,1)$	$(1,1,1,1)$

Notice that, whereas most partitions in this example have their largest hook lengths to be the upper limit 4, the partition $(2,2)$ has the largest hook length $3 < 4$.

The proof of Theorem 3.3 only requires us to interpret (3.1) with $z = 1$ as the generating function for the $\Gamma_N(n)$ numbers. To that end, we focus on the summand of the right-hand side in (3.1) with $z = 1$. Recall that the q-binomial coefficient $\left[\begin{smallmatrix} N \\ i \end{smallmatrix}\right]_q$ is the generating function of partitions that fit in a box with i rows and $N - i$ columns. We put a column of height i in front of that box. There are exactly i boxes in the leftmost column of this augmented Ferrers diagram and up to $N - i$ boxes in the topmost row. Hence, the largest hook length of this partition is $\leq N$. By summing over all possible i, we cover all possible column heights. This shows that the right-hand side of (3.1) with $z = 1$ is the generating function for $\Gamma_N(n)$ where q is keeping track of the size of the partitions.

A similar weighted theorem comes from comparing the known interpretation of (3.2) with $z = 1$.

Theorem 3.4. *Let $U_N(n)$ be the number of partitions $\pi = (\lambda_1, \lambda_2, \dots)$ with $\mathcal{O}(\pi) = n$ and $\lambda_1 \leq N$. Let $T_N(n)$ be the number of 2-color partitions (red and green) of n such that the number of parts in red plus the size of the largest green part does not exceed N. Then,*

$$U_N(n) = T_N(n).$$

We give an example of this theorem with $n = 4$ and $N = 3$. We will use subscripts r and g to indicate the colors of the parts of partitions while listing the partitions related to $T_3(4)$.

$U_3(4) = 15$		$T_3(4) = 15$	
$(3, 3, 1, 1)$,	$(3, 3, 1)$,	(4_r)	$(3_r, 1_r)$
$(3, 2, 1, 1)$,	$(3, 2, 1)$,	$(3_r, 1_g)$	$(3_g, 1_g)$
$(3, 1, 1, 1)$,	$(3, 1, 1)$,	$(2_r, 2_g)$,	$(2_g, 2_g)$,
$(2, 2, 2, 2)$,	$(2, 2, 2)$,	$(2_r, 1_r, 1_r)$,	$(2_r, 1_r, 1_g)$,
$(2, 2, 1, 1, 1, 1)$,	$(2, 2, 1, 1, 1)$,	$(2_r, 1_g, 1_g)$,	$(2_g, 1_r, 1_g)$,
$(2, 2, 2, 1)$	$(2, 1, 1, 1, 1, 1)$,	$(2_g, 1_g, 1_g)$,	$(1_r, 1_r, 1_r, 1_g)$
$(2, 1, 1, 1, 1)$,	$(1, 1, 1, 1, 1, 1, 1, 1)$,	$(1_r, 1_r, 1_g, 1_g)$,	$(1_r, 1_g, 1_g, 1_g)$,
$(1, 1, 1, 1, 1, 1, 1, 1)$.		$(1_g, 1_g, 1_g, 1_g)$.	

Note that the count of $T_N(n)$ is not symmetric in colors red and green. In our example, partitions such as (4_g), $(2_g, 1_r, 1_r)$ and $(1_r, 1_r, 1_r, 1_r)$ are not counted.

The equation (3.1) can also be presented as a refined Schmidt-type-problem-like result.

Theorem 3.5. *Let $S_{N,j}(n)$ be the number of partitions $\pi \in \mathcal{D}_{\leq N}$ such that $\mathcal{O}(\pi) = n$ and $\gamma(\pi) = j$ and let $\Gamma_{N,j}(n)$ be the number of partitions into exactly j parts where the largest hook length is $\leq N$. Then,*

$$S_{N,j}(n) = \Gamma_{N,j}(n).$$

We give examples of Theorems 3.5 and 1.5 in the next table.

$S_{4,2}(4) = 2$	$\Gamma_{4,2}(4) = 2$	$U_{3,1}(4) = 6$	$T_{3,1}(4) = 6$
$(4, 2)$	$(3, 1)$	$(3, 3, 1)$	(4_r)
$(3, 2, 1)$	$(2, 2)$	$(3, 2, 1, 1)$	$(3_r, 1_g)$
		$(2, 2, 1, 1, 1)$	$(2_r, 1_g, 1_g)$
		$(2, 2, 2, 1)$	$(2_r, 2_g)$
		$(2, 1, 1, 1, 1, 1)$	$(1_r, 1_g, 1_g, 1_g)$
		$(1, 1, 1, 1, 1, 1, 1)$	$(2_g, 1_r, 1_g)$

The proof of this result comes from comparing and interpreting the coefficients of z^j in (3.1). Similarly, we can get an analogous result for ordinary partitions by comparing the z^j terms in (3.2), which yields Theorem 1.5.

We can also prove Theorem 1.5 using the Sylvester bijection [9]. We know that $q^j \begin{bmatrix} N \\ j \end{bmatrix}_q$ is the generating function for the number of partitions with parts $\leq N-j+1$ and the number of parts exactly j. Let π be one of such partitions into j parts where its largest part is $\leq N-j+1$ (hence, the maximum hook length $\leq N$). Next, we construct a partition π_o into j parts by first drawing the Ferrers diagram of the partition π, and then gluing this Ferrers diagram's reflection π^* excluding the largest column on the left of the original Ferrers diagram. When it is read line-by-line, this is a partition into odd parts, and $|\pi_o| = 2|\pi| - j$. The $-j$ comes from the ignored largest column in the reflection. This construction from π to π_o is bijective and can be easily reversed. Now we apply the Sylvester bijection to π_o. This takes π_o to a partition into distinct parts π_d, where the largest part is $\leq N$. In the Sylvester bijection, the original Ferrers diagram corresponds to positive contribution towards the alternating sums of parts of π_d whereas the reflections' side always contributes negatively. Since there is only a column of j elements difference between these two diagrams, we see that $\gamma(\pi_d) = j$. Finally, one needs to observe that $\mathcal{O}(\pi_d) = |\pi|$. One example of this construction is given in Figure 1.

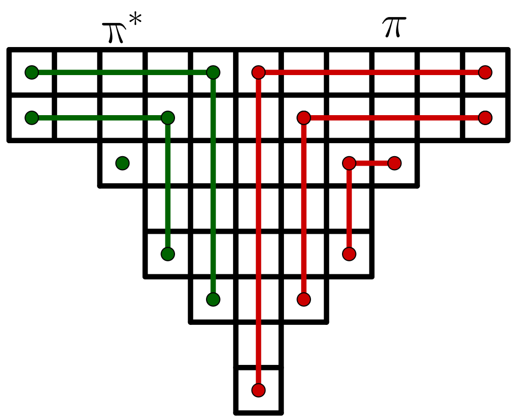

FIGURE 1. Example of the combinatorial proof of Theorem 3.5 when $N = 13$, $j = 8$, and $\pi = (5,5,3,2,2,1)$. Then the partition π_o is $(11,11,7,5,5,3,1,1)$ (read row by row). The partition we get after the Sylvester bijection is $\pi_d = (13,10,9,7,4,1)$. The red lines (right-bending lines) are the odd-indexed parts of π_d and the green lines (left-bending lines) are the even-indexed parts of π_d. It is clear that $\mathcal{O}(\pi_d) = |\pi|$ and that the $\gamma(\pi) = j = 8$.

Finally, letting N tend to infinity in Theorem 1.5 we arrive at [4, Corollary 2]. This corollary was implicit in Mork's original solution [17] of the Schmidt problem.

4. Schmidt-type results

For a given partition π, choices of variables in Boulet's and subsequently the bounded analogs of Boulet's theorem can give us great insight into Schmidt-type partition theorems. For example, by the choice $(a,b,c,d) = (q,q,-1,-1)$ in Theorem 1.2, we directly get the following theorem.

Theorem 4.1. *We have*

$$\sum_{\pi\in\mathcal{D}} (-1)^{\mathcal{E}(\pi)} q^{\mathcal{O}(\pi)} = (-q;q^2)_\infty, \tag{4.1}$$

$$\sum_{\pi\in\mathcal{P}} (-1)^{\mathcal{E}(\pi)} q^{\mathcal{O}(\pi)} = \frac{1}{(q^2;q^2)_\infty}. \tag{4.2}$$

Theorem 4.1 translates into the following two weighted combinatorial results.

Theorem 4.2. *The number of partitions π into distinct parts counted with weight $(-1)^{\mathcal{E}(\pi)}$ such that $\mathcal{O}(\pi) = n$ is equal to the number of partitions of n into distinct odd parts.*

Theorem 4.3. *The number of partitions π counted with weight $(-1)^{\mathcal{E}(\pi)}$ such that $\mathcal{O}(\pi) = n$ is equal to the number of partitions of n into even parts.*

The finite analogs of Boulet's theorem can directly be used to get refinements of these results.

Theorem 4.4. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}} (-1)^{\mathcal{E}(\pi)} q^{\mathcal{O}(\pi)} = (-q;q^2)_{\lceil N/2\rceil}, \tag{4.3}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}} (-1)^{\mathcal{E}(\pi)} q^{\mathcal{O}(\pi)} = \frac{1}{(q^2;q^2)_{\lfloor N/2\rfloor}}, \tag{4.4}$$

where $\lceil\cdot\rceil$ is the standard ceiling function.

With the choice $c=-1$, the sums in Theorem 2.1 collapse to the $i=0$ term only and this yields a direct proof of Theorem 4.4.

For example, let $N=4$. Then the explicit weighted count of the partitions related to (4.3) is as follows:

$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{E}(\pi)}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{E}(\pi)}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{E}(\pi)}q^{\mathcal{O}(\pi)}$
(4)	q^4	$(4,2,1)$	q^5	(2)	q^2
$(4,3)$	$-q^4$	$(4,3,2,1)$	q^6	$(2,1)$	$-q^2$
$(4,2)$	q^4	(3)	q^3	(1)	q
$(4,1)$	$-q^4$	$(3,2)$	q^3	$\emptyset$	1
$(4,3,2)$	$-q^6$	$(3,1)$	$-q^3$		
$(4,3,1)$	$-q^5$	$(3,2,1)$	q^4		

$$\Psi_4(q,q,-1,-1) = \sum_{\pi\in\mathcal{D}_{\leq 4}} (-1)^{\mathcal{E}(\pi)} q^{\mathcal{O}(\pi)} = q^4+q^3+q+1 = (1+q)(1+q^3) = (-q;q^2)_2.$$

We may also choose $(a,b,c,d) = (q,q,-1,1)$ and get another theorem.

Theorem 4.5.

$$\sum_{\pi\in\mathcal{D}}(-1)^{\lceil\mathcal{E}\rceil(\pi)}q^{\mathcal{O}(\pi)}=(-q;-q^2)_\infty=(-q,q^3;q^4)_\infty, \tag{4.5}$$

$$\sum_{\pi\in\mathcal{P}}(-1)^{\lceil\mathcal{E}\rceil(\pi)}q^{\mathcal{O}(\pi)}=\frac{1}{(-q^2;-q^2)_\infty}=\frac{1}{(-q^2,q^4;q^4)_\infty}, \tag{4.6}$$

where $\lceil\mathcal{E}\rceil(\lambda_1,\lambda_2,\lambda_3,\dots)=\lceil\lambda_2/2\rceil+\lceil\lambda_4/2\rceil+\lceil\lambda_6/2\rceil+\cdots$.

The refinement of Theorem 4.5 can also be found in a similar manner.

Theorem 4.6. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\le N}}(-1)^{\lceil\mathcal{E}(\pi)\rceil}q^{\mathcal{O}(\pi)}=(-q;-q^2)_{\lceil N/2\rceil}, \tag{4.7}$$

$$\sum_{\pi\in\mathcal{P}_{\le N}}(-1)^{\lceil\mathcal{E}\rceil(\pi)}q^{\mathcal{O}(\pi)}=\frac{1}{(-q^2;-q^2)_{\lfloor N/2\rfloor}}. \tag{4.8}$$

For example, (4.7) with $N=4$ counts the following partitions with the weights associated with them:

$\pi\in\mathcal{D}_{\le4}$	$(-1)^{\lceil\mathcal{E}(\pi)\rceil}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\le4}$	$(-1)^{\lceil\mathcal{E}(\pi)\rceil}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\le4}$	$(-1)^{\lceil\mathcal{E}(\pi)\rceil}q^{\mathcal{O}(\pi)}$
(4)	q^4	$(4,2,1)$	$-q^5$	(2)	q^2
$(4,3)$	q^4	$(4,3,2,1)$	$-q^6$	$(2,1)$	$-q^2$
$(4,2)$	$-q^4$	(3)	q^3	(1)	q
$(4,1)$	$-q^4$	$(3,2)$	$-q^3$	$\emptyset$	1
$(4,3,2)$	q^6	$(3,1)$	$-q^3$		
$(4,3,1)$	q^5	$(3,2,1)$	$-q^4$		

$$\begin{aligned}\Psi_4(q,q,-1,1)&=\sum_{\pi\in\mathcal{D}_{\le4}}(-1)^{\lceil\mathcal{E}(\pi)\rceil}q^{\mathcal{O}(\pi)}\\&=-q^4-q^3+q+1=(1+q)(1-q^3)=(-q;-q^2)_2.\end{aligned}$$

A modulo 3 related group of Schmidt-type results comes from the choice $(a,b,c,d)=(q,q,-1,-q)$.

Theorem 4.7. *We have*

$$\sum_{\pi\in\mathcal{D}}(-1)^{\mathcal{E}(\pi)}q^{\mathcal{O}(\pi)+\lfloor\mathcal{E}\rfloor(\pi)}=(-q;q^3)_\infty, \tag{4.9}$$

$$\sum_{\pi\in\mathcal{P}}(-1)^{\mathcal{E}(\pi)}q^{\mathcal{O}(\pi)+\lfloor\mathcal{E}\rfloor(\pi)}=\frac{1}{(q^3;q^3)_\infty}, \tag{4.10}$$

where $\lfloor\mathcal{E}\rfloor(\lambda_1,\lambda_2,\lambda_3,\dots)=\lfloor\lambda_2/2\rfloor+\lfloor\lambda_4/2\rfloor+\lfloor\lambda_6/2\rfloor+\cdots$.

Once again, a refinement of Theorem 4.7 similar to the refinements above can easily be found.

Theorem 4.8. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}} (-1)^{\lfloor\mathcal{E}(\pi)\rfloor} q^{\mathcal{O}(\pi)} = (-q;q^3)_{\lceil N/2\rceil}, \tag{4.11}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}} (-1)^{\lfloor\mathcal{E}\rfloor(\pi)} q^{\mathcal{O}(\pi)} = \frac{1}{(q^3;q^3)_{\lfloor N/2\rfloor}}. \tag{4.12}$$

We add one explicit example for (4.11) with $N=4$:

$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\lfloor\mathcal{E}(\pi)\rfloor}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\lfloor\mathcal{E}(\pi)\rfloor}q^{\mathcal{O}(\pi)}$	$\pi\in\mathcal{D}_{\leq 4}$	$(-1)^{\lfloor\mathcal{E}(\pi)\rfloor}q^{\mathcal{O}(\pi)}$
(4)	q^4	$(4,2,1)$	q^6	(2)	q^2
$(4,3)$	$-q^5$	$(4,3,2,1)$	q^7	$(2,1)$	$-q^2$
$(4,2)$	q^5	(3)	q^3	(1)	q
$(4,1)$	$-q^4$	$(3,2)$	q^4	$\emptyset$	1
$(4,3,2)$	$-q^7$	$(3,1)$	$-q^3$		
$(4,3,1)$	$-q^6$	$(3,2,1)$	q^5		

$$\begin{aligned}\Psi_4(q,q,-1,-q) &= \sum_{\pi\in\mathcal{D}_{\leq 4}} (-1)^{\lfloor\mathcal{E}(\pi)\rfloor} q^{\mathcal{O}(\pi)}\\ &= q^5+q^4+q+1 = (1+q)(1+q^4) = (-q;q^3)_2.\end{aligned}$$

In general, the choice of $c=-1$ in Theorem 1.2 always leads to the cancelation of terms in the products (1.1) and (1.2).

Theorem 4.9. *Let $(a,b,c,d)=(q^r,q^t,-1,\varepsilon q^s)$ for some non-negative integers r, t, s, (with $t+r>0$) and $\varepsilon=\pm 1$. Then we have*

$$\sum_{\pi\in\mathcal{D}} (-1)^{\lceil\mathcal{E}\rceil(\pi)} \varepsilon^{\lfloor\mathcal{E}\rfloor(\pi)} q^{r\lceil\mathcal{O}\rceil(\pi)+t\lfloor\mathcal{O}\rfloor(\pi)+s\lfloor\mathcal{E}\rfloor(\pi)} = (-q^r;\varepsilon q^{r+t+s})_\infty, \tag{4.13}$$

$$\sum_{\pi\in\mathcal{P}} (-1)^{\lceil\mathcal{E}\rceil(\pi)} \varepsilon^{\lfloor\mathcal{E}\rfloor(\pi)} q^{r\lceil\mathcal{O}\rceil(\pi)+t\lfloor\mathcal{O}\rfloor(\pi)+s\lfloor\mathcal{E}\rfloor(\pi)} = \frac{1}{(\varepsilon q^{r+t+s};\varepsilon q^{r+t+s})_\infty}, \tag{4.14}$$

where $\lceil\mathcal{O}\rceil(\pi)$ and $\lfloor\mathcal{O}\rfloor(\pi)$ are defined in a similar fashion as $\lceil\mathcal{E}\rceil(\pi)$ and $\lfloor\mathcal{E}\rfloor(\pi)$.

Similarly, the refinement of Theorem 4.9 is as follows.

Theorem 4.10. *Let $(a,b,c,d)=(q^r,q^t,-1,\varepsilon q^s)$ for some non-negative integers r, t, s, (with $t+r>0$) and $\varepsilon=\pm 1$. Then we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}} (-1)^{\lceil\mathcal{E}\rceil(\pi)} \varepsilon^{\lfloor\mathcal{E}\rfloor(\pi)} q^{r\lceil\mathcal{O}\rceil(\pi)+t\lfloor\mathcal{O}\rfloor(\pi)+s\lfloor\mathcal{E}\rfloor(\pi)} = (-q^r;\varepsilon q^{r+t+s})_{\lceil N/2\rceil}, \tag{4.15}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}} (-1)^{\lceil\mathcal{E}\rceil(\pi)} \varepsilon^{\lfloor\mathcal{E}\rfloor(\pi)} q^{r\lceil\mathcal{O}\rceil(\pi)+t\lfloor\mathcal{O}\rfloor(\pi)+s\lfloor\mathcal{E}\rfloor(\pi)} = \frac{1}{(\varepsilon q^{r+t+s};\varepsilon q^{r+t+s})_{\lfloor N/2\rfloor}}. \tag{4.16}$$

Theorems 4.1, 4.5 and 4.7 are special cases of Theorem 4.9. Similarly, Theorems 4.4, 4.6 and 4.8 are special cases of Theorem 4.10.

5. Partition identities for uncounted odd-indexed parts

In Sections 3 and 4, we made sure that our substitutions to Boulet weights are never $a = b = 1$ or $a = b = -1$. These choices clearly introduce a pole to the product representations of the generating functions (1.1) and (1.2). However, these substitutions can be entertained in the finite analog of these generating functions.

For example, substituting $a = b = -1$ and $c = d = q$ in (2.3) and (2.4) yields the two equations presented in the following theorem, respectively.

Theorem 5.1. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}} (-1)^{\mathcal{O}(\pi)} q^{\mathcal{E}(\pi)} = \begin{cases} (-q;q^2)_{N/2}, & \text{if } N \text{ is even,} \\ 0, & \text{if } N \text{ is odd.} \end{cases} \tag{5.1}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}} (-1)^{\mathcal{O}(\pi)} q^{\mathcal{E}(\pi)} = \begin{cases} \dfrac{1}{(q^2;q^2)_{N/2}}, & \text{if } N \text{ is even,} \\ 0, & \text{if } N \text{ is odd.} \end{cases} \tag{5.2}$$

We list partitions and their respective weights below to demonstrate (5.1) with $N = 4$ and $N = 3$.

$\pi \in \mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{O}(\pi)}q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{O}(\pi)}q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$(-1)^{\mathcal{O}(\pi)}q^{\mathcal{E}(\pi)}$
(4)	1	$(4,2,1)$	$-q^2$	(2)	1
$(4,3)$	q^3	$(4,3,2,1)$	q^4	$(2,1)$	q
$(4,2)$	q^2	(3)	-1	(1)	-1
$(4,1)$	q	$(3,2)$	$-q^2$	$\emptyset$	1
$(4,3,2)$	q^3	$(3,1)$	$-q$		
$(4,3,1)$	$-q^3$	$(3,2,1)$	q^2		

$$\begin{aligned}\sum_{\pi\in\mathcal{D}_{\leq 4}} (-1)^{\mathcal{O}(\pi)} q^{\mathcal{E}(\pi)} &= q^4 + q^3 + q + 1 = (1+q)(1+q^3) \\ &= (-q;q^2)_2, \text{ and } \textstyle\sum_{\pi\in\mathcal{D}_{\leq 3}} (-1)^{\mathcal{O}(\pi)} q^{\mathcal{E}(\pi)} = 0.\end{aligned}$$

By letting $z \mapsto z/q$ in Theorem 2.3, we get a result analogous to Theorem 3.1.

Theorem 5.2. *For a non-negative integer N, we have*

$$\sum_{\pi\in\mathcal{D}_{\leq N}} q^{\mathcal{E}(\pi)} z^{\gamma(\pi)} = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_q z^i, \tag{5.3}$$

$$\sum_{\pi\in\mathcal{P}_{\leq N}} q^{\mathcal{E}(\pi)} z^{\gamma(\pi)} = \sum_{i=0}^{N} \frac{z^i}{(q;q)_i (q;q)_{N-i}}. \tag{5.4}$$

We demonstrate (5.3) when $z = 1$ with $N = 4$ below.

$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$
(4)	1	$(4,2,1)$	q^2	(2)	1
$(4,3)$	q^3	$(4,3,2,1)$	q^4	$(2,1)$	q
$(4,2)$	q^2	(3)	1	(1)	1
$(4,1)$	q	$(3,2)$	q^2	$\emptyset$	1
$(4,3,2)$	q^3	$(3,1)$	q		
$(4,3,1)$	q^3	$(3,2,1)$	q^2		

$$\begin{aligned}\sum_{\pi\in\mathcal{D}_{\leq 4}} q^{\mathcal{E}(\pi)} &= 5+3q+4q^2+3q^3+q^4\\ &= 1+(1+q+q^2+q^3)+(1+q+2q^2+q^3+q^4)+(1+q+q^2+q^3)+1\\ &= \begin{bmatrix}4\\0\end{bmatrix}_q+\begin{bmatrix}4\\1\end{bmatrix}_q+\begin{bmatrix}4\\2\end{bmatrix}_q+\begin{bmatrix}4\\3\end{bmatrix}_q+\begin{bmatrix}4\\4\end{bmatrix}_q.\end{aligned}$$

Then, by extracting the coefficient of z^j in Theorem 5.2, we get the following theorem.

Theorem 5.3. *For a partition $\pi = (\lambda_1, \lambda_2, \dots)$, we write $\gamma(\pi) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots$, as before. Then,*

$$\sum_{\substack{\pi\in\mathcal{D}_{\leq N}\\ \gamma(\pi)=j}} q^{\mathcal{E}(\pi)} = \begin{bmatrix}N\\j\end{bmatrix}_q, \tag{5.5}$$

$$\sum_{\substack{\pi\in\mathcal{P}_{\leq N}\\ \gamma(\pi)=j}} q^{\mathcal{E}(\pi)} = \frac{1}{(q;q)_j(q;q)_{N-j}}. \tag{5.6}$$

We also note a direct combinatorial proof for (5.5) using the Sylvester bijection [9]. We know that $\left[\begin{smallmatrix}N\\j\end{smallmatrix}\right]_q$ is the generating function for the number of partitions with parts $\leq N-j$ and number of parts $\leq j$. Let π be a partition into $\leq j$ parts where its largest part is $\leq N-j$. Next, we construct a partition π_o into j odd parts by first putting j boxes in a column followed by putting the Ferrers diagram of the partition π on the right of the j-box column, and its reflection π on to the left of the column. When it is read line-by-line, this is a partition into odd parts, and $|\pi_o| = 2|\pi| + j$. This construction from π to π_o is bijective and can easily be reversed. Now we apply the Sylvester bijection to π_o. This takes π_o to a partition into distinct parts π_d. Obviously, the composition of the bijections $\pi \mapsto \pi_o \mapsto \pi_d$ is a bijection. The largest part of π_d is the sum of the main column length (exactly j) and the size of the largest part of π (which is $\leq N-j$), hence $\leq N$. It is clear that $\gamma(\pi_d) = j$. Finally, $\mathcal{E}(\pi_d)$ is equal to the number of boxes in the reflected copy in the Ferrers diagram, which is exactly $|\pi|$. An example of this construction is given in Figure 2.

Theorem 5.3 is elegant and it has elementary combinatorial corollaries as N tends to infinity.

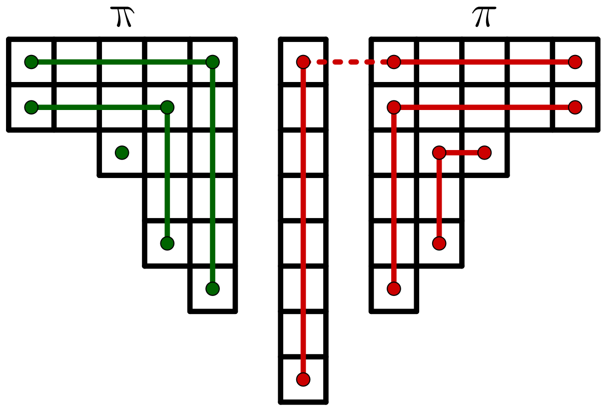

FIGURE 2. Example of the combinatorial proof of (5.5) for $N = 14$, $j = 8$, and $\pi = (5, 5, 3, 2, 2, 1)$. The partition π_o is $(11, 11, 7, 5, 5, 3, 1, 1)$ (read row by row). The partition we get after the Sylvester bijection is $\pi_d = (13, 10, 9, 7, 4, 1)$. The red lines (right-bending lines) are the odd-indexed parts of π_d and the green lines (left-bending lines) are the even-indexed parts of π_d. Only the green lines are counted under the statistic $\mathcal{E}(\pi_d)$.

Corollary 5.4. *We have*

$$\sum_{\substack{\pi\in\mathcal{D}\\ \gamma(\pi)=j}} q^{\mathcal{E}(\pi)} = \frac{1}{(q;q)_j}, \tag{5.7}$$

$$\sum_{\substack{\pi\in\mathcal{P}\\ \gamma(\pi)=j}} q^{\mathcal{E}(\pi)} = \frac{1}{(q;q)_j(q;q)_\infty}. \tag{5.8}$$

A combinatorial interpretation of (5.7) is Theorem 1.6. In the same spirit, we present a combinatorial interpretation of (5.8) below.

Theorem 5.5. *The number of partitions π where $\mathcal{E}(\pi) = n$ and $\gamma(\pi) = j$ is equal to the number of partitions of n where parts are in two colors red and green such that red parts are at most j.*

6. Identities for weighted Rogers–Ramanujan partitions

In [19], the second author demonstrated how some of these theorems can be turned into weighted identities using multiplicative weights. In that paper, it was shown that Schmidt's problem is equivalent to Alladi's weighted relation between Rogers–Ramanujan partitions and ordinary partitions (see [19, Equation (5.2)]).

Theorem 6.1. *Let $\#(\pi)$ be the number of parts in π and let $\mathcal{RR}$ be the set of partitions with gaps between parts ≥ 2. Then,*

$$\sum_{\pi\in\mathcal{D}} q^{\mathcal{O}(\pi)} = \sum_{\pi\in\mathcal{RR}} \omega(\pi) q^{|\pi|}, \tag{6.1}$$

where

$$\omega(\pi) := \lambda_{\#(\pi)} \cdot \prod_{i=1}^{\#(\pi)-1} (\lambda_i - \lambda_{i+1} - 1). \tag{6.2}$$

Recently the same observation was made by Alladi [2].

A brief explanation of the equality in (6.1) in the spirit of [19] is as follows. Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{D}$, and let $\pi = (\lambda_1, \lambda_3, \lambda_5, \dots)$. Notice that $\pi \in \mathcal{RR}$ and $\mathcal{O}(\lambda) = |\pi|$. There are multiple $\lambda \in \mathcal{D}$ that can give rise to the same π. This is the source of the multiplicative weight (6.2). For λ to be in $\mathcal{D}$, there are $\lambda_1 - \lambda_3 - 1$ possible λ_2 values. Similarly, there are $\lambda_{2i-1} - \lambda_{2i+1} - 1$ possible values for λ_{2i} in order for λ to be a partition into distinct parts, for all i such that $2i + 1 \leq \#(\lambda)$. Let k be the positive integer such that $2k - 1 \leq \#(\lambda) \leq 2k$. Then, including 0, there are a total of λ_{2k-1} viable possibilities for λ_{2k} that can form a $\lambda \in \mathcal{D}$. Renaming the parts of π as $(\pi_1, \pi_2, \dots) = (\lambda_1, \lambda_3, \dots)$ to simplify the notation is enough to get (6.1).

It is clear that (3.1) implies a refinement of Theorem 6.1 with an extra bound on the largest part of partitions.

Theorem 6.2. *Let $\mathcal{RR}_{\leq N}$ be the set of partitions into parts $\leq N$ with gaps between parts ≥ 2. Then,*

$$\sum_{\pi \in \mathcal{D}_{\leq N}} q^{\mathcal{O}(\pi)} = \sum_{\pi \in \mathcal{RR}_{\leq N}} \omega(\pi) q^{|\pi|}. \tag{6.3}$$

We can also interpret (5.3) with a weight similar to (6.2).

Theorem 6.3. *For a non-negative integer N, we have*

$$\sum_{\pi \in \mathcal{D}_{\leq N}} q^{\mathcal{E}(\pi)} = \sum_{\pi \in \mathcal{RR}_{\leq N-1}} \omega(\pi) q^{|\pi|}, \tag{6.4}$$

where

$$\hat{\omega}(\pi) := (N - \lambda_1) \cdot \lambda_{\#(\pi)} \cdot \prod_{i=1}^{\#(\pi)-1} (\lambda_i - \lambda_{i+1} - 1) \quad \text{and} \quad \hat{\omega}(\emptyset) = N + 1. \tag{6.5}$$

There are $N+1$ partitions π in $D_{\leq N}$ ($\emptyset$ and (k) for $1 \leq k \leq N$) that yield $\mathcal{E}(\pi) = 0$. This is the reasoning behind the special definition of $\hat{\omega}(\emptyset)$ in (6.5).

We give an example for the above theorem with $N = 4$. The sum of all the second columns in the first three blocks is equal to the sum of the weights in the last column.

$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{D}_{\leq 4}$	$q^{\mathcal{E}(\pi)}$	$\pi \in \mathcal{RR}_{\leq 3}$	$\hat{\omega}(\pi) q^{\lvert\pi\rvert}$
(4)	1	$(4,2,1)$	q^2	(2)	1	$(3,1)$	q^4
$(4,3)$	q^3	$(4,3,2,1)$	q^4	$(2,1)$	q	(3)	$3q^3$
$(4,2)$	q^2	(3)	1	(1)	1	(2)	$4q^2$
$(4,1)$	q	$(3,2)$	q^2	$\emptyset$	1	(1)	$3q$
$(4,3,2)$	q^3	$(3,1)$	q			$\emptyset$	5
$(4,3,1)$	q^3	$(3,2,1)$	q^2				

Acknowledgment

We would like to thank George E. Andrews for his kind interest and encouragement. We thank the anonymous referees and the editor for their valuable comments and suggestions.

The second author thanks EPSRC and FWF for partially supporting his research through grants EP/T015713/1 and P-34501N, respectively.

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