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SKEW SHAPE ASYMPTOTICS, A CASE-BASED INTRODUCTION

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Abstract. We discuss various tools in the emerging area of Asymptotic Algebraic Combinatorics, as they apply to one running example of thick ribbons. Connections to other areas, exercises and open problems are also included.

1. Introduction

1.1. **Foreword.** This paper is a short introduction to some ideas in Asymptotic Algebraic Combinatorics. We do not intend to be broad or thorough, but rather give a cross section of the area concentrated around a single example, which turned out to have rich connections to many different results and open problems. We also include a number of exercises for the curious reader. Of course, this is not a substitute of a serious survey, but in the absence of such we envision this paper as a quick guide to the literature, and an easy entry point to the area.

1.2. **Thick ribbons.** Let $\delta_k = (k-1, k-2, \ldots, 2, 1) \vdash \binom{k}{2}$ be a *staircase shape* of size k. Let $\tau_k = \big(\delta_{2k}/\delta_k\big) \vdash n$ be a *thick ribbon shape* of size k, see Figure 1. Here and below, we have $n = \frac{k(3k-1)}{2}$.

Denote by $a_k := |\operatorname{SYT}(\tau_k)|$ the number of standard Young tableaux of shape τ_k. The sequence $\{a_k\}$ is rapidly growing:

$$1, \quad 16, \quad 101376, \quad 1190156828672, \quad 68978321274090930831360, \ \ldots$$

For example, $a_2 = |\operatorname{SYT}(321/1)| = 16$. For larger values, see [OEIS, A278289]. The main goal of the paper is to give lower and upper bounds on a_k. Roughly, $a_k \approx \sqrt{n!}$. More precisely, it is known and easy to see that

$$\log a_k \, = \, \frac{3}{2}\, k^2 \log k \, + \, O(k^2) \, = \, \frac{1}{2}\, n \log n \, + \, O(n) \quad \text{as} \quad k \to \infty,$$

see [MPP4]. The following result frames the answer in an asymptotic language.

Theorem 1.1 ([MPT])**.** *There exists a universal constant ϕ, such that*

$$\log a_k \, = \, \frac{1}{2}\, n \log n \, + \, \phi\, n \, + \, o(n) \qquad \textit{as} \qquad n \, = \, \frac{1}{2}\, k(3k-1) \to \infty.$$

The rest of the paper is concerned with the following problem.

Main Problem 1.2. *Find ϕ, i.e., give sharp* rigorous *estimates for ϕ.*

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The best bounds we present are:

$$-0.2368 \leq \phi \leq -0.1648, \tag{1.1}$$

given by Lower Bound 9.2 and Upper Bound 6.1. The result in [MPT] does not determine the exact value of ϕ, but rather presents it as a solution of a variational problem (cf. [Gor1] and Exercise 12.4). It is unlikely that it can be computed exactly other than numerically.

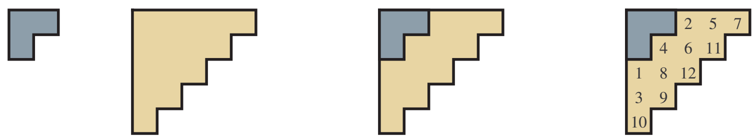

FIGURE 1. Staircases δ_3 and δ_6, thick ribbon $\tau_3 = \delta_6/\delta_3$, and a standard Young tableau of shape τ_3.

The bounds (1.1) are remarkably close to each other, as opposed to the way these things usually go. In fact, the calculations by Pantone suggest that $\phi \approx -0.18$, see [OEIS, A278289], leaving very little room for improvement on either side. Let us emphasize that getting best bounds is not really the point of this paper as we present a number of relatively weak bounds. The idea is to review the tools which in this particular case can give weaker bounds, but stronger in other cases perhaps, and often best used in combination.

1.3. **Style and structure of the paper.** In Section 2, we begin with an informal general discussion of how to obtain bounds for combinatorial numbers, notably how to approach them when some easy ideas fail.o Then, one by one, we introduce tools of the area, starting with the classical and more established ideas and leading to the most recent work. We largely restrict ourselves to the running example of the skew shape τ_k, leaving only a trail of crumbs for the interested reader to recover the full story from the references. We also heavily use figures, examples and exercises in place of formal general statements.

We do not present most definitions, standard results and notation, but instead assume that the reader is familiar with them or is able to quickly catch up using [Sag] and [S2, Ch. 7]. In every section, we supplement the results with exercises, which we believe will be helpful. In fact, the upper and lower bounds are never carefully proved and can be viewed as exercises in their own right. Additional exercises are included in Section 11. We expect the reader to be committed to doing the exercises as this is probably the only way to get a grasp of the area.

In Section 12, we state several conjectures and open problems directly related to the subject. Although natural, they show both the power of tools in Algebraic Combinatorics, and their limitations in larger setting. We conclude with Section 13, where we give brief historical remarks, mostly aimed as a guide to the references.

Finally, let us relate the style of this paper to general goals of Asymptotic Algebraic Combinatorics. In the context of "two cultures in mathematics" [Gow], one often assumes that Combinatorics belong to the second, "problem solving" culture. This is far from the truth. Like other broad fields, Combinatorics spans both cultures, even if some areas in it are more at home in one than the other. Traditionally, the culture of ever improving estimates aiming towards the true value for a key benchmark problem (as in [Ber]) was foreign to the whole area of Algebraic Combinatorics. While the specific benchmark problem we chose is not of utmost importance, it serves as a convenient battleground for several competing tools and techniques which can later be applied to other problems. With this style of exposition, we are aiming to lend further support to this important culture in the area.

2. Finding bounds

2.1. **The basics.** Suppose one is given an integer sequence $\{c_k\}$ to investigate. The first thing to do is to check whether c_k has a nice product formula. This often works, e.g. for binomial coefficients, Catalan numbers, number of boxed plane partitions, alternating sign matrices, etc. When a product formula exists and can be proved, it is relatively straightforward to compute the exact asymptotics to everyone's satisfaction. On the other hand, when there are some relatively large prime divisors appearing in $\{c_k\}$, one should look elsewhere. For example, in our sequence $\{a_k\}$, we have primes $251\,|\,a_4$ and $327317328039199\,|\,a_8$, making the prospects of a product formula rather unpromising.

The second approach is to look for a nice recurrence relation, convert it into a closed form generating function (GF), and then use various analytic tools. This works well for numerous sequences, e.g. Fibonacci, Bell, Bernoulli, Apéry, and partition numbers, see e.g. [FS, Odl, PW]. Theorem 1.1 rules out

$$\mathcal{A}(t) \;:=\; \sum_{k=1}^{\infty} \frac{a_k\, t^k}{\big(k(3k-1)/2\big)!} \tag{2.1}$$

being rational or algebraic, making it unlikely that progress can be made in this direction. However, the asymptotics do not rule out $\mathcal{A}(t)$ from being D-finite and ADE (cf. §11.1); it would be interesting to prove that. We refer to [P2] for more on this approach.

The next thing to try is to look for a determinant formula, and this is where one finds an early success. Indeed, the following *Aitken–Feit determinant formula* [Ait, Feit] is a standard result in the area, and applies to all skew shapes:

$$f^{\lambda/\mu} \;=\; n!\,\det\left(\frac{1}{(\lambda_i-\mu_j-i+j)!}\right)_{i,j=1}^{\ell(\lambda)}, \tag{2.2}$$

where $f^{\lambda/\mu} := \big|\mathrm{SYT}(\lambda/\mu)\big|$ denotes the number of standard Young tableaux of shape λ/μ, see e.g. [S2, Cor. 7.16.3]. The alternating sign nature of the formula allows only mediocre upper bounds in our case. To understand this, consider the leading (diagonal) term in the Laplace expansion of (2.2):

$$n!\,\prod_{i=1}^{\ell(\lambda)} \frac{1}{(\lambda_i-\mu_i)!}\,. \tag{2.3}$$

We show in §4.2, that this product gives the right order of magnitude. Since we have only $\ell(\lambda)! = \exp\Theta(k\log k)$ terms in the Laplace expansion, the product (2.3) rigorously implies an asymptotic upper bound (see Exercise 11.1). This also shows the limitations of the determinant approach in this case. Indeed, it is very hard to see how any nontrivial lower bound can be obtained from (2.2) in view of the signs in the Laplace expansion. Nor do other determinant tricks seem directly applicable, see [K1, K2].

2.2. **Asymptotic thinking.** While it is hard to give a broad description of what kind of arguments lead to good bounds, one natural approach is clear: clever use of identities and other summation formulas to bound one term in the summation (cf. [TV]). Let us give one important example to illustrate this approach.

Let $D_n = \max\bigl\{f^\lambda, \lambda \vdash n\bigr\}$ denote the maximal dimension of the irreducible S_n-module, where $f^\lambda := \chi^\lambda(1) = |\operatorname{SYT}(\lambda)|$. For more on $\{D_n\}$, see [OEIS, A003040]. To get a bound on D_n, recall the Burnside formula:

$$\sum_{\lambda \vdash n} \bigl(f^\lambda\bigr)^2 \, = \, n!$$

This gives an upper bound $D_n \le \sqrt{n!}$, and a lower bound $D_n \ge \sqrt{n!/p(n)}$, where $p(n) = e^{O(\sqrt{n})}$ is the number of partitions. Putting these together and using Stirling's formula, we obtain

$$\log D_n \, = \, \frac{1}{2}\, n \log n \, - \, \frac{1}{2}\, n \, - \, O(\sqrt{n})\,. \tag{2.4}$$

Since $\sqrt{n!}$ is the scale on which D_n is lying, it is the next term of the asymptotics which is most interesting. In other words, the ratio $D_n/\sqrt{n!}$ is the right quantity to consider. Given (2.4), it is natural to conjecture that in fact we have

$$\log D_n \, = \, \frac{1}{2}\, n \log n \, - \, \frac{1}{2}\, n \, - \, c\sqrt{n} \, + \, o(\sqrt{n})\,. \tag{2.5}$$

This is the celebrated *Vershik–Kerov–Pass* (*VKP*) *conjecture* which remains open [KP, VK2].

To bring this discussion back to $\{a_k\}$, one can make several conclusions. First, a_k lies on the same scale of $\sqrt{n!}$, where $n = |\tau_k|$, so the asymptotics of $a_k/\sqrt{n!}$ is exactly the right quantity to consider. Second, the success of using the Burnside identity suggests one should consider numerous summation formulas involving $f^{\lambda/\mu}$. In fact, this is the approach that works best for both the upper and the lower bounds. Third, the fact that $\phi > -0.5$ implies that a_k is exponentially larger than D_n, making the summations very large and involving exponentially large terms. Finally, the existence of the limit ϕ given by Theorem 1.1, suggests that this is an easier problem than the VKP conjecture. This turns out to be true as the problem is amenable to a variety of techniques and ideas.

2.3. **Probabilistic thinking.** In place of identities as above, one can ask a more delicate question: what is the shape of λ which attains the maximum $f^\lambda = D_n$. This may seem vague, but it turns out that the *limit shape* of maximal λ is well defined after scaling the maximal shapes by $\frac{1}{\sqrt{n}}$. The answer was computed by Vershik and Kerov [VK1] and by Logan and Shepp [LS], see also [Rom]. Note that when the row/column lengths of λ are constrained to $t\sqrt{n}$, the maximal dimension f^λ is given by a function $D_n(t)$, which is exponentially smaller than D_n, for all $1 \le t < 2$, see [LS]. The proof is based on the variational principle and uses the hook-length formula (see §6.1) in an essential way.

For our setting, one can ask for the *limit shape* of random $A \in \operatorname{SYT}(\lambda/\mu)$, where one considers scaled partitions $\frac{1}{\sqrt{n}}\bigl\{(i,j) \in \lambda/\mu,\, A(i,j) \le \alpha n\bigr\} \to \mathcal{L}_\alpha$. See Figure 2 for an example of limit curves in a $k \times 2k$ rectangle. The existence of such limit curves is proved by [Sun], see also [Gor1, MPT]. If one knows the exact shape of $\{\mathcal{L}_\alpha, 0 < \alpha < 1\}$, one can estimate

$$\log f^{\lambda/\mu} \approx \int_0^1 \log |\mathcal{L}_\alpha|!\, d\alpha\,,$$

and an even better estimate can be obtained by taking adjacent ribbon hooks, see below.

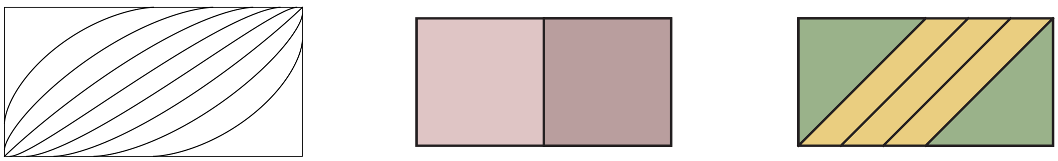

FIGURE 2. The limit curves in a $k \times 2k$ rectangle (created by Dan Romik, April 2020), and two partitions of the rectangle.

Of course, getting closed formulas for the limit curves is a difficult problem, which often involves asymptotics of determinants of multivariate functions, see [BP2, Gor2]. In the absence of closed formulas for $\{\mathcal{L}_\alpha\}$, one can still use this approach in selecting which bounds fit the problem best. For example, for $\lambda = (2k)^k$, we have $|\operatorname{SYT}(\lambda)| > |\operatorname{SYT}(k^k)|^2$, but this is a rather poor lower bound as the square boundary cuts sharply across the limit shape curves. Indeed, one can check that the right-hand side is smaller than the product $(k!)^{k+1}|\operatorname{SYT}(\delta_k)|^2$, which comes from a partition that is better aligned with the limit shape curves, see Figure 2.

3. Notation and basic asymptotics

As we mentioned earlier, we employ the standard notations in Algebraic Combinatorics and Representation Theory of S_n, see e.g. [Sag] and [S2, Ch. 7]. We refer to [S2, Ch. 3] and [Tro] for poset notation and standard results.

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$, $[n] = \{1, \ldots, n\}$ and $\mathbb{R}_+ = \{x \geq 0\}$. We use the standard asymptotics notations $f \sim g$, $f = o(g)$, $f = O(g)$ and $f = \Omega(g)$, see e.g. [FS, §A.2]. We use $c \approx c'$ to approximate their numerical value with the usual rounding rules, e.g. $\pi \approx 3.14$ and $\pi \approx 3.1416$.

Throughout the paper, we make heavy use of *Stirling's formula* $\log n! = n \log n - n + O(\log n)$. Here and everywhere below log denotes the natural logarithm. We need four more products:

$$(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1), \qquad \Phi(n) := 1! \cdot 2! \cdots n!\,,$$
$$\Psi(n) := 1! \cdot 3! \cdot 5! \cdots (2n-1)!\,, \qquad \Lambda(n) := 1!! \cdot 3!! \cdot 5!! \cdots (2n-1)!!$$

These products have similar asymptotic formulas:

$$\log (2n-1)!! = n \log n + (\log 2 - 1)n + O(1) \quad \text{[OEIS, A001147]},$$
$$\log \Phi(n) = \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + O(n \log n) \quad \text{[OEIS, A000178]},$$
$$\log \Psi(n) = n^2 \log n + \left(\log 2 - \frac{3}{2}\right) n^2 + O(n \log n) \quad \text{[OEIS, A168467]},$$
$$\log \Lambda(n) = \frac{1}{2} n^2 \log n + \left(\frac{\log 2}{2} - \frac{3}{4}\right) n^2 + O(n \log n) \quad \text{[OEIS, A057863]}.$$

4. Linear extensions of posets

4.1. Antichain partition. Observe that

$$a_k \,\geq\, k! \,\cdot\, (k+1)! \,\cdots\, (2k-1)! \,=\, \frac{\Phi(2k-1)}{\Phi(k-1)}\,. \tag{4.1}$$

This follows from counting standard Young tableaux with numbers in the first antidiagonal smaller than the numbers in the second antidiagonal, etc., see Figure 3. These are in bijection with permutations in each antidiagonal, of sizes k, $k+1$, $\ldots$, $2k-1$, respectively. This gives the following.

Lower Bound 4.1 ([MPP4, §8.2])**.**

$$\phi \,\geq\, \frac{11\log 2}{6} - \frac{\log 3}{2} - \frac{3}{2} \,\approx\, -0.7785\,.$$

It is easy to see that this approach works for all posets, see e.g. [MPP4, §2]. It was shown in [BP1] that one can consider any antichain partition, or even slightly more involved *Greene–Kleitman–Fomin* (GKF) *parameters*.

Exercise 4.2. Prove that lower bound (4.1) is optimal over all antichain partitions.

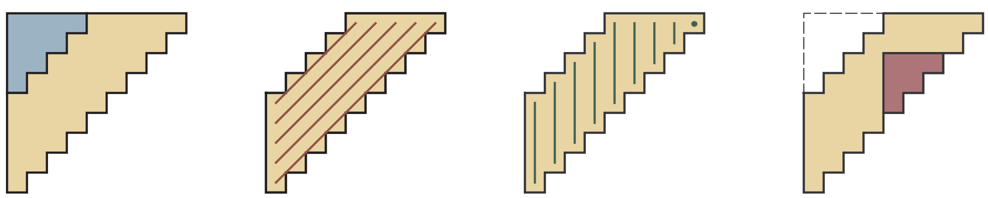

Figure 3. Skew shape $\tau_5 = \delta_{10}/\delta_5$, antichain partition, chain partition, and a lower order ideal of $(3,5) \in \tau_5$ with $\mathrm{br}(3,5) = 6$.

4.2. Chain partition. Observe that

$$a_k \,\leq\, \binom{n}{1,2,\ldots,k-1,k,k,\ldots,k} \,=\, \frac{n!}{(k!)^{k-1}\,\Phi(k)}\,. \tag{4.2}$$

To prove this, simply observe that $\mathrm{SYT}(\tau_k)$ is a subset of column-strict tableaux, and that the columns have lengths $1,2,\ldots,k-1,k,k,\ldots,k$ (k columns of length k).

Upper Bound 4.3 ([MPP4, §8.2])**.**

$$\phi \,\leq\, \frac{1}{6} - \frac{\log 2}{2} + \frac{\log 3}{2} \,\approx\, 0.3694\,.$$

It is easy to see that this approach works for all chain partitions of posets, see e.g. [MPP4, §2]. It was shown in [BP1] that one can also consider somewhat more involved GKF parameters.

Exercise 4.4. Prove that lower bound (4.2) is optimal over all chain partitions.

4.3. **Lower order ideals.** Observe that

$$a_k \;\geq\; \frac{n!}{1^{2k-1}\,\cdot\, 3^{2k-3}\,\cdot\, 6^{2k-5}\,\cdots\, \binom{k}{2}^1}\,. \tag{4.3}$$

For general posets, this was first observed by Stanley [S2, Exc. 3.57] and proved by Hammett and Pittel [HP, Eq. (1.1)]. Heuristically, to see this, observe that $\operatorname{br}(i,j) := \binom{2k-i-j+2}{2}$ is the number of squares $(p,q)\in\tau_k$ such that $i\leq p$, $j\leq q$. Thus, $\operatorname{br}(i,j)^{-1}$ is the probability that (i,j) is the smallest of these squares in a random permutation. While these events are not independent, it is known that they do have positive correlations, giving (4.3).

Exercise 4.5. Give a formal proof of (4.3). Check that it does not give a nontrivial lower bound on ϕ.

Exercise 4.6. Reverse the role of upper and lower ideals by rotating τ_k by 180 degrees. Does this give a better or worse lower bound?

5. Thin ribbons and slim diagrams

5.1. **Alternating permutations.** Denote by E_m the m-th *Euler number*, defined as the number of *alternating permutations* in S_m :

$$E_m \;:=\; \bigl|\{\sigma\in S_m,\ \sigma(1)<\sigma(2)>\sigma(3)<\sigma(4)>\ldots\}\bigr|\,,$$

see [S3] and [OEIS, A000111]). Now observe that for $k\equiv 0 \bmod 2$, we have

$$a_k \;\geq\; E_{4k-3}\cdot E_{4k-7}\cdot E_{4k-11}\;\cdots\; E_{2k+1}\,. \tag{5.1}$$

To see this, break the ribbon shape τ_k into thin ribbons as in Figure 4 and consider standard Young tableaux with numbers in smaller ribbons smaller than numbers in larger ribbons. Recall that

$$E_m \;\sim\; \frac{4}{\pi}\left(\frac{2}{\pi}\right)^m m!$$

(see e.g. [FS, S3]). This and (5.1) gives the following.

Lower Bound 5.1.

$$\phi \;\geq\; \alpha_2 \;:=\; -\frac{3}{2} - \frac{23\log 2}{6} - \frac{\log 3}{2} - \log\pi \;\approx\; -0.5370\,.$$

Exercise 5.2. Compute the lower bound corresponding to the ribbon partition in Figure 4. Prove that of all patterns $\sigma(1)*\sigma(2)*\sigma(3)*\ldots$, where $*\in\{<,>\}$, the alternating permutations form the largest class. Use Exercise 4.2 to show that Lower Bound 5.1 cannot be improved by a better partition of τ_k into ribbons.

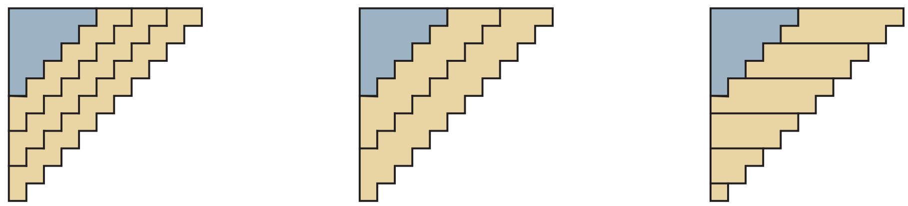

FIGURE 4. Partitions of $\tau_6 = \delta_{12}/\delta_6$ into zigzag ribbons, 3-ribbons and 2-row diagrams.

5.2. r**-ribbons.** Denote by $F_m^{(r)} = \operatorname{SYT}\bigl(\delta_m/\delta_{m-r}\bigr)$ the number of standard Young tableaux of the *r-ribbon shape*. For example, δ_m/δ_{m-2} are the usual zigzag ribbons, so $F_m^{(2)} = E_{2m-1}$. It follows from [BR] that

$$F_m^{(3)} \, = \, \frac{(3m)!\, E_{2m}\, m^{\Theta(1)}}{(2m)!\, 2^{2m}} \qquad \text{and} \qquad F_m^{(5)} \, = \, \frac{(5m)!\, (E_{2m})^2\, m^{\Theta(1)}}{\bigl((2m)!\bigr)^2\, 2^{6m}}\,. \tag{5.2}$$

Now observe that for $k \equiv 0 \bmod 3$, we have

$$a_k \, \geq \, F^{(3)}_{2k-1} \cdot F^{(3)}_{2k-4} \cdot F^{(3)}_{2k-7} \, \cdots \, F^{(3)}_{k+2}\,. \tag{5.3}$$

To see this, consider a partition of τ_k, $k \equiv 0 \bmod 3$, into 3-ribbons as Figure 4. Now formulas (5.2) and (5.3) give the following.

Lower Bound 5.3.

$$\phi \, \geq \, \alpha_3 \, := \, -\frac{3}{2} - \frac{\log 3}{2} + \frac{11 \log 2}{6} - \frac{2 \log \pi}{3} \, \approx \, -0.4431\,.$$

Exercise 5.4. Find the analogue of (5.3) for $F^{(5)}$. Use it to derive the following.

Lower Bound 5.5.

$$\phi \, \geq \, \alpha_5 \, := \, -\frac{3}{2} + \frac{43 \log 2}{30} - \frac{\log 3}{2} + \log 5 - \frac{4 \log \pi}{5} \, \approx \, -0.3621\,.$$

Exercise 5.6. Denote by α_r the lower bound on ϕ obtained from taking r-ribbons. Prove that $\{\alpha_r\}$ is strictly increasing and thus has a limit $\alpha := \lim_{r\to\infty} \alpha_r$, such that $\alpha \leq \phi$. Prove or disprove that $\alpha = \phi$.

Exercise 5.7. Find the closed form exponential generating function (EGF) for E_m to conclude that it is *D-algebraic* (ADE). Use the formulas in [BR] for conclude the same result for r-ribbons, $r = 3, 4, 5$.

5.3. **Slim diagrams.** Observe that a partition of τ_k into 2-row diagrams in Figure 4 gives

$$a_k \leq \binom{n}{1, 5, 9, \ldots, 2k, 2k, \ldots, 2k} \cdot \big[C_1 \cdot C_3 \cdot C_5 \cdots\big] \cdot \big(C_{k+1}\big)^{\lceil \frac{k}{2} \rceil}, \tag{5.4}$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the m-th *Catalan number*. To see this, follow the argument in §4.2.

Exercise 5.8. Check that (5.4) improves the upper bound in (4.2), but does not improve the bound on ϕ over that in the Upper Bound 4.3. Prove that the same holds for partitions of τ_k into r-row slim diagrams, for every fixed r. Explain formally why these slim diagrams are less effective for upper bounds compared to ribbons for lower bounds.

6. Hook-length formula

6.1. **The setup.** Let $\lambda \vdash n$. Denote by $f^\lambda = |\operatorname{SYT}(\lambda)|$ the number of standard Young tableaux of shape λ. The *hook-length formula* (HLF) states that

$$f^\lambda = n! \prod_{(i,j)\in\lambda} \frac{1}{h_\lambda(i,j)}, \tag{HLF}$$

where $h_\lambda(i,j) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the square (i,j). See [NPS, P1] for some of our favorite proofs.

6.2. **Staircase shape.** Denote by $b_m := f^{\delta_m} = |\operatorname{SYT}(\delta_m)|$ the number of standard Young tableaux of staircase shape, see [OEIS, A005118]. It follows from the HLF that

$$b_m = \frac{\binom{m}{2}!}{1^{m-1} \cdot 3^{m-2} \cdot 5^{m-3} \cdots (2m-1)} = \frac{\binom{m}{2}!}{\Lambda(m)}. \tag{6.1}$$

Observe that

$$a_k \cdot b_k \leq b_{2k}, \tag{6.2}$$

since together two tableaux of shape δ_k and δ_{2k}/δ_k can form a single tableau of shape δ_{2k}. This gives the following.

Upper Bound 6.1.

$$\phi \leq \frac{1}{2} - \frac{\log 2}{6} - \frac{\log 3}{2} \approx -0.1648.$$

Exercise 6.2. Prove or disprove: the exponential generating function (EGF) for b_m is D-finite.

6.3. **Three staircases.** Observe that

$$a_k \;\geq\; b_k^2\, b_{k+1} \binom{2\binom{k}{2}}{\binom{k}{2}}. \tag{6.3}$$

This follows immediately from the partition of τ_k into shapes δ_k, δ_k and δ_{k+1} as in Figure 5. Combined with (6.1), this gives the following.

Lower Bound 6.3.

$$\phi \;\geq\; \frac{1}{2} - \frac{5\log 2}{6} - \frac{\log 3}{2} \;\approx\; -0.6269.$$

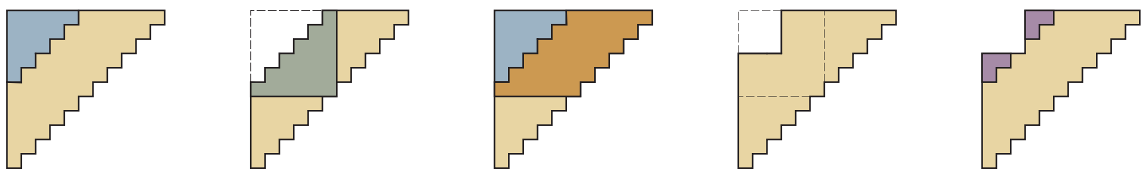

FIGURE 5. Partition of $\tau_6 = \delta_{12}/\delta_6$ into three staircases δ_6, δ_6 and δ_7. Partition of τ_6 into δ_6 and ζ_6. DeWitt shape $\varsigma_3 = \delta_{12}/\rho_3$ and its partition into δ_3, δ_3 and τ_6.

Exercise 6.4. Let $\xi_k = (2k-1,\ldots,k)$. Partition δ_{2k} into two staircases δ_k and one skew shape $\zeta_k := \xi_k/\delta_k$, see Figure 5. Define $z_k := |\operatorname{SYT}(\zeta_k)|$. Use the HLF applied to shape $(2k-1,\ldots,k)$ to get an upper bound on z_k. Note that

$$a_k \;\leq\; b_k\, z_k \binom{k^2+\binom{k}{2}}{\binom{k}{2}}.$$

Use this and (6.1) to obtain an upper bound on a_k.

6.4. **DeWitt shape.** Denote by $\rho_k = (k^k)$ the $k\times k$ *square diagram*, and define the *DeWitt shape* $\varsigma_k := \delta_{4k}/\rho_k$, see Figure 5. It was shown in [DeW] (see also [H+, KS, MPP3]), that

$$d_k \;:=\; \bigl|\operatorname{SYT}(\varsigma_k)\bigr| \;=\; n!\, \frac{\Phi(k)^3\,\Phi(3k)\;\Psi(k)\,\Psi(3k)}{\Phi(2k)^3\;\Psi(2k)^2\,\Psi(4k)}\,,$$

where $n = 7k^2 = |\varsigma_k|$. Now observe that

$$a_{2k}\cdot b_k^2 \;\leq\; d_k\,. \tag{6.4}$$

Using this and (6.1), we get the following.

Upper Bound 6.5.

$$\phi \;\leq\; -\frac{17\log 2}{3} + \log 3 + \frac{7\log 7}{6} + \frac{1}{2} \;\approx\; -0.0590.$$

7. Littlewood–Richardson coefficients

7.1. **Basic formulas.** Let $\lambda \vdash n$, $\mu \vdash m$, $\nu \vdash n-m$. Recall two properties of the Littlewood–Richardson coefficients (LR-coefficients for short):

$$f^{\mu} f^{\nu} \binom{n}{m} \, = \, \sum_{\lambda \vdash n} c^{\lambda}_{\mu\nu} f^{\lambda} \quad \text{and} \quad f^{\lambda/\mu} \, = \, \sum_{\nu \vdash n-m} c^{\lambda}_{\mu\nu} f^{\nu}. \tag{7.1}$$

It was observed in [MPP4, Prop. 2.4], that

$$f^{\lambda/\mu} \, \le \, \frac{n!\, f^{\mu}}{m!\, f^{\lambda}}. \tag{7.2}$$

Indeed, putting together equations in (7.1), we obtain

$$f^{\lambda/\mu} \, = \, \sum_{\nu \vdash n-m} c^{\lambda}_{\mu\nu} f^{\nu} \, \le \, \sum_{\nu \vdash n-m} \binom{n}{m} \frac{f^{\mu} f^{\nu}}{f^{\lambda}} f^{\nu} \, = \, \frac{f^{\mu}}{f^{\lambda}} \binom{n}{m} \sum_{\nu \vdash n-m} (f^{\nu})^2 \, = \, \frac{n!\, f^{\mu}}{m!\, f^{\lambda}}.$$

Now, applying (7.2) to $\tau_k = \delta_{2k}/\delta_k$, we get the following.

Upper Bound 7.1.

$$\phi \, \le \, \frac{3}{2} - \frac{\log 2}{2} \, \approx \, 1.1534.$$

Exercise 7.2. Explain why this upper bound is so poor.

7.2. **Evaluations.** Let $\lambda = \delta_{2k}$, $\mu = \delta_k$, $\nu = \xi_k := (2k-1, \ldots, k)$ and $x_k := |\operatorname{SYT}(\xi_k)|$. Note that $\lambda/\nu = \mu$, which implies $c^{\lambda}_{\mu\nu} = 1$. Observe that (7.1) gives $a_k \ge x_k$. Applying (HLF), we have the following.

Upper Bound 7.3.

$$\phi \, \ge \, \frac{1}{2} - \frac{\log 2}{6} - \log 3 \, \approx \, -0.7141.$$

Exercise 7.4. Recall from [Sag, S2, vL] the combinatorial interpretation of LR-coefficients $c^{\lambda}_{\mu\nu}$ as the number $x_k := |\operatorname{LR}(\lambda/\mu, \nu)|$ of *lattice tableaux* of shape λ/μ and weight ν. Observe that there is a unique lattice tableau in $\operatorname{LR}(\tau_k, \nu)$, see Figure 6. Deduce that $c^{\lambda}_{\mu\nu} = 1$, for $\lambda/\mu = \tau_k$ as above.

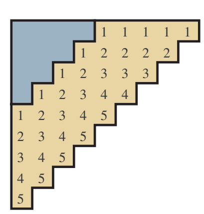

Figure 6. The unique lattice tableau in $\operatorname{LR}(\tau_5, \xi_5)$.

7.3. **Upper bound on LR-coefficients.** Recall from [PPY, Thm. 1.5] that

$$c^{\lambda}_{\mu\nu} \le \sqrt{\binom{n}{m}}\,, \quad \text{for all} \quad \lambda \vdash n\,,\ \mu \vdash m\,,\ \nu \vdash n-m. \tag{7.3}$$

To see this, use (7.1) to obtain

$$\sum_{\lambda \vdash n} \big(c^{\lambda}_{\mu,\nu}\big)^2 \le \sum_{\lambda \vdash n} c^{\lambda}_{\mu,\nu} \, \frac{f^{\lambda}}{f^{\mu} f^{\nu}} = \frac{1}{f^{\mu} f^{\nu}} \cdot f^{\mu} f^{\nu} \binom{n}{k} = \binom{n}{k}.$$

This implies

$$f^{\lambda/\mu} = \sum_{\nu \vdash n-m} c^{\lambda}_{\mu\nu} \, f^{\nu} \le p(n-m) \sqrt{\binom{n}{m}} \cdot \sqrt{(n-m)!} < p(n) \sqrt{\frac{n!}{m!}}\,,$$

where $p(n)$ denotes the number of partitions of n. Applying this to $\lambda/\mu = \tau_k$ and using $p(n) = e^{O(\sqrt{n})}$, we get the following.

Upper Bound 7.5.

$$\phi \le \frac{4 \log 2}{3} - \frac{\log 3}{2} - \frac{1}{2} \approx -0.1251.$$

Exercise 7.6 ([PPY, §4.1])**.** In the opposite direction, prove that for every $0 \le k \le n$, we have

$$\sum_{\lambda \vdash n} \sum_{\mu \vdash k,\, \nu \vdash n-k} \big(c^{\lambda}_{\mu,\nu}\big)^2 \ge \binom{n}{k}.$$

What does this formula say about how sharp (7.3) is?

8. Naruse hook-length formula

8.1. **The setup.** Let λ/μ be a skew shape and D be a subset of the Young diagram of λ. A square $(i,j) \in D$ is called *active* if $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$ are all in $\lambda \setminus D$. An *excited move* is a replacement of an active $(i,j) \in D$ with $(i+1,j+1)$. An *excited diagram* of λ/μ is a subset of squares in λ obtained from the Young diagram μ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of λ/μ, see Figure 7. The *Naruse hook-length formula* (NHLF) states that for every skew shape λ/μ we have

$$\big|\mathrm{SYT}(\lambda/\mu)\big| = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h_{\lambda}(i,j)}\,, \tag{NHLF}$$

where $n = |\lambda/\mu|$.

Exercise 8.1 ([MPP4, §12.1], see also [PPS])**.** Denote by λ^* the skew shape obtained by rotating the diagram λ by 180 degrees. Apply the NHLF to λ^*. Conclude the inequality

$$\prod_{(i,j) \in \lambda} h_{\lambda}(i,j) \le \prod_{(i,j) \in \lambda} h^*_{\lambda}(i,j),$$

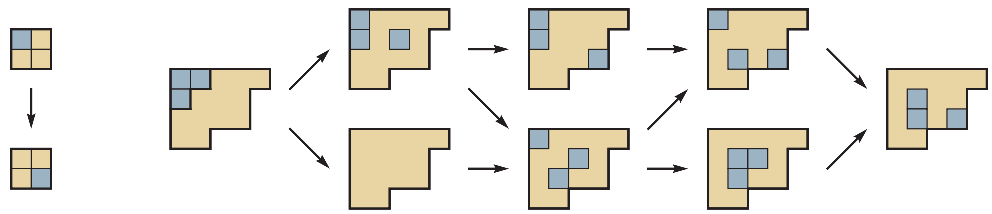

FIGURE 7. Example of an excited move. Set $\mathcal{E}(\lambda/\mu)$, for $\lambda = (5,4,4,1)$ and $\mu = (2,1)$. The arrows indicate excited moves.

where $h^*_\lambda(i,j) = (i+j-1)$. Compare this with

$$\sum_{(i,j)\in\lambda} h_\lambda(i,j) \;=\; \sum_{(i,j)\in\lambda} h^*_\lambda(i,j),$$

and explain the discrepancy.

8.2. **Basic bounds.** Observe that since $\mu \in \mathcal{E}(\lambda/\mu)$, we have

$$\bigl|\mathrm{SYT}(\lambda/\mu)\bigr| \;\geq\; F(\lambda/\mu) \;:=\; n! \prod_{(i,j)\in\lambda/\mu} \frac{1}{h(i,j)}\,. \tag{8.1}$$

Taking $\lambda = \delta_{2k}$ and $\mu = \delta_k$, we obtain the following.

Lower Bound 8.2.

$$\phi \;\geq\; \frac{1}{6} - \frac{3\log 2}{2} + \frac{\log 3}{2} \;\approx\; -0.3237.$$

Now, for the upper bound observe that the hooks decrease under excited moves. This gives

$$\bigl|\mathrm{SYT}(\lambda/\mu)\bigr| \;\leq\; F(\lambda/\mu)\cdot\bigl|\mathcal{E}(\lambda/\mu)\bigr|. \tag{8.2}$$

For $\lambda = \delta_{2k}$, $\mu = \delta_k$, and k even, observe that the excited diagrams are in bijection with non-intersecting paths as shown in Figure 8. Thus,

$$\bigl|\mathcal{E}(\tau_k)\bigr| \;\leq\; \binom{2k-2}{k-1}^{k/2}.$$

Combined with (8.2), this gives the following.

Upper Bound 8.3.

$$\phi \;\leq\; \frac{1}{6} - \frac{5\log 2}{6} + \frac{\log 3}{2} \;\approx\; 0.1384.$$

Exercise 8.4. Before moving on, show that both the lower and the upper bounds are not exact, i.e., they can be improved by some $\varepsilon > 0$, using better counting over excited diagrams.

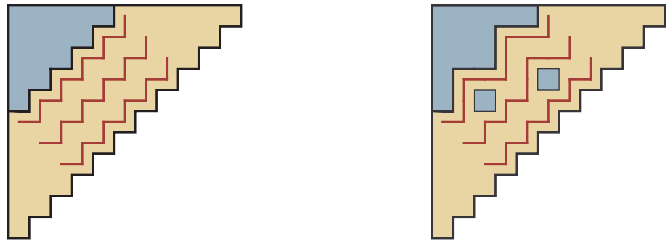

Figure 8. Non-intersecting paths in bijection with excited diagrams, in the beginning and after three moves.

8.3. **Non-intersecting paths.** Recall that non-intersecting paths with fixed start and end points are given by a determinant. For thick ribbons, this determinant was computed by Proctor:

$$(8.3)\quad \bigl|\mathcal{E}(\tau_k)\bigr| \,=\, \prod_{1\le i<j\le k} \frac{k+i+j-1}{i+j-1} \,=\, \left[\frac{\Phi(3k-1)\,\Phi(k-1)^3\,(2k-1)!!\,(k-1)!!}{\Phi(2k-1)^3\,(3k-1)!!}\right]^{1/2},$$

see [MPP4, Lemma 8.1] and [OEIS, A181119]. Combined with (8.2), this gives the following.

Upper Bound 8.5.

$$\phi \,\le\, \frac{1}{6} - \frac{7\log 2}{2} + 2\log 3 \,\approx\, -0.0621.$$

Exercise 8.6 (see [MPP3, §7])**.** Prove that the excited diagrams in this case are in bijection with lozenge tilings of half of a $(k-1)\times(k-1)\times k$ hexagon. Recall the *MacMahon box formula* for the number $M(a,b,c)$ of lozenge tilings of an $a\times b\times c$ hexagon. Use this to get an upper bound $|\mathcal{E}(\tau_k)|^2 \le M(k-1,k-1,k)$. Show that this inequality gives the same bound on ϕ as in Upper Bound 8.5. Explain why.

9. Flipped HLF

9.1. **The setup.** A skew shape λ/μ is called *slim* if $\lambda_\ell \ge \mu_1+\ell-1$, where $\ell=\lambda_1'$ is the number of parts in λ. For a subset D of Young diagram λ. Define $D^\diamond\subset\lambda$ to be a subset of elements $(\ell+1-i,j)$, for all $(i,j)\in D$. We refer to $D^\diamond$ as *vertical flipping* and consider it only when it is well defined.

A subset D of λ is called a *flipped excited diagram* if after vertical flipping it is a usual excited diagram, see Figure 9. Let $\mathcal{E}^\diamond(\lambda/\mu)$ be the set of flipped excited diagrams of λ/μ. Note that if λ/μ is slim and $D\in\mathcal{E}(\lambda/\mu)$, then $D^\diamond\in\mathcal{E}^\diamond(\lambda/\mu)$ is well defined. Thus, $|\mathcal{E}^\diamond(\lambda/\mu)|=|\mathcal{E}(\lambda/\mu)|$. In [MPP3, §3.4], we show that

$$\text{(Flipped-HLF)}\qquad \bigl|\mathrm{SYT}(\lambda/\mu)\bigr| \,=\, n!\sum_{D\in\mathcal{E}^\diamond(\lambda/\mu)}\ \prod_{(i,j)\in\lambda\setminus D}\frac{1}{h_\lambda(i,j)},$$

where $n=|\lambda/\mu|$.

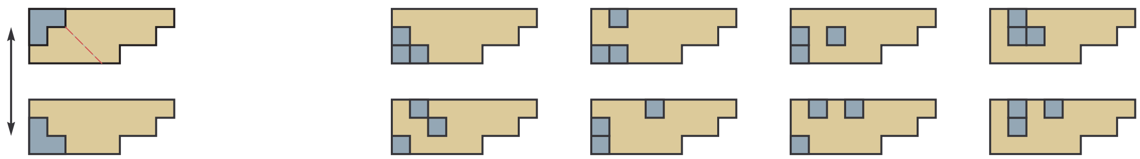

FIGURE 9. Slim skew shape λ/μ, where $\lambda = (9, 8, 6)$, $\mu = (2, 1)$, and the set $\mathcal{E}^\diamond(\lambda/\mu)$ of flipped excited diagrams.

Exercise 9.1. In the context of the lower bound (8.1), prove that

$$n! \prod_{(i,j)\in\lambda/\mu} \frac{1}{h_\lambda(i,j)} \geq n! \prod_{(i,j)\in\lambda/\mu^\diamond} \frac{1}{h_\lambda(i,j)},$$

where $\mu^\diamond \subset \lambda$ denotes the subset of squares of λ obtained by the vertical flip of μ. Conclude that (Flipped-HLF) gives a better upper bound than (8.2).

9.2. **The staircase phenomenon.** In this section, we follow [MPP5]. Clearly, diagram τ_k is not slim. However, we can create two smaller slim diagrams by partitioning δ_k into a square $\rho_{k/2}$ and two smaller staircases $\delta_{k/2}$, where k is even. In the right-hand side of (NHLF), consider only excited diagrams which do not move $\rho_{k/2}$, which are thus restricted to slim shapes. Since the sum over excited diagrams in each slim shape is equal to the sum over flipped excited diagrams, this gives a lower bound on a_k.

There are two additional properties of thick ribbons $\tau_k = \delta_{2k}/\delta_k$ which play a role here. First, because the hooks in δ_{2k} are invariant under flipped excited moves, we conclude that the terms we are summing in the right-hand side of (Flipped-HLF) are all equal. Second, by Exercise 9.3 below, the number of such terms is equal to $2^{\binom{k/2}{2}}$, for each of the two slim diagrams. Putting these together, we obtain

$$a_{2k} \geq \frac{n!\; 2^{k(k-1)}}{\Psi(2k+1)\left[(2k+1)!!\,(2k+5)!! \cdots (6k-3)!!\right]^2},$$

where $n = |\nu_{2k}| = k(6k+1)$. This gives the following.

Lower Bound 9.2 ([MPP5]).

$$\phi \geq \frac{1}{2} - \frac{2\log 2}{3} - \frac{\log 3}{4} \approx -0.2368.$$

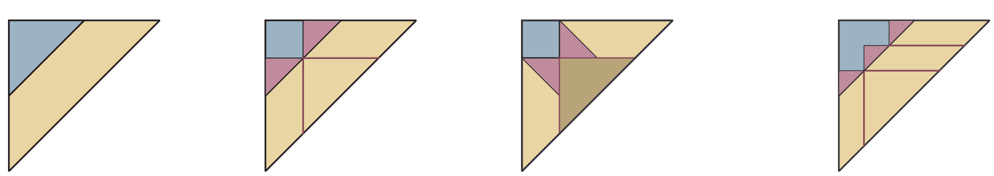

FIGURE 10. Making two smaller slim diagrams out of thick ribbon τ_k, where k is even. Same with three slim diagrams.

Exercise 9.3. Prove that for every slim shape λ/μ, such that $\mu = \delta_\ell$, we have $|\mathcal{E}(\lambda/\mu)| = 2^{\binom{\ell}{2}}$, as in Figure 9. *Hint:* Use non-intersecting paths as above to give a bijection with domino tilings of the Aztec diamond, see [OEIS, A006125].

Exercise 9.4. Make three slim smaller diagrams as in Figure 10 to obtain another lower bound for ϕ. Is this a better bound?

10. Slanted HLF

10.1. **The setup.** For a skew shape λ/μ, define a *slanted shape* $\lambda^\triangledown/\mu^\triangledown$ as in Figure 11. The *slanted excited moves* are now vertical moves as in Figure 11, while the squares of λ/μ move diagonally as before. *Slanted excited diagrams* are defined similarly, see an example in Figure 12, where slanted diagrams are in dark blue. We use $\mathcal{E}^\triangledown(\lambda/\mu)$ to denote the set of slanted excited diagrams of $\lambda^\triangledown/\mu^\triangledown$.

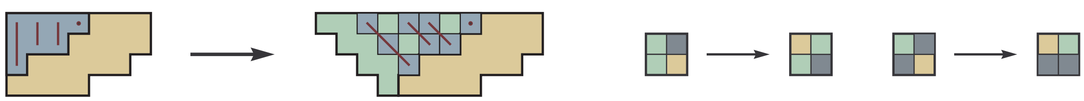

Figure 11. Transforming skew shape λ/μ into a slanted shape $\lambda^\triangledown/\mu^\triangledown$, where $\lambda = (7,7,6,4)$ and $\mu = (4,3,1)$. Two types of slanted excited moves.

In [MZ] and earlier in a different language in [KT, OO], the authors show that

$$\text{(Slanted-HLF)} \qquad \big|\mathrm{SYT}(\lambda/\mu)\big| \;=\; \frac{n!\, f^\lambda}{m!} \sum_{D\in\mathcal{E}^\triangledown(\lambda/\mu)} \prod_{(i,j)\in D} a_\lambda(i,j)\,,$$

where $m = |\lambda|$, $n = |\lambda/\mu|$, $f^\lambda = |\,\mathrm{SYT}(\lambda)|$ as above, and $a_\lambda(i,j) := \lambda_i - j + 1$ is the *arm length* of the square $(i,j) \in \lambda$.

Exercise 10.1. Suppose the smallest part λ_ℓ is at least μ_1. Show that in this case $\mathcal{E}^\triangledown(\lambda/\mu)$ is in bijection with $\mathcal{E}^\diamond(\lambda/\mu)$. Then compare (Slanted-HLF) vs. (Flipped-HLF).

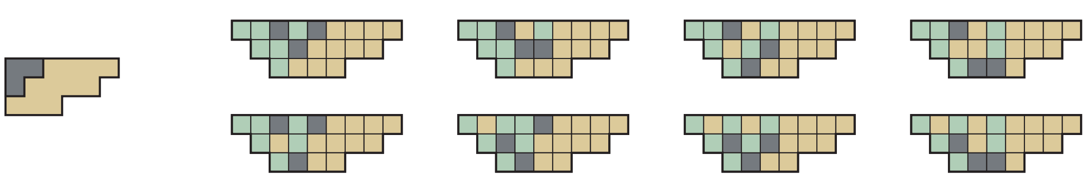

Figure 12. Slanted excited diagrams in $\mathcal{E}^\triangledown(\lambda/\mu)$, where $\lambda = (6,5,3)$ and $\mu = (2,1)$.

Exercise 10.2. Let $\lambda = \delta_{2k}$ and $\mu = \delta_k$. For a slanted excited diagram $D = \mu^\triangledown$, suppose square $(i,j) \in D$ is an image of $(p,q) \in \mu$. From (Slanted-HLF), we have

$$\big|\mathrm{SYT}(\lambda/\mu)\big| \;\ge\; G(\lambda/\mu) \;:=\; \frac{n!\, f^\lambda}{m!} \prod_{(i,j)\in\mu^\triangledown} a_\lambda(i,j)\,. \tag{10.1}$$

Use this to obtain an explicit lower bound for ϕ.

Exercise 10.3. Check that the arm lengths $a_\lambda(i,j)$ are non-increasing under slanted excited moves. This gives

$$\big|\mathrm{SYT}(\lambda/\mu)\big| \,\le\, G(\lambda/\mu)\,\cdot\,\big|\mathcal{E}^\triangledown(\lambda/\mu)\big|. \tag{10.2}$$

Exercise 10.4. Using the above notation, observe that $a_\lambda(i,j)\le h_\lambda(p,q)$. Conclude that $G(\lambda/\mu)\le F(\lambda/\mu)$ for every λ/μ, i.e., the lower bound in (10.1) is *never better* than the lower bound in (8.1).

10.2. **Lower and upper bounds.** We follow [MZ, §9.6] in this section. Let $\lambda=\delta_{2k+1}$ and $\mu=\delta_{k+1}$, where k is even. As before, let $n=|\tau_{k+1}|$. Define $u_k := \big|\mathcal{E}^\triangledown(\lambda/\mu)\big|$. It follows from (Slanted-HLF) that

$$a_{k+1} \,=\, \big|\mathrm{SYT}(\lambda/\mu)\big| \,=\, n!\,2^n\,u_k\,\frac{\Phi(3k)\ \Psi(2k)\,\Psi(k/2)}{\Phi(4k)\,\Psi(3k/2)}\,. \tag{10.3}$$

By employing ad hoc estimates on u_k, see Exercise 10.7 below, one can get the following.

Lower Bound 10.5 ([MZ, Thm. 1.4])**.**

$$\phi \,\ge\, \frac{1}{2}-\frac{9\log 2}{2}\,+\,2\log 3\,\approx\,-0.4219\,.$$

Upper Bound 10.6 ([MZ, Thm. 1.4])**.**

$$\phi \,\le\, \frac{1}{2}-\frac{13\log 2}{2}+\frac{7\log 3}{2}\,\approx\,-0.1603\,.$$

Exercise 10.7 ([MZ, §7.2])**.** Let

$$w(\mu,\ell) \,:=\, \big|\mathrm{SSYT}(\mu,\le\ell)\big| \,=\, s_\mu(1,\ldots,1)\,,\quad \ell\text{ ones},$$

where $\mathrm{SSYT}(\mu,\le\ell)$ denotes the set of *semistandard Young tableaux* of shape μ with entries $\le\ell$. Give an explicit injection to show that

$$w\big(\delta_{k+1},k\big)\,\le\,u_k\,\le\,w\big(\delta_{k+1},2k\big).$$

Use Stanley's *hook–content formula* [S2, Thm. 7.21.2] to obtain product formulas for $w\big(\delta_{k+1},\ell\big)$. Conclude that

$$\liminf_{k\to\infty}\,\frac{1}{n}\,\log u_k \,\ge\, \frac{\log 2}{3}\approx 0.2310\,,\qquad\text{and}$$
$$\limsup_{k\to\infty}\,\frac{1}{n}\,\log u_k \,\le\, \frac{3\log 3}{2}-\frac{5\log 2}{3}\approx 0.4927\,.$$

Use these to derive the Upper Bound 10.6 and the Lower Bound 10.5. Note that $\lim_{k\to\infty}\frac{1}{n}\log u_k$ exists by Exercise 12.4, but this result is not needed to prove the bounds.

Exercise 10.8. Let $\lambda=\rho_{2k}$ and $\mu=\rho_k$ be two square shapes. Compare the bounds given by (Slanted-HLF) vs. (NHLF) for the *thick hook* $\varrho_k=\lambda/\mu$.

11. Additional exercises

11.1. General bounds.

Exercise 11.1. Follow up on the discussion in §2.1 and prove an upper bound for general $f^{\lambda/\mu}$ from the Aitken–Feit determinant formula (2.2).

Exercise 11.2 ([McK])**.** Recall the identity

$$\sum_{\lambda \vdash n} f^\lambda \, = \, v_n \, ,$$

where $v_n \, = \, \bigl|\bigl\{\sigma \in S_n, \sigma^2 = 1\bigr\}\bigr|$ is the number of involutions in S_n, see e.g. [OEIS, A000085]. Use this identity to conclude (2.4). Which approach gives a better upper bound for the constant c in (2.5)? Explain why.

Exercise 11.3. Find a closed form EGF for $\{v_n\}$. Prove that it is D-finite.

Exercise 11.4 ([VK2])**.** Prove that

$$D_n \, \le \, \sqrt{n!}\, e^{-c\sqrt{n}} \quad \text{for some} \ \ c > 0,$$

cf. (2.5). Conclude that "most" dimensions f^λ are much smaller than D_n.

Exercise 11.5. In the context of §2.3, check that

$$\bigl|\mathrm{SYT}\bigl(k^{2k}\bigr)\bigr| \, > \, (k!)^{k+1} \bigl|\mathrm{SYT}(\delta_k)\bigr|^2 \, > \, \bigl|\mathrm{SYT}(k^k)\bigr|^2.$$

Exercise 11.6. Prove that for every series parallel poset $\mathcal{P} = (\mathcal{P}, \prec)$, we have

$$e(\mathcal{P}) \, = \, \prod_{x \in X} \frac{1}{\mathrm{br}(x)} \, ,$$

where $e(\mathcal{P})$ is the number of linear extensions of $\mathcal{P}$. Conclude the HLF for trees.

Exercise 11.7 ([SK])**.** Let $\mathcal{B}_n$ denote the *Boolean lattice*, the poset of all subsets of $[n]$ ordered by inclusion. Use induction to prove that $\mathcal{B}_n$ has a partition into $\binom{n}{\lfloor n/2 \rfloor}$ saturated chains. Use this partition to obtain an upper bound for $e(\mathcal{B}_n)$. Compare this bound with the lower bound obtained via the natural partition of $\mathcal{B}_n$ into $(n+1)$ antichains.

Exercise 11.8. In §7.3, we used a simple $f^\nu \le \sqrt{(n-m)!}$ inequality to obtain an upper bound. Using the fact that we must have $\nu_1, \ell(\nu) \le 2k$ and the result in [LS], give a better upper bound for f^ν, and then for ϕ.

Exercise 11.9. Let ν be a collection of squares in $\mathbb{N}^2$, such that $\nu \subset [k] \times [k]$ and $n := |\nu| > \epsilon k^2$ for some fixed $\epsilon > 0$. Use chains and antichains bounds to prove that $\log|\,\mathrm{SYT}(\nu)| = \frac{1}{2} n \log n + O(n)$.

Exercise 11.10. Fix $d \geq 3$, and let ν be a collection of squares in $[k]^d$, and $n := |\nu| > \epsilon k^d$ for some fixed $\epsilon > 0$. Generalize standard Young tableaux to d-dimensional space, and prove that $\log|\operatorname{SYT}(\nu)| = \frac{d-1}{d} n \log n + O(n)$ bound.

Exercise 11.11. Let $H_k \subset \mathbb{N}^3$ be the $k \times k \times k$ cube. Use partition of H_k into k layers of $k \times k$ squares ρ_k to give an upper bound for the constant implied by the $O(n)$ notation in the previous exercise. Compare it with the chain upper bound.

Exercise 11.12. What is the cost of computing a_k using the Aitken–Feit determinant formula? Do this carefully.

11.2. **More skew shapes.**

Exercise 11.13. Note that $c_k := |\operatorname{SYT}(\rho_{2k-1}/\delta_k)|$ can be computed by the HLF. Show that $a_k b_{2k-1} \leq c_k$. *Hint:* See Figure 13. Use this inequality to derive an explicit upper bound on ϕ.

Exercise 11.14. Recall (or compute directly) the asymptotics for $|\operatorname{SYT}(\rho_k|$, see e.g. [OEIS, A039622]. Consider two more partitions of a square as in Figure 13 and find the upper bounds on ϕ. Compare them with the bound in the previous exercise. Is it clear a priori which bound is better?

Exercise 11.15. Use the approach in (6.2) to get an upper bound for $|\operatorname{SYT}(\varrho_k)|$ for the *thick hook* $\varrho_k := \rho_{2k}/\rho_k$. Compare with the true value given by the HLF.

Exercise 11.16. Use the approach in (6.2) to get an upper bound for $|\operatorname{SYT}(\varsigma_k)|$ for the DeWitt shape $\varsigma_k = \delta_{4k}/\rho_k$. Compare with the actual asymptotics.

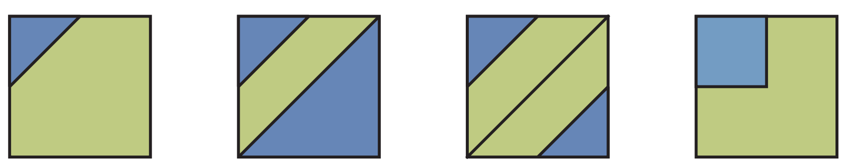

FIGURE 13. Three partitions of a square and a thick hook $\varrho_k := \rho_{2k}/\rho_k$.

Exercise 11.17 (see [MPP4, §9.2])**.** Use (8.1) for the *thick hook* $\varrho_k := \rho_{2k}/\rho_k$. How good is the resulting lower bound? Compare with upper bound first using non-intersecting paths, and then with the exact asymptotics given by the *MacMahon box formula* for $M(k, k, k)$, see e.g. [OEIS, A008793].

Exercise 11.18. Explain the similarity of asymptotics in Exercises 8.6 and 11.17 on the level of lozenge tilings of the hexagon, and then on the level of standard Young tableaux. Can you make the connection formal, at least in one direction?

Exercise 11.19. A remarkable result by Lam, Postnikov and Pylyavskyy [LPP, Thm. 5] gives an inequality for LR-coefficients. Translate this result into asymptotic language for unions and intersections of skew shapes.

11.3. **Other shapes in the plane.**

Exercise 11.20. Recall the shifted analogue of the HLF for shifted shapes, see e.g. [S2, §7.21] and [Sag, p. 139]. Use it to compute the asymptotics of the shifted staircase as in Figure 14. Check your calculations here: [OEIS, A003121]

Exercise 11.21. Define a θ-truncated square $\eta_{k,\theta}$ as in Figure 14, obtained by removing a shifted triangle of size θk from a k-square. Use the chain and the antichain partitions to prove that

$$\log\left|\mathrm{SYT}\big(\eta_{k,\theta}\big)\right| \;=\; \frac{1}{2}\,n\log n \,+\, O(n),$$

where $n = k^2 - \frac{\theta k}{2}$ is the size of $\eta_{k,\theta}$, and the constants implied by the $O(\cdot)$ notation can depend on θ. Now use formulas in [Pan] (see also [AR, §8.2]), to prove that $O(n)$ can be replaced with $c(\theta)n + o(n)$, and compute the exact formula for $c(\theta)$. Explain why $c(\theta)$ is monotone on $[0,1]$.

Exercise 11.22. Define a θ-truncated triangle and a double θ-truncated square $\eta_{k,\theta}$ as in Figure 14. Give upper and lower bounds for the corresponding asymptotic constant.

Exercise 11.23. Partition the square into a shifted staircase and a θ-truncated square, see Figure 14. Use this to get an upper bound for $c(\theta)$. Explain why this upper bound is sharper than the corresponding chain partition bound.

Exercise 11.24. Consider a skew shifted staircase as in Figure 14. Use the shifted analogue of the NHLF, see [MPP3, §9.5] and [NO], to get the lower and upper bounds on the corresponding asymptotic constant. Partition the shifted staircase into a smaller shifted staircase and a skew shifted staircase. Use the shifted HLF for the shifted staircase, see Exercise 11.20, to derive an upper bound from this outpartition. Which bound is better?

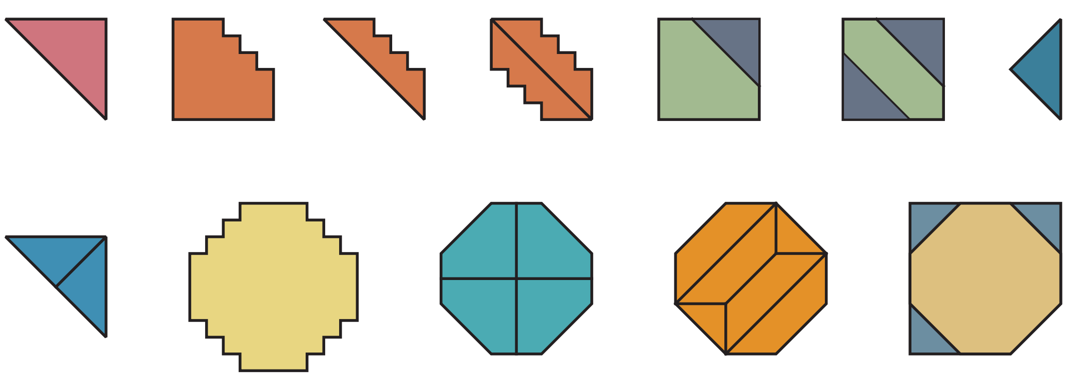

FIGURE 14. First row: shifted staircase, truncated square, truncated triangle, double truncated square, two partitions of a square, and a skew shifted staircase. Second row: partition of a shifted staircase, partition of a square 4-octagon, two partitions and one outpartition of an octagon.

Exercise 11.25. Consider a k-octagon ω_k, defined as the difference of a square ρ_{3k} and four rotated staircases δ_k. Use two partitions in Figure 14 to obtain lower bounds on the number of standard Young tableaux of the k-octagon via exact formulas for shifted staircases and truncated squares, see above. Try to guess which bound is sharper before the calculation, and check if your guess is confirmed by the calculation.

Exercise 11.26. Use a 5-ribbon partition to get yet another lower bound for k-octagons ω_k. Compare with the bound in the previous exercise.

Exercise 11.27. Use an outpartition as in Figure 14 to get an upper bound for k-octagons ω_k. Compare with the chain upper bound.

Exercise 11.28. Consider the plane region B_k of squares which fit the circle of radius k. Read about the *Gauss circle problem.* Prove that

$$\log\left|\operatorname{SYT}\big(B_k\big)\right| \, = \, \frac{1}{2}\, n \log n \, + \, O(n),$$

where $n = |B_k| \sim \pi k^2$. Do you think that $O(n)$ can be replaced with $cn + o(n)$?

Exercise 11.29. Find two subsets of squares $D, D' \subset \mathbb{N}^2$ which are equal up to a permutation of rows and columns, and $|\operatorname{SYT}(D)| \neq |\operatorname{SYT}(D')|$. Generalize the *Young symmetrizer* to obtain the corresponding S_n-characters $\chi^D = \chi^{D'}$. Conclude that we do not always have $\chi^D(1) = |\operatorname{SYT}(D)|$, and the Algebraic Combinatorics technology no longer applies.

Exercise 11.30. Use the #P-completeness of $|\operatorname{SYT}(D)|$ proved in [DP], to argue why the determinant formula is unlikely to extend for general D.

12. Conjectures and open problems

12.1. **Enumeration.** Recall the definitions of classes of GFs from [P2, S2].

Open Problem 12.1 (see §5.2)**.** *Prove that the exponential generating function* (EGF) *for $F_m^{(r)}$ is D-algebraic* (*ADE*), *for every fixed $r > 1$.*

Open Problem 12.2 (see §2.1)**.** *Prove that the* EGF $\mathcal{A}(t)$ *given by* (2.1) *is not D-finite.*

12.2. **Slanted exciting diagrams and lozenge tilings.** We start with some unfinished business.

Open Problem 12.3 (see §10.2). *Improve the upper bound on* $u_k = |\mathcal{E}^\triangledown(\tau_{k+1})|$.

Note that even a relatively small improvement of the Upper Bound 10.6 can perhaps give a bound better than Upper Bound 6.1.

Exercise 12.4. Prove that u_k is equal to the number of lozenge tilings of a region Γ_k as in the Figure 15. Note that asymptotically, after the region Γ_k is scaled by $\frac{1}{k}$ in both directions, this gives a piecewise linear region. Use the technology in [Gor2] to prove the existence of $\lim_{k\to\infty} \frac{1}{k^2} \log u_k$. Use (10.3) to conclude Theorem 1.1.

Thus, Main Theorem 1.2 is equivalent to the following.

Open Problem 12.5. *Compute* $\lim_{k\to\infty} \frac{1}{k^2} \log u_k$.

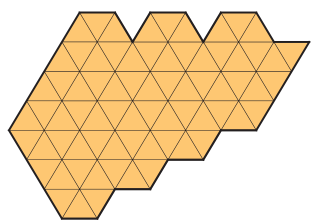

FIGURE 15. Region Γ_k with u_k lozenge tilings for $k = 3$.

12.3. **Limits for general shapes.** The main result in [MPT] of which Theorem 1.1 is a corollary, proves existence of the limit

$$\lim_{n\to\infty} \frac{1}{n} \left[\log \left|\mathrm{SYT}(\lambda/\mu)\right| - \frac{1}{2} n \log n \right], \tag{12.1}$$

where the skew shape λ/μ is obtained from a piecewise smooth region in the plane, scaled by $\sqrt{n}$. For (usual) Young diagram shapes this follows easily from the (HLF), see [MPP4].

Conjecture 12.6. *The limit* (12.1) *holds for general piecewise smooth regions, not just for skew shapes.*

This conjecture extends Exercise 11.9. It is open even in the special case of k-octagons.

Conjecture 12.7. *Prove that for k-octagons ω_k with $n = 9k^2 - 4\binom{k}{2}$ we have*

$$\log \left|\mathrm{SYT}(\omega_k)\right| = \frac{1}{2} n \log n + cn + o(n) \quad \textit{as } k \to \infty,$$

for some $c \in \mathbb{R}$.

In $\mathbb{R}^d$, the analogous conjecture is likely to hold as well but is open even for 3-dimensional Young diagram shapes (also called *solid partitions*).

Conjecture 12.8. *Let $S \subset \mathbb{R}^3_+$ be a 3-dimensional piecewise linear shape, such that $(a,b,c) \in S$ implies $(x,y,z) \in S$ for all $x \le a$, $y \le b$, $z \le c$. Then there is a limit:*

$$\lim_{n\to\infty} \frac{1}{n}\left[\log\left|\mathrm{SYT}(\Lambda_n)\right| - \frac{2}{3}\,n\log n\right], \tag{12.2}$$

where the shape Λ_n is obtained by a scaling of S by $\sqrt[3]{n}$.

Even in the special case of a 3-dimensional cube this conjecture is open, and is especially attractive.

Conjecture 12.9. *Let $H_k \subset \mathbb{N}^3$ be a $k\times k\times k$ cube. Then there is a limit:*

$$\lim_{n\to\infty} \frac{1}{k^3}\left[\log\left|\mathrm{SYT}(H_k)\right| - 2k^3\log k\right]. \tag{12.3}$$

In view of §2.3, we also conjecture the existence of the *limit surfaces* for the shapes of integers $\le \alpha n$ in random $A \in \mathrm{SYT}(\Lambda_n)$. It would be exciting to prove this even for the cube, but this goes beyond the scope of this paper.

13. Brief historical remarks

The study of *Young tableaux* goes back to the works of Alfred Young (c. 1900), and is so extensive that a quick overview would not give it justice. We refer to the extensive Ch. 7 in [S2] which gives a thorough overview of the subject, textbook [Sag] for the friendly introduction, and to [AR] for an extensive survey of enumerative results. See also [DP] for complexity aspects and the recent #P-completeness of $|\,\mathrm{SYT}(D)|$ for general $D \subset \mathbb{N}^2$ (cf. Exercise 11.30).

The *hook-length formula* goes back to Thrall (1952) and Frame–Robinson–Thrall (1954). It has been reproved and generalized in numerous ways. We refer to [CKP, §6.2] for a survey. Similarly, the *Littlewood–Richardson coefficients* are classical and go back to the 1930s. They are the subject of intensive investigation in its own right with many generalizations and variations. We refer to [vL] for a somewhat dated but helpful entry point, and to [PPY] for a recent asymptotic analysis based in part on some earlier observations by Stanley.

The study of limit shapes and the *Vershik–Kerov–Logan–Shepp theory* is in itself a subject of intensive investigation. We refer to [Rom] for a thorough treatment and connections to longest increasing subsequences in random permutations. A related followup story of random lozenge tilings, the *Arctic circle phenomenon*, etc., is well presented in [Gor2]. It relates to the subject of this paper in connection with excited diagrams, and provides the motivation for many of the limit questions and conjectures above.

The equation (NHLF) is in an unpublished work by Naruse, and is discussed at length in [MPP1, MPP2]. We refer to [Kon, MPP2] for elementary proofs, to [MPP1, §9] for a survey of other formulas for $f^{\lambda/\mu} = |\,\mathrm{SYT}(\lambda/\mu)|$, to [MPP3, §9] for a variety of other exact formulas (some conjectured and proved in later work by other authors), and to [NO] for further generalizations. The asymptotic applications of (NHLF) were first introduced in [MPP4], and advanced in [MPP3, MZ].

Despite superficial similarities, our two variations on (NHLF) have very different nature. Equation (Flipped-HLF) is given in [MPP3, §3] and further explored in [MPP5]. A simple proof via reduction to (NHLF) is given in [PP]. Equation (Slanted-HLF) is due to Morales and Zhu [MZ], and is equivalent to the Okounkov–Olshanski formula [OO] (see also [S1]). In [MZ], the authors also give a simple proof, and establish a previously conjectured equivalence to the rule in [KT].

In conclusion, let us mention that the unusual style of this paper borrows ideas from several sources. First, we are heavily influenced by the brief exercise-based presentation by Lovász [Lov], which is itself a continuation of a long tradition, see e.g. the celebrated problem books by Pólya–Szegő (1925) and Yaglom–Yaglom (1954). Second, we learned the value of worked out publishable examples clarifying the theory from the Pemantle–Wilson papers, see [PW]. Third, while the idea of a "running example" is standard, we personally were influenced by Krattenthaler's presentation [K4].

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