GENERALIZED MATRIX POLYNOMIALS OF TREE LAPLACIANS INDEXED BY SYMMETRIC FUNCTIONS AND THE GTS POSET

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ABSTRACT. Let T be a tree on n vertices with Laplacian matrix L_T and q-Laplacian $\mathcal{L}_T^q$. Let GTS_n be the generalized tree shift poset on the set of unlabelled trees on n vertices. Inequalities are known between coefficients of the immanantal polynomial of L_T and $\mathcal{L}_T^q$ as one moves up the poset GTS_n. Using the Frobenius characteristic, this can be thought as a result involving the Schur symmetric function s_λ. In this paper, we use an arbitrary symmetric function to define a *generalized matrix function* of an $n \times n$ matrix. When the symmetric function is the monomial and the forgotten symmetric function, we generalize such inequalities among coefficients of the generalized matrix polynomial of $\mathcal{L}_T^q$ as one moves up the GTS_n poset.

1. INTRODUCTION

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the symmetric group on the set $[n]$. We denote partitions λ of the number n as $\lambda \vdash n$. We write partitions using the exponential notation, with multiplicities of parts written as exponents. For $\lambda \vdash n$, let χ_λ be the irreducible character of $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by λ. We think of χ_λ as a function $\chi_\lambda : \mathfrak{S}_n \mapsto \mathbb{Z}$. With respect to an irreducible character χ_λ, define the immanant of the $n \times n$ matrix $A = (a_{i,j})_{1 \le i,j \le n}$ as

$$d_\lambda(A) = \sum_{\psi \in \mathfrak{S}_n} \chi_\lambda(\psi) \prod_{i=1}^{n} a_{i,\psi(i)}. \tag{1}$$

Let $\Lambda_{\mathbb{Q}}^n$ denote the vector space of degree n symmetric functions with coefficients from $\mathbb{Q}$. The set of monomial symmetric functions $\{m_\lambda\}_{\lambda \vdash n}$ is one of the well known bases of $\Lambda_{\mathbb{Q}}^n$. We refer the reader to the books by Stanley [14] and by Mendes and Remmel [9] for background on symmetric functions. $\Lambda_{\mathbb{Q}}^n$ actually has an inner product structure as well. Another inner product space often studied is CF_n, the space of class functions from $\mathfrak{S}_n \mapsto \mathbb{Q}$. Further, there is a well known isometry between these two spaces called the Frobenius characteristic, denoted $\mathsf{ch} : \mathsf{CF}_n \to \Lambda_{\mathbb{Q}}^n$ (see Stanley [14] for more details).

Let $\gamma \in \Lambda_{\mathbb{Q}}^n$ and consider $\Gamma_\gamma = \mathsf{ch}^{-1}(\gamma)$ its inverse Frobenius image. Clearly, $\Gamma_\gamma \in \mathsf{CF}_n$ is a class function indexed by γ. Define the generalized matrix function (GMF henceforth) of an $n \times n$ matrix A with respect to γ as

$$d_\gamma(A) = \sum_{\psi \in \mathfrak{S}_n} \Gamma_\gamma(\psi) \prod_{i=1}^{n} a_{i,\psi(i)}. \tag{2}$$

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Define the generalized matrix polynomial $\zeta_\gamma^A(x)$ of A in a new variable x with respect to a symmetric function γ as follows: $\zeta_\gamma^A(x) = d_\gamma(xI - A)$. It is well known that the inverse Frobenius image $\mathsf{ch}^{-1}(s_\lambda)$ of the Schur symmetric function s_λ is χ_λ, the irreducible character of $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by λ (see [14]). Thus, from (1) and (2), we can see that $d_{s_\lambda}(A) = d_\lambda(A)$ and $\zeta_{s_\lambda}^A(x) = d_\lambda(xI - A)$.

Csikvári in [5] defined a poset on the set of unlabelled trees with n vertices that we denote in this paper as GTS_n. Among other results, he showed that if one moves up along the GTS_n poset, all coefficients of the characteristic polynomial of the Laplacian matrix L_T of a tree T decrease in absolute value. This result was generalized by Nagar and Sivasubramanian in [12] to immanantal polynomials of $\mathcal{L}_T^q$ indexed by any $\lambda \vdash n$, where $\mathcal{L}_T^q$ is the q-Laplacian matrix of T (see Theorem 7).

Let $\mathbb{R}^+$ denote the set of non-negative real numbers and $\mathbb{R}^+[q^2]$ denote the set of polynomials in q^2 with coefficients in $\mathbb{R}^+$. Let $m_\lambda \in \Lambda_{\mathbb{Q}}^n$ be the monomial symmetric function indexed by $\lambda \vdash n$. In Section 4 of this paper, we prove the following result which shows monotonicity of the coefficient of $(-1)^r x^{n-r}$ for $0 \leq r \leq n$ in $\zeta_{m_\lambda}^{\mathcal{L}_T^q}(x)$ when we move up along GTS_n.

Theorem 1. *Let T_1 and T_2 be two trees with n vertices such that T_2 covers T_1 in GTS_n. Let $\mathcal{L}_{T_1}^q$ and $\mathcal{L}_{T_2}^q$ be the q-Laplacians of T_1 and T_2 respectively. For $\lambda \vdash n$, let*

$$\zeta_{m_\lambda}^{\mathcal{L}_{T_1}^q}(x) = d_{m_\lambda}(xI - \mathcal{L}_{T_1}^q) = \sum_{r=0}^{n} (-1)^r c_{m_\lambda,r}^{\mathcal{L}_{T_1}^q}(q) x^{n-r} \text{ and}$$

$$\zeta_{m_\lambda}^{\mathcal{L}_{T_2}^q}(x) = d_{m_\lambda}(xI - \mathcal{L}_{T_2}^q) = \sum_{r=0}^{n} (-1)^r c_{m_\lambda,r}^{\mathcal{L}_{T_2}^q}(q) x^{n-r}.$$

Then for all $\lambda \vdash n$, we have $c_{m_\lambda,r}^{\mathcal{L}_{T_1}^q}(q) - c_{m_\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \in \mathbb{R}^+[q^2]$, where $0 \leq r \leq n$. Further if $\lambda \neq 2^k, 1^{n-2k} \vdash n$, then, $d_{m_\lambda}(xI - \mathcal{L}_{T_1}^q) = d_{m_\lambda}(xI - \mathcal{L}_{T_2}^q) = 0$.

Recall that for $\gamma \in \Lambda_{\mathbb{Q}}^n$, $\Gamma_\gamma = \mathsf{ch}^{-1}(\gamma)$. For the proof of Theorem 1, we show the following lemma involving $\Gamma_{m_\lambda} = \mathsf{ch}^{-1}(m_\lambda)$ and binomial coefficients. Let $\Gamma_\gamma(j)$ denote the class function $\Gamma_\gamma(\cdot)$ evaluated at a permutation $\psi \in \mathfrak{S}_n$ with cycle type $2^j, 1^{n-2j}$. For $0 \leq i \leq \lfloor n/2 \rfloor$, define

$$\alpha_i(\gamma) = \sum_{j=0}^{i} \binom{i}{j} \Gamma_\gamma(j). \tag{3}$$

In Section 4, we prove the following lemma which we believe is of independent interest.

Lemma 2. *For all $\lambda \vdash n$ and for $0 \leq i \leq \lfloor n/2 \rfloor$, $\alpha_i(m_\lambda) = 2^i$ if $\lambda = 2^i, 1^{n-2i}$ and 0 otherwise.*

In Section 5 we consider the generalized matrix polynomial of $\mathcal{L}_T^q$ with respect to the *forgotten symmetric function*. Our main result of that section is Theorem 20, where we show similar monotonicity results as we move on the GTS_n poset.

2. Preliminaries

We now give some motivational background for our results and place it in its context. The normalized immanant of a matrix A corresponding to a partition λ is defined as $\overline{d}_\lambda(A) = \dfrac{d_\lambda(A)}{\chi_\lambda(\mathsf{id})}$. Here, $\chi_\lambda(\mathsf{id})$ equals the dimension of the irreducible representation of $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by λ.

Schur [13] showed that among normalized immanants, the determinant is the smallest normalized immanant for any positive semidefinite Hermitian matrix. This shows that for any positive semidefinite Hermitian matrix A, all its normalized immanants (and hence immanants) are non-negative. Though the immanant $d_\lambda(A)$ is non-negative, (1) is not a non-negative expression for $d_\lambda(A)$ as $\chi_\lambda(\psi)\prod_{i=1}^n a_{i,\psi(i)}$ is not necessarily non-negative for all $\psi \in \mathfrak{S}_n$ that contribute to $d_\lambda(A)$.

Recall that the Laplacian matrix L_G of a graph G is defined as $L_G = D - A$, where D is the diagonal matrix with degrees of G on the diagonal and A is the adjacency matrix of G. It is well known that L_G is positive semidefinite for all graphs G (see [8]). When the matrix A is the Laplacian L_T of a tree T, then Chan and Lam in [4] gave the following two results which give an alternate positive expression for the immanant $d_\lambda(L_T)$.

We denote the number of parts of a partition λ of n as $l(\lambda)$. For $\lambda \vdash n$, let $\chi_\lambda(j)$ denote the character value $\chi_\lambda(\cdot)$ evaluated at a permutation $\psi \in \mathfrak{S}_n$ with cycle type $2^j, 1^{n-2j}$. For $0 \leq i \leq \lfloor n/2 \rfloor$ and $\lambda \vdash n$, define

$$\alpha_{i,\lambda} = \sum_{j=0}^{i} \binom{i}{j} \chi_\lambda(j). \tag{4}$$

Lemma 3 (Chan and Lam, [3])**.** *For all $\lambda \vdash n$ and for $0 \leq i \leq \lfloor n/2 \rfloor$, the quantity $\alpha_{i,\lambda}$ is a non-negative integral multiple of 2^i. Moreover for $1 \leq i \leq \lfloor n/2 \rfloor$, $\alpha_{i,\lambda} = 0$ if and only if $l(\lambda) > n - i$.*

Theorem 4 (Chan and Lam, [4])**.** *Let L_T be the Laplacian matrix of a tree T on n vertices. Then, for all $\lambda \vdash n$, $d_\lambda(L_T) = \sum_{i=0}^{\lfloor n/2 \rfloor} \alpha_{i,\lambda} a_i(T)$, where $a_i(T)$ equals the number of vertex orientations with exactly i bidirected edges (and is hence a non-negative integer for all i).*

Lemma 3 and Theorem 4 make it clear that all immanants of L_T are non-negative. Similar results are known for the q-Laplacian of T (see Theorem 5). Define $\mathcal{L}_G^q = I + q^2(D - I) - qA$ as the q-Laplacian of a graph G, where D and A are as before and I is the identity matrix. Here q is a variable. It is easy to see that setting $q = 1$ in $\mathcal{L}_G^q$ gives us the usual combinatorial Laplacian L_G. The matrix $\mathcal{L}_G^q$ has appeared in the contexts of Ihara–Selberg zeta functions of graphs G (see Bass [2] and Foata and Zeilberger [7]). When the graph G is a tree T, $\mathcal{L}_T^q$ has connections with the inverse of the exponential distance matrix of T (see Bapat, Lal and Pati [1] and Nagar [10]). Nagar and Sivasubramanian in [11] gave the following q-analogue of Theorem 4 involving $\mathcal{L}_T^q$. To state it, we need a q-analogue of the term $a_i(T)$. We describe it briefly and refer the reader to [11] for more details. Let $\mathcal{O}_i(T)$ denote the set of vertex orientations in T with i bidirected edges, and let $a_i(T) = |\mathcal{O}_i(T)|$. A statistic $\mathsf{Lexaway} : \mathcal{O}_i(T) \mapsto \mathbb{Z}_{\geq 0}$ was defined in [11]. Using it, when $i \geq 1$, the following q-analogue $a_i(T, q)$ of $a_i(T)$ was defined (see [11, Corollary 9]):

$$a_i(T, q) = \sum_{O \in \mathcal{O}_i} q^{\mathsf{Lexaway}(O)}. \tag{5}$$

In [11], we had also defined $a_0(T,q) = 1 - q^2$. It turns out that for all $i \geq 0$, the $a_i(T,q)$ is a polynomial in q^2 (see [11, Remark 15]). With this definition, we can state a q-analogue of Theorem 4.

Theorem 5 (Nagar and Sivasubramanian)**.** *Let $\mathcal{L}_T^q$ be the q-Laplacian matrix of a tree T on n vertices. Then, for all $\lambda \vdash n$, $d_\lambda(\mathcal{L}_T^q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \alpha_{i,\lambda} a_i(T,q)$, where the $a_i(T,q)$ is defined in* (5).

Clearly, setting $q = 1$ in $\mathcal{L}_T^q$ gives L_T and setting $q = 1$ in $a_i(T,q)$ gives $a_i(T)$ for i with $0 \leq i \leq \lfloor n/2 \rfloor$. Extensions of Theorem 5 to the bivariate q,t-Laplacian denoted as $\mathcal{L}_T^{q,t}$ were also given in [11]. Later, in [12], the following more general result about the coefficient of $(-1)^r x^{n-r}$ in the immanantal polynomial $d_\lambda(xI - \mathcal{L}_T^q)$ was proved. To state it, we need polynomials $a_{i,r}(T,q)$ which generalize $a_i(T,q)$. Let $B \subset V(T)$ be a subset of the vertex set of T with $|B| = r$, where $r \leq n$. Let $\mathcal{O}_{B,i}^T$ be the set of vertex orientations of vertices in B that have i bidirected edges. There is a statistic $\mathsf{Aw}_B^T : \mathcal{O}_{B,i}^T \mapsto \mathbb{Z}_{\geq 0}$ with respect to which we define

$$a_{i,r}(T,q) = \sum_{B \subseteq V(T), |B| = r} \sum_{O \in \mathcal{O}_{B,i}^T} q^{\mathsf{Aw}(_B^T(O)}. \tag{6}$$

With this definition, we have the following [12, Lemma 7, Corollary 12].

Lemma 6 (Nagar and Sivasubramanian)**.** *Let T be a tree on n vertices with q-Laplacian $\mathcal{L}_T^q$. For $\lambda \vdash n$, let $d_\lambda(xI - \mathcal{L}_T^q) = \sum_{r=0}^n (-1)^r c_{\lambda,r}^{\mathcal{L}_T^q}(q) x^{n-r}$. Then for $0 \leq r \leq n$, we have $c_{\lambda,r}^{\mathcal{L}_T^q}(q) = \sum_{i=0}^{\lfloor r/2 \rfloor} \alpha_{i,\lambda} a_{i,r}(T,q)$.*

In Lemma 6, one can get the immanant $d_\lambda(\mathcal{L}_q^T)$ as the constant term of the immanantal polynomial (that is, by looking at the special case when $r = n$). Thus, Lemma 6 generalizes Theorem 5. Using Lemma 6, the following result about monotonicity for the coefficients of all immanantal polynomials of $\mathcal{L}_T^q$ along GTS_n was proved in [12, Theorem 1].

Theorem 7 (Nagar and Sivasubramanian)**.** *Let T_1 and T_2 be two trees with n vertices such that T_2 covers T_1 in GTS_n. Let $\mathcal{L}_{T_1}^q$ and $\mathcal{L}_{T_2}^q$ be the q-Laplacians of T_1 and T_2 respectively. For $\lambda \vdash n$, let*

$$\zeta_{s_\lambda}^{\mathcal{L}_{T_1}^q}(x) = d_\lambda(xI - \mathcal{L}_{T_1}^q) = \sum_{r=0}^n (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) x^{n-r} \text{ and}$$
$$\zeta_{s_\lambda}^{\mathcal{L}_{T_2}^q}(x) = d_\lambda(xI - \mathcal{L}_{T_2}^q) = \sum_{r=0}^n (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) x^{n-r}.$$

Then for all $\lambda \vdash n$, we have $c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) - c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \in \mathbb{R}^+[q^2]$, where $0 \leq r \leq n$.

The proof of Theorem 7 required to give a combinatorial expression for the coefficients $c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q)$ and $c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q)$ (see Lemma 6) and then to give an injection between the objects counting $c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q)$ and those counting $c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q)$ (see Lemma 11).

3. GMF OF q-LAPLACIANS ARISING FROM SYMMETRIC FUNCTIONS

Let $\Gamma_\gamma = \mathsf{ch}^{-1}(\gamma)$ be the inverse Frobenius image of the symmetric function $\gamma \in \Lambda_{\mathbb{Q}}^n$. Note that Γ_γ is a class function of $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by γ. Plugging the matrix $\mathcal{L}_T^q = (l_{i,j}^q)_{1\le i,j\le n}$ in (2), we get

$$d_\gamma(\mathcal{L}_T^q) = \sum_{\psi\in\mathfrak{S}_n} \Gamma_\gamma(\psi) \prod_{i=1}^n l_{i,\psi(i)}^q. \tag{7}$$

For the matrix $\mathcal{L}_T^q$, it is very easy to show the following counterpart of Theorem 5, where we change the Frobenius inverse of the Schur symmetric function to the Frobenius inverse of an arbitrary symmetric function. We recall the polynomial $a_i(T,q)$ from (5) and $\alpha_i(\gamma)$ from (3). We start with the following result. As its proof is a verbatim copy of the proof of Theorem 5, with the only change being that we replace $\alpha_{i,\lambda}$ by $\alpha_i(\gamma)$, we omit it.

Theorem 8. *Let $\mathcal{L}_T^q$ be the q-Laplacian matrix of a tree T on n vertices. Then, for all $\gamma \in \Lambda_{\mathbb{Q}}^n$, $d_\gamma(\mathcal{L}_T^q) = \sum_{i=0}^{\lfloor n/2\rfloor} \alpha_i(\gamma) a_i(T,q)$.*

It is very simple to show the following counterpart of Lemma 6. Recall the polynomial $a_{i,r}(T,q)$ from (6). Since the proof is identical, we omit it and merely state the result.

Lemma 9. *Let T be a tree on n vertices with q-Laplacian $\mathcal{L}_T^q$. Then for all $\gamma \in \Lambda_{\mathbb{Q}}^n$, we have $d_\gamma(xI - \mathcal{L}_T^q) = \sum_{r=0}^n (-1)^r c_{\gamma,r}^{\mathcal{L}_T^q}(q) x^{n-r}$, where $c_{\gamma,r}^{\mathcal{L}_T^q}(q) = \sum_{i=0}^{\lfloor r/2\rfloor} \alpha_i(\gamma) a_{i,r}(T,q)$ for $0 \le r \le n$.*

Every element $\gamma \in \Lambda_{\mathbb{Q}}^n$ is written as a linear combination of basis vectors of $\Lambda_{\mathbb{Q}}^n$. The vector space $\Lambda_{\mathbb{Q}}^n$ has six standard bases. We use standard terminology to denote each of the usual bases of $\Lambda_{\mathbb{Q}}^n$. Thus, s_λ, p_λ, e_λ, h_λ, m_λ and f_λ denote the Schur, power sum, elementary, homogeneous, monomial and forgotten symmetric functions respectively. The inverse Frobenius map of each of these will be denoted by the same letter, but in capital font and with the partition λ as a superscript rather than a subscript. Thus, $E^\lambda = \mathsf{ch}^{-1}(e_\lambda)$, $\chi_\lambda = S^\lambda = \mathsf{ch}^{-1}(s_\lambda)$ and so on.

As mentioned earlier, the inverse Frobenius image $\mathsf{ch}^{-1}(s_\lambda)$ of the Schur symmetric function s_λ is χ_λ, the irreducible character of $\mathfrak{S}_n$ over $\mathbb{C}$ indexed by λ. Therefore the identity given in (4) is a special case of the identity given in (3) when $\gamma = s_\lambda$. Thus, if any symmetric function $\gamma \in \Lambda_{\mathbb{Q}}^n$ is Schur-positive (that is, $\gamma = \sum_{\lambda\vdash n} a_\lambda s_\lambda$ where $a_\lambda \in \mathbb{R}^+$ for all $\lambda \vdash n$), then, by linearity, Lemma 3 will be true with χ_λ replaced by $\mathsf{ch}^{-1}(\gamma)$ in (4). Unfortunately, the monomial symmetric function m_λ indexed by λ is not Schur-positive. Thus, if $M^\lambda = \mathsf{ch}^{-1}(m_\lambda)$, it is not clear that Lemma 3 with χ_λ replaced by M^λ in (4) is true.

For $\gamma \in \Lambda_{\mathbb{Q}}^n$, let $\Gamma_\gamma = \mathsf{ch}^{-1}(\gamma)$. Thus, if $\Gamma_\gamma(j) \ge 0$ for all j, then from (3), we get $\alpha_i(\gamma) \ge 0$. Since the inverse Frobenius image Γ_{p_λ} of p_λ is a scalar multiple of the indicator function of the conjugacy class C_λ indexed by λ (see [14]), it follows that $\alpha_i(p_\lambda) \ge 0$ for all $\lambda \vdash n$ and for all $i = 0, 1, \ldots, \lfloor n/2\rfloor$.

As mentioned earlier, if $\gamma \in \Lambda_{\mathbb{Q}}^n$ is Schur-positive, then $\alpha_i(\gamma) \ge 0$. Since h_λ is Schur-positive (see [14, Corollary 7.12.4]), it follows from Lemma 3 that $\alpha_i(h_\lambda) \ge 0$ for all $\lambda \vdash n$. Similarly, it is well known that e_λ is also Schur-positive. Thus $\alpha_i(e_\lambda) \ge 0$ for all $\lambda \vdash n$. We record these facts below for future use.

Lemma 10. *For all $\lambda \vdash n$ and for $0 \le i \le \lfloor n/2\rfloor$, we have $\alpha_i(\gamma) \ge 0$ for $\gamma \in \{p_\lambda, h_\lambda, e_\lambda\}$.*

Let the tree T have n vertices and, for $0 \leq i, r \leq n$, let $a_{i,r}(T,q) = a_{i,r}^T(q)$ be the polynomial in q^2 with non-negative real coefficients, defined by Nagar and Sivasubramanian in [12, page 7]. They showed the following result (see [12, Lemma 19, 23 and Corollary 22]).

Lemma 11 (Nagar and Sivasubramanian)**.** *Let T_1 and T_2 be two trees with n vertices such that T_2 covers T_1 in GTS_n. Then for all $0 \leq i, r \leq n$, we have $a_{i,r}(T_1, q) - a_{i,r}(T_2, q) \in \mathbb{R}^+[q^2]$.*

With Lemma 10 and Lemma 11 in place, we can get the following Corollary. Its proof mimics the proof of Theorem 7.

Corollary 12. *Theorem 7 is true when we replace the immanantal polynomial $d_\lambda(xI - \mathcal{L}_{T_j}^q)$ by the generalized matrix polynomials $d_\gamma(xI - \mathcal{L}_{T_j}^q)$ for $\gamma \in \{p_\lambda, h_\lambda, e_\lambda\}$ and for $j = 1, 2$.*

By Theorem 7 and Corollary 12, we thus have monotonicity results about four of the six standard bases, as we move up on GTS_n. This paper plugs the gaps left by considering the generalized matrix polynomials of $\mathcal{L}_T^q$ arising from the last two bases m_λ and f_λ. In other words, we consider the cases when we replace the immanantal polynomial $d_\lambda(xI - \mathcal{L}_{T_j}^q)$ by $d_g(xI - \mathcal{L}_{T_j}^q)$ when $g \in \{m_\lambda, f_\lambda\}$ in Theorem 7.

4. GMF of tree Laplacians arising from m_λ, the monomial symmetric function

We prove Theorem 1 in this section. We need the notion of λ-brick tabloids of shape μ which is used to give a combinatorial interpretation of the quantity $\Gamma_{m_\lambda}(\psi) = M^\lambda(\psi)$, where $\lambda, \mu \vdash n$ and $\psi \in \mathfrak{S}_n$. We recall the following definition of λ-brick tabloid of shape μ defined by Eğecioğlu and Remmel in [6].

Definition 13. *Let $\lambda \vdash n$ with $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$. Let $\mu \vdash n$ and let F_μ be its Ferrers diagram. A λ-brick tabloid $\mathcal{B}_{\lambda,\mu}$ of shape μ is a filling of F_μ with bricks $b_1, b_2, \ldots, b_{l(\lambda)}$ of size $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$ respectively such that brick b_i covers exactly λ_i squares of F_μ all of which lie in a single row of F_μ and no two bricks overlap. Here, bricks of the same size are indistinguishable.*

We refer the reader to the book by Mendes and Remmel [9, Chapter 2] for an introduction to λ-brick tabloids. Given a brick b in $\mathcal{B}_{\lambda,\mu}$, let $|b|$ denote the length of b. Define $\mathsf{wt}_{\mathcal{B}_{\lambda,\mu}}(b)$ to be $|b|$ if b is at the end of a row in $\mathcal{B}_{\lambda,\mu}$ and 1 otherwise. We next define a weight $w(\mathcal{B}_{\lambda,\mu})$ for each λ-brick tabloid $\mathcal{B}_{\lambda,\mu}$ of shape μ as follows:

$$w(\mathcal{B}_{\lambda,\mu}) = \prod_{b \in \mathcal{B}_{\lambda,\mu}} \mathsf{wt}_{\mathcal{B}_{\lambda,\mu}}(b).$$

In other words, $w(\mathcal{B}_{\lambda,\mu})$ is the product of the lengths of the rightmost brick in each row of $\mathcal{B}_{\lambda,\mu}$. For $\lambda, \mu \vdash n$, let $\mathsf{BT}_{\lambda,\mu}$ be the set of all λ-brick tabloids of shape μ. For $\lambda \vdash n$ and for $\psi \in \mathfrak{S}_n$, Eğecioğlu and Remmel in [6, Theorem 1] gave the following combinatorial interpretation of $M^\lambda(\psi)$ which we need.

Theorem 14 (Eğecioğlu and Remmel)**.** *For $\lambda \vdash n$, let $M^\lambda = \mathsf{ch}^{-1}(m_\lambda)$. When a permutation $\psi \in \mathfrak{S}_n$ has cycle type μ, then*

$$M^\lambda(\psi) = (-1)^{l(\lambda)-l(\mu)} \sum_{\mathcal{B}_{\lambda,\mu} \in \mathsf{BT}_{\lambda,\mu}} w(\mathcal{B}_{\lambda,\mu}).$$

We say λ is a refinement of μ if the parts μ_i of μ can be obtained as a disjoint union of the parts λ_j of λ. From the definition of $\mathcal{B}_{\lambda,\mu}$, it is easy to see that if λ is not a refinement of μ, then $\mathsf{BT}_{\lambda,\mu} = \emptyset$. Thus $M^\lambda(\psi) = 0$, unless when λ refines the cycle type μ of ψ. Let $M^\lambda(j)$ denote the value of the function $M^\lambda(\cdot)$ evaluated at a permutation $\psi \in \mathfrak{S}_n$ with cycle type $2^j, 1^{n-2j}$. We start with the following simple consequence of Theorem 14,

Corollary 15. *Let $\lambda \vdash n$ and let $\psi \in \mathfrak{S}_n$ be a permutation with cycle type $\mu = 2^j, 1^{n-2j}$. Then,*

$$M^\lambda(j) = \begin{cases} (-1)^{j-k} 2^k \binom{j}{k}, & \text{if } \lambda = 2^k, 1^{n-2k} \text{ and } k \le j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We divide the proof into two cases when λ is a refinement of μ and when it is not. If λ is a refinement of μ, then we must have $\lambda = 2^k, 1^{n-2k}$ for some k with $0 \le k \le j$. In this case, all the k bricks of length 2 must be placed in some of the k rows from the first j rows of F_μ. Thus, the number of such λ-brick tabloids $\mathcal{B}_{\lambda,\mu}$ of shape μ is $\binom{j}{k}$, and each $\mathcal{B}_{\lambda,\mu}$ contributes the weight 2^k in the summation of Theorem 14. Thus, the total contribution in $M^\lambda(j)$ is $2^k\binom{j}{k}$ and $l(\lambda) - l(\mu) = j - k$. Hence $M^\lambda(j) = (-1)^{j-k} 2^k \binom{j}{k}$.

When λ is not a refinement of μ, by Theorem 14, $M^\lambda(j) = 0$, completing the proof. □

We next prove Lemma 2 which says that for all $\lambda \vdash n$ and for $0 \le i \le \lfloor n/2 \rfloor$, the quantity $\alpha_i(m_\lambda)$ is a non-negative integral multiple of 2^i. We will need the following very easy identity involving binomial coefficients $\binom{j}{k}\binom{i}{j} = \binom{i-k}{j-k}\binom{i}{k}$.

Proof of Lemma 2. By Corollary 15, when $\lambda \ne 2^k, 1^{n-2k}$, $M^\lambda(j) = 0$ for all j. Thus, from (3), $\alpha_i(m_\lambda) = 0$ for $0 \le i \le \lfloor n/2 \rfloor$. When $\lambda = 2^k, 1^{n-2k} \vdash n$ for some k with $0 \le k \le \lfloor n/2 \rfloor$, by Corollary 15, we see that

$$\begin{aligned} \alpha_i(m_\lambda) &= \sum_{j=0}^{i} M^\lambda(j) \binom{i}{j} = \sum_{j=0}^{i} (-1)^{j-k} 2^k \binom{j}{k}\binom{i}{j} \\ &= 2^k \binom{i}{k} \sum_{j=k}^{i} (-1)^{j-k} \binom{i-k}{j-k} = \begin{cases} 2^i, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The proof is complete. □

Using Lemmas 2 and 11, we can now prove Theorem 1.

Proof of Theorem 1. From Lemmas 2 and 9, for $0 \le r \le n$ and for $j = 1, 2$, it is simple to see that the coefficient of $(-1)^r x^{n-r}$ in $d_{m_\lambda}(xI - \mathcal{L}_{T_j}^q)$ is given by

$$c_{m_\lambda,r}^{\mathcal{L}_{T_j}^q}(q) = \sum_{i=0}^{\lfloor r/2 \rfloor} \alpha_i(m_\lambda) a_{i,r}(T_j, q) = \begin{cases} 2^k a_{k,r}(T_j, q), & \text{if } \lambda = 2^k, 1^{n-2k} \text{ and } k \le \lfloor r/2 \rfloor, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

By Lemma 11 and (8), we get $c_{m_\lambda,r}^{\mathcal{L}_{T_1}^q}(q) - c_{m_\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \in \mathbb{R}^+[q^2]$ for all $\lambda \vdash n$ and for $0 \le r \le n$. Furthermore, if $\lambda \ne 2^k, 1^{n-2k} \vdash n$ for all k, then from (8), the coefficient of $(-1)^r x^{n-r}$ in $d_{m_\lambda}(xI - \mathcal{L}_{T_j}^q)$ is zero for $0 \le r \le n$ and $j = 1, 2$. The proof is complete. □

Thus, for all $\lambda \vdash n$ and for all $q \in \mathbb{R}$, moving up on GTS_n decreases the coefficient of $(-1)^r x^{n-r}$ in $d_{m_\lambda}(xI - \mathcal{L}_T^q)$, where $0 \leq r \leq n$. Consequently, for all positive integers n, this monotonicity result on GTS_n shows that the max-min pair of these coefficients is (P_n, S_n), where P_n and S_n are the path tree and the star tree on n vertices respectively. The following example illustrates Lemma 2.

Example 16. *Let $n = 15$. For all $\lambda \vdash n$ and for all $i \in \{0,1,2,3,4,5,6,7\}$, the values of the quantity $\alpha_i(m_\lambda)$ are tabulated in Table 1.*

$\lambda \vdash n = 15$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$\lambda = 1^{15}$	1	0	0	0	0	0	0	0
$\lambda = 2, 1^{13}$	0	2	0	0	0	0	0	0
$\lambda = 2^2, 1^{11}$	0	0	4	0	0	0	0	0
$\lambda = 2^3, 1^9$	0	0	0	8	0	0	0	0
$\lambda = 2^4, 1^7$	0	0	0	0	16	0	0	0
$\lambda = 2^5, 1^5$	0	0	0	0	0	32	0	0
$\lambda = 2^6, 1^3$	0	0	0	0	0	0	64	0
$\lambda = 2^7, 1$	0	0	0	0	0	0	0	128
$\lambda \neq 2^k, 1^{n-2k}$	0	0	0	0	0	0	0	0

TABLE 1. The value of $\alpha_i(m_\lambda)$.

5. GMF of tree Laplacians arising from f_λ, the forgotten symmetric function

Let f_λ denote the forgotten symmetric function indexed by $\lambda \vdash n$ and let $F^\lambda = \mathsf{ch}^{-1}(f_\lambda)$ denote its inverse Frobenius image. This section is devoted to show monotonicity of the coefficient of $(-1)^r x^{n-r}$ in $\zeta_{f_\lambda}^{\mathcal{L}_T^q}(x) = d_{f_\lambda}(xI - \mathcal{L}_T^q)$ when we move up along GTS_n. For this, we need the following combinatorial interpretation of $F^\lambda(\psi)$ by Eğecioğlu and Remmel in [6, Theorem 8].

Theorem 17 (Eğecioğlu and Remmel)**.** *Let $\lambda \vdash n$ and let $\psi \in \mathfrak{S}_n$ be a permutation with cycle type μ. Then,*

$$F^\lambda(\psi) = (-1)^{n-l(\mu)} \sum_{\mathcal{B}_{\lambda,\mu} \in \mathsf{BT}_{\lambda,\mu}} w(\mathcal{B}_{\lambda,\mu}).$$

For $\lambda \vdash n$, let $F^\lambda(j)$ denote the value of the function $F^\lambda(\cdot)$ evaluated at a permutation $\psi \in \mathfrak{S}_n$ with cycle type $2^j, 1^{n-2j}$. By Theorem 17, the proof of the following corollary is identical to the proof of Corollary 15, we omit it and merely state the result.

Corollary 18. *For $\lambda \vdash n$ and for $0 \leq j \leq \lfloor n/2 \rfloor$, we have*

$$F^\lambda(j) = \begin{cases} (-1)^j 2^k \binom{j}{k}, & \textit{if } \lambda = 2^k, 1^{n-2k} \textit{ and } k \leq j, \\ 0, & \textit{otherwise.} \end{cases}$$

For a fixed $\lambda \vdash n$, let $\gamma = f_\lambda$ in (3). We next calculate the quantity $\alpha_i(f_\lambda)$ in the following lemma which will be used later in the determination of the coefficient of $(-1)^r x^{n-r}$ in $d_{f_\lambda}(xI - \mathcal{L}_T^q)$.

Lemma 19. *Let $\lambda \vdash n$. Then for $0 \leq i \leq \lfloor n/2 \rfloor$, we have*

$$\alpha_i(f_\lambda) = \begin{cases} (-1)^i 2^i, & \text{if } \lambda = 2^i, 1^{n-2i}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 18 when $\lambda \neq 2^k, 1^{n-2k}$, $F^\lambda(j) = 0$ for all j. By (3), $\alpha_i(f_\lambda) = 0$ for $0 \leq i \leq \lfloor n/2 \rfloor$. We next assume $\lambda = 2^k, 1^{n-2k} \vdash n$ for some k with $0 \leq k \leq \lfloor n/2 \rfloor$. In this case, by Corollary 18, we see that

$$\begin{aligned} \alpha_i(f_\lambda) &= \sum_{j=0}^{i} F^\lambda(j) \binom{i}{j} = \sum_{j=0}^{i} (-1)^j 2^k \binom{j}{k} \binom{i}{j} \\ &= 2^k \binom{i}{k} \sum_{j=k}^{i} (-1)^j \binom{i-k}{j-k} = \begin{cases} (-1)^i 2^i, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

All equalities above follow by simple manipulations and hence the proof is complete. □

Theorem 20. *Let T be a tree on n vertices with q-Laplacian matrix $\mathcal{L}_T^q$. Then for $0 \leq r \leq n$, the coefficient of $(-1)^r x^{n-r}$ in $d_{f_\lambda}(xI - \mathcal{L}_T^q)$ is given by*

$$c_{f_\lambda,r}^{\mathcal{L}_T^q}(q) = \begin{cases} (-1)^k 2^k a_{k,r}(T,q), & \text{if } \lambda = 2^k, 1^{n-2k} \text{ with } k \leq \lfloor r/2 \rfloor, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for all $\lambda \vdash n$ and all $q \in \mathbb{R}$, moving up on GTS_n decreases $c_{f_\lambda,r}^{\mathcal{L}_T^q}(q)$ in absolute value.

Proof. From Lemmas 9 and 19, it is simple to see that

$$c_{f_\lambda,r}^{\mathcal{L}_T^q}(q) = \sum_{i=0}^{\lfloor r/2 \rfloor} \alpha_i(f_\lambda) a_{i,r}(T,q) = \begin{cases} (-1)^k 2^k a_{k,r}(T,q), & \text{if } \lambda = 2^k, 1^{n-2k} \text{ and } k \leq \lfloor r/2 \rfloor, \\ 0, & \text{otherwise.} \end{cases}$$

Let T_1 and T_2 be two trees with n vertices such that T_2 covers T_1 in GTS_n. By Lemma 11, $\left| c_{f_\lambda,r}^{\mathcal{L}_{T_1}^q}(q) \right| - \left| c_{f_\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \right| \in \mathbb{R}^+[q^2]$ completing the proof. □

Thus, for all $\lambda \vdash n$ when $0 \leq r \leq n$, by Theorem 1, Theorem 20 and Corollary 12, we get that the coefficient of x^r in $d_\gamma(xI - \mathcal{L}_T^q)$ decreases as we move up along GTS_n in absolute value for each $\gamma \in \{m_\lambda, s_\lambda, p_\lambda, h_\lambda, e_\lambda, f_\lambda\}$. Plugging in $q = 1$, we get the following corollary of Theorem 1, Theorem 20 and Corollary 12.

Corollary 21. *For all $\lambda \vdash n$ and for all $\gamma \in \{m_\lambda, s_\lambda, p_\lambda, h_\lambda, e_\lambda, f_\lambda\}$, moving up on GTS_n decreases the coefficient of x^r in $d_\gamma(xI - L_T)$ in absolute value, for $0 \leq r \leq n$. Here L_T is the usual combinatorial Laplacian of the tree T.*

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