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MODULAR NEKRASOV–OKOUNKOV FORMULAS

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To Christian Krattenthaler on the occasion of his 60th birthday

Abstract. Using Littlewood's map, which decomposes a partition into its r-core and r-quotient, Han and Ji have shown that many well-known hook-length formulas admit modular analogues. In this paper we present a variant of the Han–Ji 'multiplication theorem' based on a new analogue of Littlewood's decomposition. We discuss several applications to hook-length formulas, one of which leads us to conjecture a modular analogue of the q,t-Nekrasov–Okounkov formula.

1. Introduction

Hook-length formulas abound in combinatorics and representation theory. Perhaps the most famous example is the formula for f^{λ}, the number of standard Young tableaux of shape λ, which was discovered in 1954 by Frame, Robinson and Thrall [22]. If λ is a partition of n and $\mathscr{H}(\lambda)$ denotes the multiset of hook-lengths of λ (see Section 3 for notation and definitions), then

$$f^{\lambda}=\frac{n!}{\prod_{h\in\mathscr{H}(\lambda)} h}. \tag{1.1}$$

A much more recent identity in the spirit of (1.1) is the Nekrasov–Okounkov formula. It was discovered independently by Nekrasov and Okounkov [44] in their work on random partitions and Seiberg–Witten theory, and by Westbury [56] in his work on universal characters for $\mathfrak{sl}_n$. The form in which the formula is commonly stated is that of Nekrasov and Okounkov (see [44, Equation (6.12)])

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}\Big(1-\frac{z}{h^2}\Big)=\prod_{k\geqslant 1}(1-T^k)^{z-1} \tag{1.2}$$

(where $\mathscr{P}$ is the set of all partitions) rather than Westbury's hook-length formula for the D'Arcais polynomials $P_n(z)$, defined by the expansion [17]

$$\prod_{k\geqslant 1}\frac{1}{(1-T^k)^z}=\sum_{n=0}^{\infty}P_n(z)T^n,$$

and implied by Propositions 6.1 and 6.2 of his paper [56].

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The Nekrasov–Okounkov formula has attracted significant attention in a number of different areas of mathematics and physics, including algebraic geometry, combinatorics, number theory and string theory. It has seen q-generalisations [19, 36], q,t-generalisations [5, 6, 12, 13, 32, 33, 35, 48, 49], an elliptic analogue [40, 49, 55], modular analogues [19, 27, 28], and generalisations to the affine Lie algebras $\mathrm{C}_n^{(1)}$ [46] and $\mathrm{D}_{n+1}^{(2)}$ [47]. It has also sparked the study of several combinatorial problems on partitions and hook-length statistics, see e.g., [2, 4, 20, 23, 30, 31, 34, 45, 53], and has given new impetus to the study of the arithmetical properties of Euler-type products, see e.g., [14, 16, 24, 29].

Let

$$(a_1,a_2,\dots,a_k;q_1,q_2,\dots,q_m)_\infty:=\prod_{i=1}^k\prod_{j_1,\dots,j_m\geqslant 0}\big(1-a_iq_1^{j_1}q_2^{j_2}\cdots q_m^{j_m}\big)$$

be a multiple q-shifted factorial and, for λ a partition and r a positive integer, let $\mathscr{H}_r(\lambda)$ denote the multiset of hook-lengths of λ that are congruent to 0 modulo r. Then one particularly interesting generalisation of the Nekrasov–Okounkov formula is Han's modular analogue [27, Theorem 1.3]

$$\sum_{\lambda\in\mathscr{P}}T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{h\in\mathscr{H}_r(\lambda)}\Big(1-\frac{z}{h^2}\Big)=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty(ST^r;ST^r)_\infty^{r-z/r}}. \tag{1.3}$$

He proved this identity by combining (1.2) with Littlewood's decomposition, which is a generalisation of Euclidean division to integer partitions that has played a key role in the modular representation theory of the symmetric group. The question of modular analogues of hook-length formulas was further pursued by Han in subsequent papers with Dehaye [19] and Ji [28]. The most general statement was formulated in this last paper.

Theorem 1.1 ('Multiplication theorem' [28, Theorem 1.5])**.** *For r a positive integer and ρ a function on the positive integers, let $f_r(T)$ be the formal power series defined by*

$$f_r(T):=\sum_{\lambda\in\mathscr{P}}T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}\rho(rh). \tag{1.4}$$

Then

$$\sum_{\lambda\in\mathscr{P}}T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{h\in\mathscr{H}_r(\lambda)}\rho(h)=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty}\big(f_r(ST^r)\big)^r. \tag{1.5}$$

Whenever ρ is chosen such that $f_r(T)$ admits a closed-form expression, the above theorem immediately implies a modular analogue of the hook-length formula

$$\sum_{\lambda\in\mathscr{P}}T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}\rho(h)=f_1(T).$$

In this paper we describe a variant of Littlewood's decomposition which implies an analogue of the Han–Ji multiplication theorem for hook-length formulas involving the hook-lengths of squares of partitions that have trivial leg-length. In combination with the Han–Ji multiplication theorem, this suggests a modular analogue of the q,t-analogue of (1.2), see e.g., [12, Theorem 1.0.2] or [49, Theorem 1.3]

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}\prod_{s\in\lambda}\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}=\frac{(uqT,u^{-1}tT;q,t,T)_\infty}{(T,tT;q,t,T)_\infty}, \tag{1.6}$$

where $a(s)$ and $l(s)$ are the arm-length and leg-length of the square $s\in\lambda$. In its simplest form this conjecture can be stated as follows.

Conjecture 1.2. *For r a positive integer*

$$\begin{aligned}
&\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{\substack{s\in\lambda\\ h(s)\equiv 0 \ (\mathrm{mod}\ r)}}\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}\\
&\quad=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty(ST^r;ST^r)_\infty^r}\prod_{\substack{i,j\geqslant 1\\ i+j\equiv 1 \ (\mathrm{mod}\ r)}}\frac{(uq^it^{j-1}ST^r,u^{-1}q^{i-1}t^jST^r;ST^r)_\infty}{(q^it^{j-1}ST^r,q^{i-1}t^jST^r;ST^r)_\infty}\\
&\quad=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty(ST^r;ST^r)_\infty^r}\prod_{i=1}^{r}\frac{(uq^it^{r-i}ST^r,u^{-1}q^{r-i}t^iST^r;q^r,t^r,ST^r)_\infty}{(q^it^{r-i}ST^r,q^{r-i}t^iST^r;q^r,t^r,ST^r)_\infty}.
\end{aligned}$$

The remainder of this paper is organised as follows. After some preliminary discussions on formal power series and integer partitions in the next two sections, Section 4 reviews Littlewood's classical decomposition of a partition into its r-core and r-quotient. This is used in Section 5 in our discussion of the Han–Ji multiplication theorem. Then, in Section 6, we propose an analogue of Littlewood's decomposition. This new decomposition is applied to prove an analogue of Theorem 1.1 by Han and Ji. Section 7 contains a number of applications of this new multiplication theorem to hook-length formulas. Finally, in Section 8 we present some conjectures and open problems. This includes a refinement of Conjecture 1.2, some problems pertaining to elliptic q,t-Nekrasov–Okounkov formulas (Proposition 8.3 of that section does prove an elliptic q-analogue of (1.3)) and a discussion of a possible extension of our new multiplication theorem motivated by a combinatorial identity of Buryak, Feigin and Nakajima which arose in their work on quasihomogeneous Hilbert schemes.

2. Formal power series

All series and series identities considered in this paper are viewed from the point of view of formal power series, typically in the formal variable T. For example, we regard the Nekrasov–Okounkov formula (1.2) as an identity in $\mathbb{Q}[z][[T]]$, where

$$(1-T)^z:=\sum_{n\geqslant 0}\binom{z}{n}(-T)^n$$

and

$$\binom{z}{n}:=\frac{z(z-1)\cdots(z-n+1)}{n!}.$$

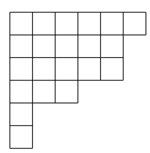
Those preferring an analytic point of view should have little trouble adding the required convergence conditions. In the case of (1.2), for example, it suffices to take $T \in \mathbb{C}$, $z \in \mathbb{R}$ such that $|T| < 1$, or $-1 < T < 1$ and $z \in \mathbb{C}$.

3. Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of nonnegative integers such that only finitely many λ_i's are strictly positive. The positive λ_i's are called the parts of λ, and the length of λ, denoted $\ell(\lambda)$, counts the number of parts. If $|\lambda| := \sum_{i\geqslant 1} \lambda_i = n$ we say that λ is a partition of n, denoted as $\lambda \vdash n$. We typically suppress the infinite tail of zeros of a partition, so that, for example, the partition $(6,5,5,3,1,1,0,\dots)$ of 21 is denoted by $(6,5,5,3,1,1)$. The set of all partitions, including the unique partition of 0 (also written as 0), is denoted by $\mathscr{P}$. This set has the well-known generating function

$$\sum_{\lambda\in\mathscr{P}} z^{\ell(\lambda)} T^{|\lambda|} = \frac{1}{(zT;T)_\infty}. \tag{3.1}$$

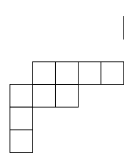
We identify a partition λ with its Young diagram, consisting of $\ell(\lambda)$ left-aligned rows of squares such that the ith row contains λ_i squares. For example, the partition $(6,5,5,3,1,1)$ corresponds to the diagram

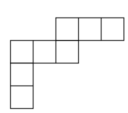


The squares of λ are indexed by coordinates $(i,j) \in \mathbb{N}^2$, with i the row and j the column coordinate, such that the top-left square corresponds to $(1,1)$.

The conjugate λ' of the partition λ is obtained by reflecting λ in the diagonal $i = j$, so that rows become columns and vice versa. The conjugate of the partition in our running example is $(6,4,4,3,3,1)$. Given a partition λ, the multiplicity of parts of size i, denoted $m_i(\lambda)$, can be expressed in terms of λ' as $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$. We alternatively write partitions using the multiplicities, so that $(6,4,4,3,3,1) = (6,4^2,3^2,1)$.

Given two partitions λ and μ such that $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$, the skew diagram λ/μ is the diagram obtained from λ by deleting the squares belonging to μ. For example, the skew diagrams $(6,5,5,3,1,1)/(5,5,1)$ and $(6,5,5,3,1,1)/(6,5,2)$ are given by





respectively. A skew diagram λ/μ is called a border strip (or ribbon) if it is connected (i.e., $\mu_i < \lambda_{i+1}$ for all i) and does not contains a 2×2 block of squares (i.e., $\lambda_{i+1} - \mu_i \leqslant 1$ for all i). The size and height of a border strip are given by the number of squares and the number of rows minus 1, respectively. In the above example, $(6,5,5,3,1,1)/(6,5,2)$ is a border strip of size 8 and height 3.

To each square $s = (i, j) \in \lambda$ we associate an arm, leg and hook, defined as the sets of squares

$$\begin{aligned} \operatorname{arm}(s) &:= \{(i, k) : j < k \leqslant \lambda_i\}, \\ \operatorname{leg}(s) &:= \{(k, j) : i < k \leqslant \lambda'_j\}, \\ \operatorname{hook}(s) &:= \operatorname{arm}(s) \cup \operatorname{leg}(s) \cup \{s\}. \end{aligned}$$

In the following two diagrams of the partition $(6, 5, 5, 3, 1, 1)$, the arm and leg of the square $s = (2, 2)$ are coloured dark and light blue on the left and the hook of s is coloured on the right:

Correspondingly, we have the three statistics $a(s)$, $l(s)$ and $h(s)$, known as arm-length, leg-length and hook-length, given by

$$\begin{aligned} a(s) &:= |\operatorname{arm}(s)| = \lambda_i - j, \\ l(s) &:= |\operatorname{leg}(s)| = \lambda'_j - i, \\ h(s) &:= |\operatorname{hook}(s)| = \lambda_i + \lambda'_j - i - j + 1. \end{aligned}$$

For r a positive integer, the multiset of hook-lengths of λ congruent to 0 modulo r is denoted by $\mathscr{H}_r(\lambda)$. When $r = 1$ we more simply write $\mathscr{H}(\lambda)$ for the multiset of all hook-lengths, omitting the subscript 1. It is often convenient to record $\mathscr{H}_r(\lambda)$ or $\mathscr{H}(\lambda)$ by writing the hook-lengths in the diagram of λ. For the partition in our example this gives

$$\mathscr{H}(6,5,5,3,1,1) = \begin{array}{llllll} 11 & 8 & 7 & 5 & 4 & {\color{red}1} \\ 9 & 6 & 5 & 3 & 2 & \\ 8 & 5 & 4 & {\color{red}2} & {\color{red}1} & \\ 5 & {\color{red}2} & {\color{red}1} & & & \\ 2 & & & & & \\ {\color{red}1} & & & & & \end{array} \qquad \mathscr{H}_2(6,5,5,3,1,1) = \begin{array}{llllll} & 8 & & & 4 & \\ & 6 & & & 2 & \\ 8 & & 4 & {\color{red}2} & & \\ & {\color{red}2} & & & & \\ 2 & & & & & \\ & & & & & \end{array}$$

where the colouring of some of the hook-lengths is to be ignored for now.

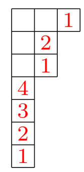
We refer to a square $s \in \lambda$ as a 'bottom square' if it has coordinates (λ'_j, j) for some $1 \leqslant j \leqslant \lambda_1$. In other words, a square $s \in \lambda$ is a bottom square if it has leg-length $l(s)$ equal to zero. (In [34] such squares are referred to as having 'trivial legs'.) In analogy with $\mathscr{H}_r(\lambda)$, we write $\mathscr{H}_r^{(\mathrm{b})}(\lambda)$ for the multiset of hook-lengths of bottom squares that are congruent to 0 modulo r, and set $\mathscr{H}^{(\mathrm{b})}(\lambda) := \mathscr{H}_1^{(\mathrm{b})}(\lambda)$. The hook-lengths coloured red in the above two diagrams correspond to those bottoms squares that contribute to $\mathscr{H}^{(\mathrm{b})}(6, 5, 5, 3, 1, 1)$ and $\mathscr{H}_2^{(\mathrm{b})}(6, 5, 5, 3, 1, 1)$, respectively. Thus

$$\mathscr{H}^{(\mathrm{b})}(6, 5, 5, 3, 1, 1) = \{1^4, 2^2\} \quad \text{and} \quad \mathscr{H}_2^{(\mathrm{b})}(6, 5, 5, 3, 1, 1) = \{2^2\}.$$

We also use the notation $\mathscr{H}_r^{(\mathrm{r})}(\lambda) := \mathscr{H}_r^{(\mathrm{b})}(\lambda')$ (r for right), so that $\mathscr{H}^{(\mathrm{r})}(\lambda)$ is the multiset of hook-lengths of squares $s \in \lambda$ which have trivial arm. We may think of

$$\ell_r(\lambda) := \lvert \mathscr{H}_r^{(\mathrm{r})}(\lambda) \rvert = \sum_{i \geqslant 1} \Big\lfloor \frac{m_i(\lambda)}{r} \Big\rfloor \tag{3.2}$$

as a modular generalisation of the ordinary length statistic on partitions, counting the number of squares $s = (i, \lambda_i)$ of λ such that $h(s) \equiv 0 \pmod{r}$. For example, for the partition $\lambda = (3, 2, 2, 1, 1, 1, 1)$ we have $\ell(\lambda) = \ell_1(\lambda) = 7$, $\ell_2(\lambda) = 3$, $\ell_3(\lambda) = \ell_4(\lambda) = 1$ and $\ell_r(\lambda) = 0$ for $r \geqslant 5$:



A partition λ is called an r-core if $\mathscr{H}_r(\lambda) = \emptyset$, i.e., if none of its hook-lengths is a multiple of r. We use $\mathscr{C}_r$ to denote the set of r-cores. Note that $\mathscr{C}_1 = \{0\}$ and $\mathscr{C}_2$ is the set of staircase partitions:

$$\mathscr{C}_2 = \{\delta_n : n \geqslant 1\},$$

where $\delta_n := (n-1, \dots, 2, 1, 0)$. In much the same way, we say that λ is an r-kernel if $\mathscr{H}_r^{(\mathrm{b})}(\lambda) = \emptyset$, and denote the set of r-kernels by $\mathscr{K}_r$. Clearly, $\mathscr{C}_r \subset \mathscr{K}_r$, where the inclusion is strict unless $r = 1$. $\mathscr{K}_r$ is given by the set of partitions λ such that the differences between consecutive λ_i are at most $r - 1$:

$$\mathscr{K}_r = \big\{\lambda \in \mathscr{P} : \lambda_i - \lambda_{i+1} < r \text{ for all } i \geqslant 1\big\}. \tag{3.3}$$

It is an elementary fact, see e.g., [3], that such partitions have generating function

$$\sum_{\lambda \in \mathscr{K}_r} z^{\lambda_1} T^{|\lambda|} = \frac{(z^r T^r; T^r)_\infty}{(zT; T)_\infty}. \tag{3.4}$$

4. Littlewood's decomposition

Littlewood's decomposition

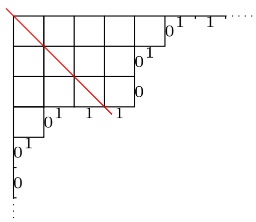
$$\begin{aligned} \phi_r : \mathscr{P} &\longrightarrow \mathscr{C}_r \times \mathscr{P}^r \\ \lambda &\longmapsto (\mu, \boldsymbol{\nu}) = \big(\mu, (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(r-1)})\big) \end{aligned}$$

is a generalisation of Euclidean division to integer partitions, and first arose in the modular representation theory of the symmetric group [41, 43]. Given a positive integer r, it decomposes a partition λ into an r-core, μ, and a sequence $\boldsymbol{\nu} = (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(r-1)})$ of r partitions, known as the r-quotient of λ. Instead of μ we will sometimes write r-core(λ) for the r-core of λ. We also use the shorthand notation

$$|\boldsymbol{\nu}| := \sum_{i=0}^{r-1} |\nu^{(i)}| \quad \text{and} \quad \mathscr{H}(\boldsymbol{\nu}) := \bigcup_{i=0}^{r-1} \mathscr{H}(\nu^{(i)}),$$

where the union is that of multisets.

There are numerous equivalent descriptions of ϕ_r, see e.g., [1, 25, 28, 37, 39, 42, 54]. Given a partition λ we form its bi-infinite edge or 0/1-sequence $c = c(\lambda)$ (also known as the code of λ) by tracing the extended boundary of λ, encoding an up step by a 0 and a right step by a 1. (When 0s are replaced by black beads and 1s by white beads, such a sequence is also known as a Maya diagram [18, § 4.1].) For example, the 0/1-sequence of the partition $(5, 4, 4, 1)$ is determined as



$$\longmapsto \quad \dots 0001011 \,|\, 10010111 \cdots = \dots c_{-3}c_{-2}c_{-1} \,|\, c_0c_1c_2 \dots$$

Here we have put a marker in the 0/1-sequence such that the number of ones to the left of the marker is equal to the number of zeros to its right. Equivalently, the marker corresponds to the 0/1-sequence crossing the main diagonal of the partition. The r subsequences $c^{(0)}, \dots, c^{(r-1)}$ defined by

$$c^{(i)} := (c_{i+rj})_{j\in\mathbb{Z}} = (\dots c_{i-2r}c_{i-r} \,|\, c_ic_{i+r} \dots) \quad \text{for } 0 \leqslant i \leqslant r-1$$

correspond to the 0/1-sequences of $\nu^{(0)}, \dots, \nu^{(r-1)}$ forming the r-quotient $\boldsymbol{\nu}$. Note that we have left the marker in its original position so that the balancing of zeros and ones will generally not hold for the individual $c^{(i)}$. For example, when $r = 3$ the subsequences $c^{(0)}, c^{(1)}, c^{(2)}$ and partitions $\nu^{(0)}, \nu^{(1)}, \nu^{(2)}$ corresponding to the partition $(5, 4, 4, 1)$ are given by

$$\begin{aligned} c^{(0)} &= \dots 0000 \,|\, 1111 \dots & \qquad \nu^{(0)} &= 0 \\ c^{(1)} &= \dots 0001 \,|\, 0011 \dots & \nu^{(1)} &= (1, 1) \\ c^{(2)} &= \dots 0011 \,|\, 0111 \dots & \nu^{(2)} &= (2). \end{aligned}$$

To also obtain the r-core of λ we move the zeros in each of the $c^{(i)}$ to the left, like the beads on an abacus [37], and then reassemble the subsequences to form a single 0/1-sequence. For our example this gives

$$\begin{aligned} c^{(0)} &= \dots 0000 \,|\, 1111 \cdots \mapsto \dots 0000 \,|\, 1111 \dots \\ c^{(1)} &= \dots 0001 \,|\, 0011 \cdots \mapsto \dots 0000 \,|\, 0111 \dots \;\longmapsto\; \dots 0001 \,|\, 10111 \dots \\ c^{(2)} &= \dots 0011 \,|\, 0111 \cdots \mapsto \dots 0001 \,|\, 1111 \dots \end{aligned}$$

so that the 3-core of $(5, 4, 4, 1)$ is (2). Hence

$$\phi_3(5, 4, 4, 1) = \Big((2), \big(0, (1, 1), (2)\big)\Big).$$

Alternatively, if we only are interested in finding the r-core of λ, we may colour the 0/1-sequence $c(\lambda)$ with r colours according to the r congruence classes formed by the position labels. Then we push all of the zeros of each of the r colours to the left, past the ones of that same colour, to obtain the 0/1-sequence of its r-core. In the case of our example this would give

$$\begin{aligned}\dots 01011|1001011\dots &\cong \dots 01011|1001011\dots \\ &\xmapsto{\phi_3} 00001|1011111\dots \cong \dots 00001|1011111\dots\end{aligned}$$

We in particular remark that if λ/η is a border strip of size r and height k then λ and η have the same r-core and their respective 0/1-sequences are of the form

$$c(\lambda) = \dots 1u_1u_2\dots u_{r-1}0\dots \quad \text{and} \quad c(\eta) = \dots 0u_1u_2\dots u_{r-1}1\dots,$$

where k of the u_i are equal to 0 and $r-k-1$ of the u_i are equal to 1 (and where the 'dots' to the left and right are the same for $c(\lambda)$ and $c(\eta)$). Applied to border strips λ/η of height zero, it follows that [1]

$$r\text{-core}(\lambda) = r\text{-core}(\eta) \quad \text{if } \lambda_i \equiv \eta_i \pmod{r}. \tag{4.1}$$

Some key properties of Littlewood's decomposition (4.2) are collected in the following proposition.

Proposition 4.1. *Littlewood's decomposition*

$$\phi_r : \mathscr{P} \longrightarrow \mathscr{C}_r \times \mathscr{P}^r \tag{4.2a}$$

$$\lambda \longmapsto (\mu, \boldsymbol{\nu}) \tag{4.2b}$$

is a bijection such that

$$|\lambda| = |\mu| + r\,|\boldsymbol{\nu}| \tag{4.3}$$

and

$$\mathscr{H}_r(\lambda) = r\mathscr{H}(\boldsymbol{\nu}), \tag{4.4}$$

where, for a set or multiset S, $rS := \{rs : s \in S\}$.

In the case of the above example the respective hook-lengths are

$$\mathscr{H}_3(5,4,4,1) = \begin{array}{|c|c|c|c|c|}\hline & 6 & & & \\ \hline 6 & & 3 & & \\ \cline{1-4} & 3 & & & \\ \cline{1-4} & & & & \\ \cline{1-1}\end{array} \qquad \mathscr{H}\big(0,(1,1),(2)\big) = \Big(\emptyset\,,\ \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\,,\ \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}\Big),$$

consistent with (4.4).

It follows from the properties of ϕ_r, in particular bijectivity and (4.3), that

$$\sum_{\substack{\lambda\in\mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} = \frac{T^{|\omega|}}{(T^r;T^r)_\infty^r}, \tag{4.5}$$

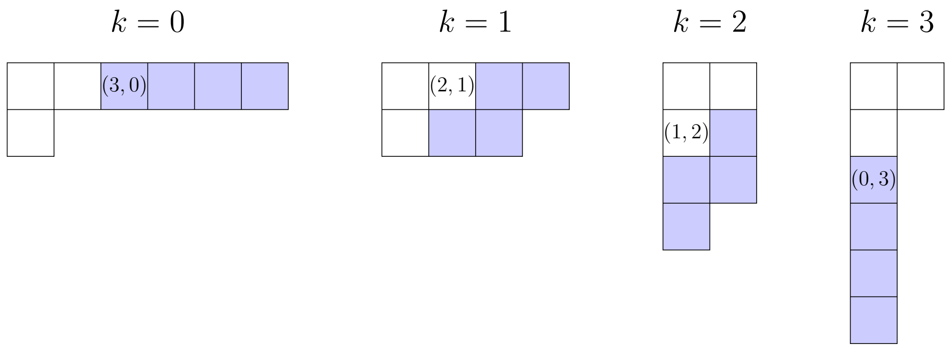
[1] More generally, for $\lambda, \eta \in \mathscr{P}$ such that $\max\{\ell(\lambda),\ell(\eta)\} \leqslant n$, $r\text{-core}(\lambda) = r\text{-core}(\eta)$ if and only if $\lambda \equiv w(\eta+\delta_n) - \delta_n \pmod{r}$ for some $w \in \mathfrak{S}_n$, see [42, page 13].

where $\omega \in \mathscr{C}_r$. Summing both sides over $\omega \in \mathscr{C}_r$, the left-hand side becomes the ordinary generating function for partitions. Solving for $\sum_{\omega\in\mathscr{C}_r} T^{|\omega|}$ we then obtain [38]

$$\sum_{\omega\in\mathscr{C}_r} T^{|\omega|} = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty}. \tag{4.6}$$

Lemma 4.2. *Let r be a positive integer, $k \in \{0,1,\dots,r-1\}$ and $\omega \in \mathscr{C}_r$. Then there exists a unique partition λ such that λ/ω is a border strip of size r and height k.*

For example, if $r = 4$ and $\omega = (2,1) \in \mathscr{C}_4$ then the four partitions λ are given by



where in each of the four diagrams on the right the blue squares mark the border strip added to $(2,1)$ to form λ. We have also included the arm and leg lengths of the square $s \in \lambda$ for which $h(s) = 4$. Obviously $l(s)$ coincides with the height of the border strip λ/ω.

Proof. From the Littlewood decomposition it is evident that there are exactly r partitions λ such that λ/ω is a border strip of size r. Indeed, the set of such partitions is simply

$$\phi^{-1}\biggl(\Bigl\{\Bigl(\omega, (\underbrace{0,\dots,0}_{k \text{ times}}, (1), \underbrace{0,\dots,0}_{r-k-1 \text{ times}})\Bigr) : k \in \{0,1,\dots,r-1\}\Bigr\}\biggr).$$

Colour-code the 0/1-sequence of the r-core ω according to the r congruence classes as explained above. For example, for $r = 4$ and $\omega = (2,1)$, this gives

$$c(2,1) = \dots 0000001010111111 \dots$$

Rank the colours (or subsequences) from 0 to $r-1$, with a colour given rank k if its right-most 0 has zeros of exactly k different colours to its right. In our example, blue has rank 0, green rank 1, magenta rank 2 and black rank 3. Then the partition λ such that the border strip λ/ω has height k corresponds to the 0/1-sequence $c(\lambda)$ obtained from $c(\omega)$ by swapping the right-most 0 of the colour ranked k with the 1 a distance r to its right. Indeed, the subword of $c(\omega)$ bounded by this 0-1 pair is of the form $0u_1u_2\dots u_{r-1}1$ with exactly k of the u_i equal to 0. To see this, note that, by the definition of rank and the fact that the u_i belong to $r-1$ different congruence classes, there are at most k of the u_i equal to zero. However, if there were fewer than k zeros among the u_i's, this would imply that at least one of the congruence classes ranked less than k has a 1 among the u_i's and, consequently, a 0 to the right of the subword. This congruence class would thus have a 1-0 pair, contradicting the fact that ω is an r-core. For example, for $r=4$ the relevant subwords of $c(2,1)$ are

$$\begin{array}{c|c|c|c|c} k & 0 & 1 & 2 & 3 \\ \hline & 01111 & 01011 & 00101 & 00001 \end{array}$$

and

$$\begin{aligned} c(6,1) &= \dots 000000101\underline{1}111\underline{0}11\dots, & c(4,3) &= \dots 0000001\underline{1}101\underline{0}1111\dots \\ c(2^3,1) &= \dots 0000\underline{1}010\underline{0}0111111\dots, & c(2,1^5) &= \dots 00\underline{1}000\underline{0}010111111\dots, \end{aligned}$$

where the swapped zero-one pairs have been underlined. $\square$

5. The multiplication theorem of Han and Ji

The Han–Ji multiplication theorem, stated as Theorem 1.1 in the introduction, provides a simple mechanism to obtain modular analogues of many known hook-length formulas. For example, by combining the hook-length formula (1.1) with the Robinson–Schensted correspondence [50, 51] between permutations and pairs of standard Young tableaux, it follows that[2]

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}(\lambda)} \frac{1}{h^2} = \mathrm{e}^T . \tag{5.1}$$

After the substitution $T\mapsto T/r^2$ this takes the form

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}(\lambda)} \frac{1}{(rh)^2} = \mathrm{e}^{T/r^2}$$

Comparing this with (1.4) it follows that for $\rho(h)=1/h^2$ we have the closed-form expression for $f_r(T)$ given by $f_r(T)=\exp(T/r^2)$. By (1.5) we thus obtain the modular analogue (see [27, Corollary 5.4])

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}_r(\lambda)} \frac{S}{h^2} = \mathrm{e}^{ST^r/r}\, \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty}. \tag{5.2}$$

Similarly, by replacing $z\mapsto z/r^2$ in (1.2), Han's modular analogue of the Nekrasov–Okounkov formula (1.3) follows from the multiplication theorem with

$$\rho(h) = 1-\frac{z}{h^2} \quad\text{and}\quad f_r(T) = (T;T)_\infty^{z/r^2-1}.$$

[2]To avoid the Robinson–Schensted correspondence one may alternatively use that (i) f^λ gives the dimension of the irreducible (complex) $\mathfrak{S}_n$-module indexed by the partition λ, (ii) the sum of the squares of the dimensions of the irreducible G-modules of a finite group G is equal to the order of G. Hence $\sum_{\lambda\vdash n}(f^\lambda)^2=n!$, which, by (1.1), is equivalent to (5.1).

The proof of Theorem 1.1 is surprisingly simple, and follows in a few elementary steps from Littlewood's decomposition. Using all of (4.2)–(4.4) (with $\mu \mapsto \omega$) and also noting that, by (4.4),

$$S^{|\mathscr{H}_r(\lambda)|} = S^{|\boldsymbol{\nu}|} = S^{\sum_{i=0}^{r-1}|\nu^{(i)}|},$$

we get

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{h\in\mathscr{H}_r(\lambda)}\rho(h) = T^{|\omega|}\Bigg(\sum_{\nu\in\mathscr{P}}(ST^r)^{|\nu|}\prod_{h\in\mathscr{H}(\nu)}\rho(rh)\Bigg)^r$$

for $\omega \in \mathscr{C}_r$. By (1.4) with $T \mapsto ST^r$ this simplifies to

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{h\in\mathscr{H}_r(\lambda)}\rho(h) = T^{|\omega|}\big(f_r(ST^r)\big)^r. \tag{5.3}$$

Summing ω over $\mathscr{C}_r$, and using (4.6) for the generating function of r-cores, (1.5) follows.

If we divide both sides of (5.3) by $T^{|\omega|}$, use that $|\mathscr{H}_r(\lambda)| = (|\lambda| - |\omega|)/r$ and finally replace ST^r by T, we obtain a variant of the multiplication theorem without the factor

$$\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty}.$$

Theorem 5.1 (Modified Han–Ji multiplication theorem)**.** *For r a positive integer and ρ a function on the positive integers, let $f_r(T)$ be defined by* (1.4)*. Then*

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\prod_{h\in\mathscr{H}_r(\lambda)}\rho(h) = \big(f_r(T)\big)^r,$$

where ω is an r-core.

The reason for stating this alternative version is that in Section 8.1 we discuss some modular Nekrasov–Okounkov-type series of the form

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\rho_r(\lambda)$$

for which the observed 'ω-independence' of the sum cannot simply be explained by the Littlewood decomposition.

6. An analogue of the Han–Ji multiplication theorem

In this section we describe a new Littlewood-like decomposition which implies the following analogue of Theorem 1.1.

Theorem 6.1.a. *For r a positive integer and ρ a function on the positive integers, let $g_r(T)$ be the formal power series defined by*

$$g_r(T) := \sum_{\lambda\in\mathscr{P}} T^{|\lambda|}\prod_{h\in\mathscr{H}^{(\mathrm{b})}(\lambda)}\rho(rh). \tag{6.1}$$

Then

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} \prod_{h\in\mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \frac{g_r(T)}{(T;T)_\infty^{r-1}}, \tag{6.2}$$

where ω is an r-core.

Equation (6.2) may be replaced by an expression which includes a factor representing r-cores. This is to be compared with Theorem 1.1.

Theorem 6.1.b. *For r a positive integer and ρ a function on the positive integers, let $g_r(T)$ be defined by* (6.1)*. Then*

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} \prod_{h\in\mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (ST^r;ST^r)_\infty^{r-1}}\, g_r(ST^r). \tag{6.3}$$

Let $\{\cdot\}:\mathbb{R}\to[0,1)$ and $\lfloor\cdot\rfloor:\mathbb{R}\to\mathbb{Z}$ be the fractional-part and floor functions, respectively, and recall that $\mathscr{K}_r$ is the set of r-kernels, see (3.3). We define the Littlewood-like map

$$\begin{aligned} \psi_r : \mathscr{P} &\longrightarrow \mathscr{K}_r \times \mathscr{P} \\ \lambda &\longmapsto (\mu,\nu) \end{aligned}$$

by

$$\mu_i = r\sum_{j\geqslant i}\{(\lambda_j-\lambda_{j+1})/r\} \tag{6.4a}$$

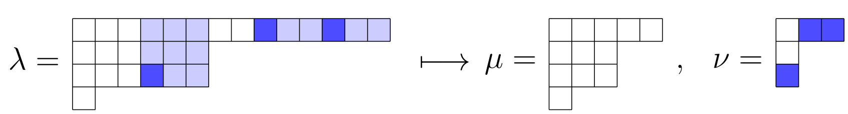
$$\nu_i = \sum_{j\geqslant i}\lfloor(\lambda_j-\lambda_{j+1})/r\rfloor \tag{6.4b}$$

for all $i\geqslant 1$. Since λ is a partition, it is clear that both μ and ν are partitions. Moreover, since

$$\mu_i-\mu_{i+1} = r\{(\lambda_i-\lambda_{i+1})/r\}\in\{0,1,\dots,r-1\},$$

it follows that μ is an r-kernel.

Graphically, μ and ν are obtained from λ by identifying the bottom squares of λ with hook-lengths congruent to 0 modulo r, and colouring the columns spanned by these squares as well as the $r-1$ columns immediately to the right of these. The coloured squares of λ then yield $r\,\nu$ and the remaining white squares form μ. For example, if $r=3$ and $\lambda=(14,6,6,1)$, then $\mu=(5,3,3,1)$ and $\nu=(3,1,1)$:



where the bottom squares of λ with hook-lengths congruent to 0 modulo 3 as well as the bottom squares of ν have been marked in dark blue.

Although ψ_r is much simpler than Littlewood's map ϕ_r, it has many properties in common with the latter.

Proposition 6.2. *The decomposition*

$$\psi_r : \mathscr{P} \longrightarrow \mathscr{K}_r \times \mathscr{P} \tag{6.5a}$$

$$\lambda \longmapsto (\mu, \nu) \tag{6.5b}$$

defined by (6.4a) *and* (6.4b) *is a bijection such that*

$$r\text{-core}(\lambda) = r\text{-core}(\mu), \tag{6.6}$$

$$|\lambda| = |\mu| + r|\nu|, \tag{6.7}$$

$$\mathscr{H}_r^{(\mathrm{b})}(\lambda) = r\mathscr{H}^{(\mathrm{b})}(\nu). \tag{6.8}$$

Equations (6.7) and (6.8) are the analogues of (4.3) and (4.4) in Littlewood's decomposition. Equation (6.6) also holds in the Littlewood case as part of the much stronger $r\text{-core}(\lambda) = r\text{-core}(\mu) = \mu$.

Proof. To see that ψ_r is a bijection we use

$$r\{n/r\} = n - r\lfloor n/r \rfloor$$

to rewrite (6.4a) as

$$\begin{aligned} \mu_i &= \sum_{j \geqslant i} \Big(\lambda_j - \lambda_{j+1} - r\lfloor (\lambda_j - \lambda_{j+1})/r \rfloor \Big) \\ &= \lambda_i - r \sum_{j \geqslant i} \lfloor (\lambda_j - \lambda_{j+1})/r \rfloor \\ &= \lambda_i - r\nu_i, \end{aligned}$$

where the final equality follows from (6.4b). This shows that the tuple (μ, ν) uniquely fixes λ, so that ψ_r is injective. Surjectivity is also clear since, for arbitrary $\mu \in \mathscr{K}_r$ and $\nu \in \mathscr{P}$, the image of $\mu + r\nu := (\mu_1 + r\nu_1, \mu_2 + r\nu_2, \dots)$ is (μ, ν). We may thus conclude that ψ_r is a bijection, with inverse

$$\begin{aligned} \psi_r^{-1} : \mathscr{K}_r \times \mathscr{P} &\longrightarrow \mathscr{P} \\ (\mu, \nu) &\longmapsto \lambda \end{aligned}$$

given by

$$\lambda_i = \mu_i + r\nu_i \quad \text{for all } i \geqslant 1.$$

This immediately implies (6.7) and, by (4.1), also implies (6.6).

Finally, since $\mu \in \mathscr{K}_r$,

$$\mathscr{H}_r^{(\mathrm{b})}(\lambda) = \mathscr{H}_r^{(\mathrm{b})}(\lambda - \mu) = \mathscr{H}_r^{(\mathrm{b})}(r\nu) = r\mathscr{H}_1^{(\mathrm{b})}(\nu) = r\mathscr{H}^{(\mathrm{b})}(\nu),$$

completing the proof. □

The generating function of r-kernels with fixed r-core admits a closed-form expression as follows.

Lemma 6.3. *Let $\omega \in \mathscr{C}_r$. Then*

$$\sum_{\substack{\mu \in \mathscr{K}_r \\ r\text{-core}(\mu)=\omega}} T^{|\mu|} = \frac{T^{|\omega|}}{(T^r;T^r)_\infty^{r-1}}.$$

Summation of the left-hand side over $\omega \in \mathscr{C}_r$ yields the generating function for all r-kernels. By (4.6), carrying out this same sum on the right we obtain $(T^r;T^r)_\infty/(T;T)_\infty$, so that we recover (3.4).

Proof. Let $\omega \in \mathscr{C}_r$. From Proposition 6.2 it follows that

$$\sum_{\substack{\lambda \in \mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} = \frac{1}{(T^r;T^r)_\infty} \sum_{\substack{\mu \in \mathscr{K}_r \\ r\text{-core}(\mu)=\omega}} T^{|\mu|}.$$

The left-hand side is the generating function of partitions with fixed r-core, which can be expressed in closed form by (4.5). The claim then follows by multiplying both sides by $(T^r;T^r)_\infty$. $\square$

We now have all the ingredients needed to prove Theorems 6.1.a and 6.1.b.

Proof. Let $\omega \in \mathscr{C}_r$. By Proposition 6.2,

$$\sum_{\substack{\lambda \in \mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} \prod_{h \in \mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \Bigg(\sum_{\substack{\mu \in \mathscr{K}_r \\ r\text{-core}(\mu)=\omega}} T^{|\mu|} \Bigg) \Bigg(\sum_{\nu \in \mathscr{P}} T^{r|\nu|} \prod_{h \in \mathscr{H}^{(\mathrm{b})}(\nu)} \rho(rh) \Bigg).$$

The first sum on the right can be carried out by Lemma 6.3, whereas the second sum is exactly $g_r(T^r)$ by (6.1) with $T \mapsto T^r$. Hence

$$\sum_{\substack{\lambda \in \mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} \prod_{h \in \mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \frac{T^{|\omega|}}{(T^r;T^r)_\infty^{r-1}}\, g_r(T^r).$$

Dividing both sides by $T^{|\omega|}$ and replacing T by $T^{1/r}$ we get (6.2). If instead we replace T by $TS^{1/r}$, then multiply both sides by $S^{-|\omega|/r}$ and finally sum over $\mu \in \mathscr{C}_r$ using (4.6), we obtain

$$\sum_{\omega \in \mathscr{C}_r} \sum_{\substack{\lambda \in \mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} S^{(|\lambda|-|\omega|)/r} \prod_{h \in \mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (ST^r;ST^r)_\infty^{r-1}}\, g_r(ST^r).$$

From (4.3) and (4.4) it follows that $(|\lambda|-|\omega|)/r = |\mathscr{H}_r(\lambda)|$, so that the left-hand side may be replaced by

$$\sum_{\omega \in \mathscr{C}_r} \sum_{\substack{\lambda \in \mathscr{P} \\ r\text{-core}(\lambda)=\omega}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} \prod_{h \in \mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h) = \sum_{\lambda \in \mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} \prod_{h \in \mathscr{H}_r^{(\mathrm{b})}(\lambda)} \rho(h).$$

This also proves (6.3). $\square$

7. Applications

While hook-length formulas abound in the combinatorics literature, identities that involve only hook-lengths of bottom squares are rare, making it more difficult to apply Theorems 6.1.a and 6.1.b than the Han–Ji multiplication theorem.

As a first example we discuss what is essentially a trivial application by taking $\rho(h) = z$, independent of h. Then the right-hand side of (6.1) simplifies to

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} z^{|\mathscr{H}^{(\mathrm{b})}(\lambda)|} = \sum_{\lambda\in\mathscr{P}} T^{|\lambda|} z^{\lambda_1} = \sum_{\lambda\in\mathscr{P}} T^{|\lambda|} z^{l(\lambda)} = \frac{1}{(zT;T)_\infty},$$

where the second equality follows from the substitution $\lambda \mapsto \lambda'$ and the third equality follows from (3.1). Hence

$$g_r(T) = \frac{1}{(zT;T)_\infty},$$

independent of r. Substitution of this into (6.2) and (6.3) yields

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} z^{|\mathscr{H}_r^{(\mathrm{b})}(\lambda)|} = \frac{1}{(zT;T)_\infty (T;T)_\infty^{r-1}} \tag{7.1a}$$

and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} z^{|\mathscr{H}_r^{(\mathrm{b})}(\lambda)|} = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (zST^r;ST^r)_\infty (ST^r;ST^r)_\infty^{r-1}}. \tag{7.1b}$$

In Section 8.2 we discuss how these results relate to a combinatorial identity of Buryak, Feigin and Nakajima which arose in their work on the quasihomogeneous Hilbert scheme of points in the plane.

In order to express (7.1a) and (7.1b) in terms of the modular length function (3.2), we replace $\lambda \mapsto \lambda'$ in both formulas. In (7.1a) we further make the substitution $\omega \mapsto \omega'$, noting that r-core$(\lambda') = \omega'$ is equivalent to r-core$(\lambda) = \omega$, and in (7.1b) we use that $|\mathscr{H}_r(\lambda')| = |\mathscr{H}_r(\lambda)|$. This leads to the following pair of partition identities, generalising (3.1).

Proposition 7.1. *For r a positive integer and ω an r-core,*

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} z^{\ell_r(\lambda)} = \frac{1}{(zT;T)_\infty (T;T)_\infty^{r-1}}$$

and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} z^{\ell_r(\lambda)} = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (zST^r;ST^r)_\infty (ST^r;ST^r)_\infty^{r-1}}.$$

To motivate our second application, we recall that the exponential generating function for the number of involutions $e_2(n)$ in the symmetric group $\mathfrak{S}_n$ is given by [52, Equation (5.32)]

$$\sum_{n\geqslant 0} e_2(n) \frac{T^n}{n!} = \exp\bigg(T + \frac{T^2}{2}\bigg). \tag{7.2}$$

As a direct consequence of the Robinson–Schensted correspondence, $e_2(n)$ is equal to the number of standard Young tableaux of size n. By (1.1) it thus follows that (7.2) can be written as

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}(\lambda)} \frac{1}{h} = \exp\left(T+\frac{T^2}{2}\right), \tag{7.3}$$

which is to be compared with (5.1).

Somewhat surprisingly, (7.3) has an analogue for $\mathscr{H}^{(\mathrm{b})}(\lambda)$.

Lemma 7.2. *We have*

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}^{(\mathrm{b})}(\lambda)} \frac{z}{h} = \exp\left(\frac{zT}{1-T}\right). \tag{7.4}$$

Proof. Let $f(z,T)$ denote the left-hand side of (7.4). Replacing λ by λ', we obtain

$$f(z,T) = \sum_{\lambda\in\mathscr{P}} T^{|\lambda|} \prod_{h\in\mathscr{H}^{(\mathrm{b})}(\lambda')} \frac{z}{h} = \sum_{\lambda\in\mathscr{P}} z^{\ell(\lambda)} T^{|\lambda|} \prod_{h\in\mathscr{H}^{(\mathrm{r})}(\lambda)} \frac{1}{h}.$$

Since $|\lambda|=\sum_{i\geqslant 1} i m_i(\lambda)$, $\ell(\lambda)=\sum_{i\geqslant 1} m_i(\lambda)$ and $\prod_{h\in\mathscr{H}^{(\mathrm{r})}(\lambda)} h = \prod_{i\geqslant 1} m_i(\lambda)!$, it follows that

$$\begin{aligned} f(z,T) = \sum_{\lambda\in\mathscr{P}} \prod_{i\geqslant 1} \frac{z^{m_i(\lambda)} T^{i m_i(\lambda)}}{m_i(\lambda)!} &= \prod_{i\geqslant 1} \sum_{m_i\geqslant 0} \frac{(zT^i)^{m_i}}{m_i!} \\ &= \prod_{i\geqslant 1} \exp(zT^i) = \exp\left(\frac{zT}{1-T}\right). \end{aligned}$$

□

After the substitution $z\mapsto z/r$ the identity (7.4) takes the form of (6.1) with

$$\rho(h)=\frac{z}{h} \quad\text{and}\quad g_r(T)=\exp\left(\frac{zT}{r(1-T)}\right).$$

By (6.2) and (6.3) we thus obtain the following modular analogues of (7.4).

Proposition 7.3. *For r a positive integer and ω an r-core,*

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} z^{\ell_r(\lambda)} \prod_{h\in\mathscr{H}_r^{(\mathrm{r})}(\lambda)} \frac{1}{h} = \frac{1}{(T;T)_\infty^{r-1}} \exp\left(\frac{zT}{r(1-T)}\right)$$

and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} z^{\ell_r(\lambda)} \prod_{h\in\mathscr{H}_r^{(\mathrm{r})}(\lambda)} \frac{1}{h} = \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (ST^r;ST^r)_\infty^{r-1}} \exp\left(\frac{zST^r}{r(1-ST^r)}\right).$$

It is not difficult to see that by the q-binomial theorem [26, Equation (II.3)]

$$\sum_{m\geqslant 0} \frac{(a;q)_m}{(q;q)_m} z^m = \frac{(az;q)_\infty}{(z;q)_\infty}$$

with $(a,z,m)\mapsto(uq,zT^i,m_i)$ the identity (7.4) admits the q-analogue[3]

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}\prod_{h\in\mathscr{H}^{(\mathrm{b})}(\lambda)}\frac{z(1-uq^h)}{1-q^h}=\frac{(uzqT;q,T)_\infty}{(zT;q,T)_\infty}. \tag{7.5}$$

We note that for $z=1$ this is the $t=0$ case of the q,t-Nekrasov–Okounkov formula (1.6). Moreover, by the substitution $(z,u)\mapsto(t,q^{-\zeta})$ followed by the limit $q\to 1$, it simplifies to

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}t^{|\mathscr{H}^{(\mathrm{b})}(\lambda)|}\prod_{h\in\mathscr{H}^{(\mathrm{b})}(\lambda)}\Big(1-\frac{z}{h}\Big)=(tT;T)_\infty^{z-1}.$$

For $t=1$ this was conjectured in [2, Conjecture 2.1] and proved in [34]. The identity (7.5) implies the most general pair (ρ,g_r) that we have been able to find for which g_r admits a simple closed form:

$$\rho(h)=\frac{z(1-uq^h)}{(1-q^h)},\qquad g_r(T)=\frac{(uzq^rT;q^r,T)_\infty}{(zT;q^r,T)_\infty}.$$

This leads to our final application, unifying the previous two propositions.

Proposition 7.4. *For r a positive integer and ω an r-core,*

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}z^{\ell_r(\lambda)}\prod_{h\in\mathscr{H}_r^{(\mathrm{r})}(\lambda)}\frac{1-uq^h}{1-q^h}=\frac{(uzq^rT;q^r,T)_\infty}{(T;T)_\infty^{r-1}(zT;q^r,T)_\infty} \tag{7.6}$$

and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}z^{\ell_r(\lambda)}\prod_{h\in\mathscr{H}_r^{(\mathrm{r})}(\lambda)}\frac{1-uq^h}{1-q^h}=\frac{(T^r;T^r)_\infty^r(uzq^rST^r;q^r,ST^r)_\infty}{(T;T)_\infty(ST^r;ST^r)_\infty^{r-1}(zST^r;q^r,ST^r)_\infty}.$$

8. Conjectures and open problems

8.1. **A modular analogue of the q,t-Nekrasov–Okounkov formula.** By setting $z=1$ in (7.6) and carrying out some elementary manipulations and rewritings we obtain the following pair of equivalent identities

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\prod_{\substack{s\in\lambda\\ h(s)\equiv 0\ (\mathrm{mod}\ r)\\ l(s)=0}}\frac{1-uq^{a(s)+1}}{1-q^{a(s)+1}}=\frac{(uq^rT;q^r,T)_\infty}{(T;T)_\infty^r(q^rT;q^r,T)_\infty}$$

and

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\prod_{\substack{s\in\lambda\\ h(s)\equiv 0\ (\mathrm{mod}\ r)\\ a(s)=0}}\frac{1-u^{-1}t^{l(s)+1}}{1-t^{l(s)+1}}=\frac{(u^{-1}t^rT;t^r,T)_\infty}{(T;T)_\infty^r(t^rT;t^r,T)_\infty}.$$

[3] To recover (7.4), set $u=0$, perform the replacement $z\mapsto z(1-q)$, and then let q tend to 1.

This should be compared with the modular analogue of the q-Nekrasov–Okounkov formula

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} \prod_{h\in\mathscr{H}_r(\lambda)} \frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2} = \bigg(\frac{(uq^rT,u^{-1}q^rT;q^r,q^r,T)_\infty}{(T,q^rT;q^r,q^r,T)_\infty}\bigg)^r,$$

which, in a slightly different form, is due to Dehaye and Han [19, Theorem 2] and follows from the $r=1$ case (see [36, p. 749] and [19, Theorem 5]) combined with Theorem 5.1.

The above three identities suggest a modular analogue of the full q,t-Nekrasov–Okounkov formula (1.6).

Conjecture 8.1. *For r a positive integer and ω an r-core,*

$$\begin{aligned}\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} &\prod_{\substack{s\in\lambda\\ h(s)\equiv 0 \ (\mathrm{mod}\ r)}} \frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}\\ &=\frac{1}{(T;T)_\infty^r}\prod_{\substack{i,j\geqslant 1\\ i+j\equiv 1 \ (\mathrm{mod}\ r)}} \frac{(uq^it^{j-1}T,u^{-1}q^{i-1}t^jT;T)_\infty}{(q^it^{j-1}T,q^{i-1}t^jT;T)_\infty}\\ &=\frac{1}{(T;T)_\infty^r}\prod_{i=1}^r \frac{(uq^it^{r-i}T,u^{-1}q^{r-i}t^iT;q^r,t^r,T)_\infty}{(q^it^{r-i}T,q^{r-i}t^iT;q^r,t^r,T)_\infty}.\end{aligned}$$

We note that it is not at all clear why

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} \prod_{\substack{s\in\lambda\\ h(s)\equiv 0 \ (\mathrm{mod}\ r)}} \frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})} \tag{8.1}$$

is independent of the choice of ω, so that there are really two parts to the conjecture. Assuming this independence, we obtain a weaker form, stated as Conjecture 1.2 in the introduction, by replacing $T\mapsto ST^r$, multiplying both sides by $T^{|\omega|}$ and then summing ω over $\mathscr{C}_r$. It appears that neither conjecture is tractable by a multiplication-type theorem. In particular, non-trivial rational function identities are behind the ω-independence of (8.1). An example of such an identity in the more general elliptic case is discussed below.

The q,t-Nekrasov–Okounkov formula admits an elliptic analogue as follows. Let $\theta(z;p)$ be the modified theta function

$$\theta(z;p):=\sum_{n\in\mathbb{Z}} z^n p^{\binom{n}{2}}, \quad z\neq 0,$$

and define the set of integers $\{C(m,\ell,n_1,n_2)\}_{m\in\mathbb{N},\,\ell,n_1,n_2\in\mathbb{Z}}$ by

$$\frac{(upq,u^{-1}pq^{-1},upt^{-1},u^{-1}pt;p)_\infty}{(pq,pq^{-1},pt^{-1},pt;p)_\infty}=1+\sum_{m\geqslant 1}\sum_{\ell,n_1,n_2\in\mathbb{Z}} C(m,\ell,n_1,n_2)p^m u^\ell q^{-n_1}t^{n_2}.$$

Note that $C(m,\ell,n_1,n_2)=C(m,\ell,n_2,n_1)=C(m,-\ell,-n_1,-n_2)$ and (set $u=1$)

$$\sum_{\ell\in\mathbb{Z}} C(m,\ell,n_1,n_2)=0. \tag{8.2}$$

This latter sum is well-defined since $C(m,\ell,n_1,n_2)=0$ if $|\ell|\geqslant\lfloor\sqrt{4m+1}\rfloor$. Then

$$\begin{aligned}\sum_{\lambda} T^{|\lambda|}\prod_{s\in\lambda}&\frac{\theta(uq^{a(s)+1}t^{l(s)};p)\theta(u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)};p)\theta(q^{a(s)}t^{l(s)+1};p)}\\ &=\frac{(uqT,u^{-1}tT;q,t,T)_\infty}{(T,qtT;q,t,T)_\infty}\\ &\quad\times\prod_{m,k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\bigg(\frac{(p^mT^ku^{\ell+1}q^{1-n_1}t^{n_2},p^mT^ku^{\ell-1}q^{-n_1}t^{1+n_2};q,t)_\infty}{(p^mT^ku^{\ell}q^{-n_1}t^{n_2},p^mT^ku^{\ell}q^{1-n_1}t^{1+n_2};q,t)_\infty}\bigg)^{C(km,\ell,n_1,n_2)},\end{aligned} \tag{8.3}$$

see [40, 49, 55]. For $p=1$ this simplifies to (1.6) and, by (8.2), for $u=1$ it simplifies to (3.1) with $z=1$.

Conjecture 8.2. *Let r be a positive integer, ω an r-core and*

$$f_{\omega;r}(u;q,t,T;p):=\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\prod_{\substack{s\in\lambda\\ h(s)\equiv 0\ (\mathrm{mod}\ r)}}\frac{\theta(uq^{a(s)+1}t^{l(s)};p)\theta(u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)};p)\theta(q^{a(s)}t^{l(s)+1};p)}.$$

Then $f_{\omega;r}(u;q,t,T;p)$ is independent of ω.

We have not yet found a (conjectural) closed-form expression for the above sum, except when $t=q$, see Proposition 8.3 below.

Conjecture 8.2 may also be stated as the claim that for all nonnegative integers n the sum

$$f_{\omega;r,n}(u;q,t;p):=\sum_{\substack{\lambda\vdash|\omega|+rn\\ r\text{-core}(\lambda)=\omega}}\prod_{\substack{s\in\lambda\\ h(s)\equiv 0\ (\mathrm{mod}\ r)}}\frac{\theta(uq^{a(s)+1}t^{l(s)};p)\theta(u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)};p)\theta(q^{a(s)}t^{l(s)+1};p)}, \tag{8.4}$$

which satisfies the quasi-periodicity

$$f_{\omega;r,n}(pu;q,t;p)=\Big(\frac{t}{u^2pq}\Big)^n f_{\omega;r,n}(u;q,t;p),$$

does not depend on the choice of $\omega\in\mathscr{C}_r$. For $n=0$ this is trivially true: $f_{\omega;r,0}(u;q,t;p)=1$ for all $\omega\in\mathscr{C}_r$. For $n=1$ the sum in (8.4) is over partitions λ such that λ/ω is a border strip of size r. By Lemma 4.2 there is exactly one such λ for which λ/ω has height k with $k\in\{0,1,\dots,r-1\}$. Since this height coincides with the leg-length of the square $s\in\lambda$ such that $h(s)=r$, it follows that

$$f_{\omega;r,1}(u;q,t;p)=\sum_{k=0}^{r-1}\frac{\theta(uq^{r-k}t^k;p)\theta(u^{-1}q^{r-k-1}t^{k+1};p)}{\theta(q^{r-k}t^k;p)\theta(q^{r-k-1}t^{k+1};p)},$$

which is independent of the r-core ω. For $n\geqslant 2$ Littlewood's decomposition alone is not enough and theta-function addition formulas come into play. For example, there are $2r+\binom{r}{2}$

partitions λ contributing to $f_{\omega;r,2}(u;q,t;p)$ since the r-quotient $\boldsymbol{\nu}$ of λ is given by one of the following:

(i) $\boldsymbol{\nu} = (\underbrace{0,\dots,0}_{i-1 \text{ times}},(2),\underbrace{0,\dots,0}_{r-i \text{ times}})$ for $1 \leqslant i \leqslant r$, contributing r terms;

(ii) $\boldsymbol{\nu} = (\underbrace{0,\dots,0}_{i-1 \text{ times}},(1^2),\underbrace{0,\dots,0}_{r-i \text{ times}})$ for $1 \leqslant i \leqslant r$, contributing r terms;

(iii) $\boldsymbol{\nu} = (\underbrace{0,\dots,0}_{i-1 \text{ times}},(1),\underbrace{0,\dots,0}_{j-i-1 \text{ times}},(1),\underbrace{0,\dots,0}_{r-j \text{ times}})$ for $1 \leqslant i < j \leqslant r$, contributing $\binom{r}{2}$ terms.

Accordingly, we symbolically write

$$f_{\omega;r,2}(u;q,t;p) = \sum_{i=1}^{r} \Big(\varphi_i^{\text{(i)}}(\omega) + \varphi_i^{\text{(ii)}}(\omega)\Big) + \sum_{1\leqslant i<j\leqslant r} \varphi_{i,j}^{\text{(iii)}}(\omega).$$

Because the terms on the right sensitively depend on the choice of $\omega \in \mathscr{C}_r$, we further restrict ourselves to a comparison of $\omega = 0$ and $\omega = (1)$ (so that we require $r \geqslant 2$). Then $\binom{r}{2} + 2r - 3$ of the terms contributing to $f_{0;r,2}(u;q,t;p)$ have a counterpart in $f_{(1);r,2}(u;q,t;p)$ and 3 terms in each of the sums are different. More precisely,

$$\varphi_i^{\text{(i)}}(0) = \begin{cases} \varphi_i^{\text{(i)}}(1), & \text{for } 2 \leqslant i \leqslant r-1, \\ \varphi_1^{\text{(i)}}(1), & \text{for } i = r, \end{cases}$$

$$\varphi_i^{\text{(ii)}}(0) = \begin{cases} \varphi_i^{\text{(ii)}}(1), & \text{for } 2 \leqslant i \leqslant r-1, \\ \varphi_r^{\text{(ii)}}(1), & \text{for } i = 1, \end{cases}$$

and

$$\varphi_{i,j}^{\text{(iii)}}(0) = \begin{cases} \varphi_{i,j}^{\text{(iii)}}(1), & \text{for } 1 < i < j < r, \\ \varphi_{j,r}^{\text{(iii)}}(1), & \text{for } i = 1,\ 1 < j < r, \\ \varphi_{1,i}^{\text{(iii)}}(1), & \text{for } j = r,\ 1 < i < r. \end{cases}$$

Hence

$$f_{0;r,2}(u;q,t;p) - f_{(1);r,2}(u;q,t;p) = \varphi_1^{\text{(i)}}(0) + \varphi_r^{\text{(ii)}}(0) + \varphi_{1,r}^{\text{(iii)}}(0) - \varphi_r^{\text{(i)}}(1) - \varphi_1^{\text{(ii)}}(1) - \varphi_{1,r}^{\text{(iii)}}(1).$$

If by slight abuse of notation we index the above by the actual partitions λ this yields

$$\begin{aligned} f_{0;r,2}(u;q,t;p) - f_{(1);r,2}(u;q,t;p) &= \varphi_{(r+1,1^{r-1})} + \varphi_{(r,1^r)} + \varphi_{(r,2,1^{r-2})} \\ &\quad - \varphi_{(r,2,1^{r-1})} - \varphi_{(r+1,2,1^{r-2})} - \varphi_{(r+1,1^r)}. \end{aligned}$$

To see that this vanishes we note that

$$\begin{aligned}
&\varphi_{(r+1,1^{r-1})}-\varphi_{(r+1,1^r)}\\
&\quad=\frac{\theta(uq^r;p)\theta(uq^{r+1}t^{r-1};p)\theta(u^{-1}q^{r-1}t;p)\theta(u^{-1}q^rt^r;p)}{\theta(q^r;p)\theta(q^{r+1}t^{r-1};p)\theta(q^{r-1}t;p)\theta(q^rt^r;p)}\\
&\qquad-\frac{\theta(uq^r;p)\theta(uqt^{r-1};p)\theta(u^{-1}q^{r-1}t;p)\theta(u^{-1}t^r;p)}{\theta(q^r;p)\theta(qt^{r-1};p)\theta(q^{r-1}t;p)\theta(t^r;p)}\\
&\quad=-q^{2r-1}t^{r+1}\,\frac{\theta(uqt^{-1};p)\theta(uq^{1-r}t^{-1};p)\theta(u^{-1}q^{-r};p)\theta(u^{-1};p)\theta(q^{r+1}t^{2r-1};p)}{\theta(q^{r-1}t;p)\theta(qt^{r-1};p)\theta(t^r;p)\theta(q^{r+1}t^{r-1};p)\theta(q^rt^r;p)}=:t_1.
\end{aligned}$$

Here the last equality follows from the addition formula [57, p. 451, Example 5]

$$\begin{aligned}
\theta(xz;p)\theta(x/z;p)\theta(yw;p)\theta(y/w;p)-\theta(xw;p)\theta(x/w;p)\theta(yz;p)\theta(y/z;p)&\\
=\frac{y}{z}\,\theta(xy;p)\theta(x/y;p)\theta(zw;p)\theta(z/w;p)&
\end{aligned}\tag{8.5}$$

as well as $\theta(z;p)=-z\theta(1/z;p)$. Similarly, we have

$$\begin{aligned}
&\varphi_{(r,1^r)}-\varphi_{(r,2,1^{r-1})}\\
&\quad=\frac{\theta(uqt^{r-1};p)\theta(uq^rt^r;p)\theta(u^{-1}t^r;p)\theta(u^{-1}q^{r-1}t^{r+1};p)}{\theta(qt^{r-1};p)\theta(q^rt^r;p)\theta(t^r;p)\theta(q^{r-1}t^{r+1};p)}\\
&\qquad-\frac{\theta(uq^{r-1}t;p)\theta(uq^rt^r;p)\theta(u^{-1}q^{r-2}t^2;p)\theta(u^{-1}q^{r-1}t^{r+1};p)}{\theta(q^{r-1}t;p)\theta(q^rt^r;p)\theta(q^{r-2}t^2;p)\theta(q^{r-1}t^{r+1};p)}\\
&\quad=\frac{\theta(uqt^{-1};p)\theta(uq^rt^r;p)\theta(u^{-1}q^{r-1}t^{r+1};p)\theta(u^{-1};p)\theta(q^{2-r}t^{r-2};p)}{\theta(q^{r-1}t;p)\theta(q^{2-r}t^{-2};p)\theta(qt^{r-1};p)\theta(t^r;p)\theta(q^rt^r;p)}=:t_2
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{(r,2,1^{r-2})}-\varphi_{(r+1,2,1^{r-2})}\\
&\quad=\frac{\theta(uq^{r-1}t;p)\theta(uq^2t^{r-2};p)\theta(u^{-1}q^{r-2}t^2;p)\theta(u^{-1}qt^{r-1};p)}{\theta(q^{r-1}t;p)\theta(q^2t^{r-2};p)\theta(q^{r-2}t^2;p)\theta(qt^{r-1};p)}\\
&\qquad-\frac{\theta(uq^2t^{r-2};p)\theta(uq^{r+1}t^{r-1};p)\theta(u^{-1}qt^{r-1};p)\theta(u^{-1}q^rt^r;p)}{\theta(q^2t^{r-2};p)\theta(q^{r+1}t^{r-1};p)\theta(qt^{r-1};p)\theta(q^rt^r;p)}\\
&\quad=-\frac{\theta(uq^2t^{r-2};p)\theta(uqt^{-1};p)\theta(u^{-1}qt^{r-1};p)\theta(u^{-1};p)\theta(q^{2r-1}t^{r+1};p)}{\theta(q^{r-1}t;p)\theta(q^{2-r}t^{-2};p)\theta(qt^{r-1};p)\theta(q^{r+1}t^{r-1};p)\theta(q^rt^r;p)}=:t_3.
\end{aligned}$$

By one more application of (8.5) it follows that $t_1+t_2+t_3=0$, and hence that

$$f_{0;r,2}(u;q,t;p)=f_{(1);r,2}(u;q,t;p)$$

for $r\geqslant 2$.

If we set $t=q$ in the elliptic Nekrasov–Okounkov formula (8.3) we obtain

$$\sum_{\lambda} T^{|\lambda|} \prod_{h\in\mathscr{H}(\lambda)} \frac{\theta(uq^h;p)\theta(u^{-1}q^h;p)}{\theta(q^h;p)\theta(q^h;p)} = \frac{(uqT,u^{-1}qT;q,q,T)_\infty}{(T,q^2T;q,q,T)_\infty} \times \prod_{m,k\geqslant 1} \prod_{\ell,n_1,n_2\in\mathbb{Z}} \bigg(\frac{(p^mT^ku^{\ell+1}q^{n_2-n_1+1},p^mT^ku^{\ell-1}q^{n_2-n_1+1};q,q)_\infty}{(p^mT^ku^{\ell}q^{n_2-n_1},p^mT^ku^{\ell}q^{n_2-n_1+2};q,q)_\infty}\bigg)^{C(km,\ell,n_1,n_2)}.$$

By Theorem 1.1 this implies our next result.

Proposition 8.3 (A p,q-Nekrasov–Okounkov formula)**.** *For r a positive integer,*

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} \prod_{h\in\mathscr{H}_r(\lambda)} \frac{\theta(uq^h;p)\theta(u^{-1}q^h;p)}{\theta(q^h;p)\theta(q^h;p)}$$
$$= \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty} \bigg(\frac{(uq^rST^r,u^{-1}q^rST^r;q^r,q^r,ST^r)_\infty}{(ST^r,q^{2r}ST^r;q^r,q^r,ST^r)_\infty}\bigg)^r$$
$$\times \prod_{m,k\geqslant 1} \prod_{\ell,n_1,n_2\in\mathbb{Z}} \bigg(\frac{(p^mS^kT^{kr}u^{\ell+1}q^{(n_2-n_1+1)r},p^mS^kT^{kr}u^{\ell-1}q^{(n_2-n_1+1)r};q^r,q^r)_\infty}{(p^mS^kT^{kr}u^{\ell}q^{(n_2-n_1)r},p^mS^kT^{kr}u^{\ell}q^{(n_2-n_1+2)r};q^r,q^r)_\infty}\bigg)^{C(km,\ell,n_1,n_2)r}.$$

This naturally leads to the following open problem.

Problem 8.4. *For $r\geqslant 2$, identify the integers $B_r(m,\ell,n_1,n_2)$ in the expansion*

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|} S^{|\mathscr{H}_r(\lambda)|} \prod_{\substack{s\in\lambda \\ h(s)\equiv 0 \ (\mathrm{mod}\ r)}} \frac{\theta(uq^{a(s)+1}t^{l(s)},u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)},q^{a(s)}t^{l(s)+1};p)}$$
$$= \frac{(T^r;T^r)_\infty^r}{(T;T)_\infty(ST^r;ST^r)_\infty^r} \prod_{i=1}^r \frac{(uq^it^{r-i}ST^r,u^{-1}q^{r-i}t^iST^r;q^r,t^r,ST^r)_\infty}{(q^it^{r-i}ST^r,q^{r-i}t^iST^r;q^r,t^r,ST^r)_\infty}$$
$$\times \prod_{m,k\geqslant 1} \prod_{\ell,n_1,n_2\in\mathbb{Z}} \big(1-p^m(ST^r)^ku^\ell q^{n_1}t^{n_2}\big)^{B_r(mk,\ell,n_1,n_2)}.$$

8.2. **The Buryak–Feigin–Nakajima formula.** Let $(\mathbb{C}^2)^{[n]}$ be the Hilbert scheme of points in the plane, parametrising the ideals I of $\mathbb{C}[x,y]$ of colength n. The action of the torus $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$ given by $(s,t)\cdot(x,y)=(sx,ty)$ lifts to an action on $(\mathbb{C}^2)^{[n]}$. For α,β nonnegative integers such that $\alpha+\beta\geqslant 1$, define the one-dimensional subtorus $T_{\alpha,\beta}$ of $(\mathbb{C}^*)^2$ by $T_{\alpha,\beta}:=\{(t^\alpha,t^\beta):\ t\in\mathbb{C}^*\}$. When both α and β are strictly positive, the set of fixed points

$$(\mathbb{C}^2)^{[n]}_{\alpha,\beta} := \big((\mathbb{C}^2)^{[n]}\big)^{T_{\alpha,\beta}}$$

parametrises quasi-homogeneous ideals $I\subset\mathbb{C}[x,y]$ of colength n, and is known as the quasi-homogeneous Hilbert scheme [10, 21]. Let $P(X;z):=\sum_{i\geqslant 0}\dim H_i(X;\mathbb{Q})z^{i/2}$ denote the Poincaré polynomial of a manifold X, where H_i is the ith homology group (over $\mathbb{Q}$) and

$\dim H_i$ the ith Betti number. Proving an earlier conjecture of Buryak [9, Conjecture 1.4], Buryak and Feigin [10, Theorem 1] proved the following beautiful identity for the generating function of the Poincaré polynomial of the quasihomogeneous Hilbert scheme.

Theorem 8.5. *For α, β positive integers such that $\gcd(\alpha,\beta)=1$, let $r:=\alpha+\beta$. Then*

$$\sum_{n=0}^{\infty} P\big((\mathbb{C}^2)^{[n]}_{\alpha,\beta};z\big)T^n = \frac{(T^r;T^r)_\infty}{(T;T)_\infty(zT^r;T^r)_\infty}.$$

Subsequently, Buryak, Feigin and Nakajima [11] obtained a more general result which eliminates the need for the restriction that α and β are coprime. To this end the torus $T_{\alpha,\beta}$ is replaced by $T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}$, where

$$\Gamma_r := \big\{(\mathrm{e}^{2\pi\,\mathrm{i}\,k/r}, \mathrm{e}^{-2\pi\,\mathrm{i}\,k/r}) \in (\mathbb{C}^*)^2 : 0 \leqslant k \leqslant r-1\big\},$$

and singular homology is replaced by Borel–Moore (BM) homology.

Theorem 8.6. *For α, β nonnegative integers such that $\alpha+\beta\geqslant 1$, let $r:=\alpha+\beta$. Then*

$$\sum_{n=0}^{\infty} P^{\mathrm{BM}}\Big(\big((\mathbb{C}^2)^{[n]}_{\alpha,\beta}\big)^{\Gamma_{\alpha+\beta}};z\Big)T^n = \frac{(T^r;T^r)_\infty}{(T;T)_\infty(zT^r;T^r)_\infty},$$

where

$$P^{\mathrm{BM}}(X;z) := \sum_{i\geqslant 0} \dim H_i^{\mathrm{BM}}(X;\mathbb{Q})z^{i/2}.$$

For positive, coprime α and β, $\Gamma_{\alpha+\beta}\subset T_{\alpha+\beta}$, so that $\big((\mathbb{C}^2)^{[n]}_{\alpha,\beta}\big)^{\Gamma_{\alpha+\beta}} = (\mathbb{C}^2)^{[n]}_{\alpha,\beta}$. Positivity also implies compactness, in which case both homology theories are equivalent. Hence the second theorem contains the first as special case.

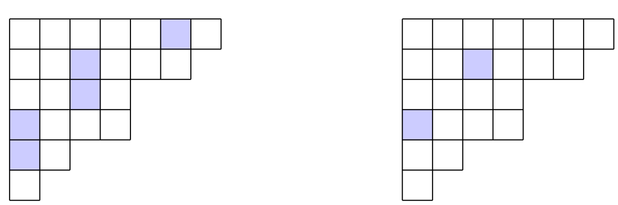
Both theorems admit a purely combinatorial description in terms of a statistic on partitions introduced by Buryak and Feigin in [10] (and refined in [11] to $\gcd(\alpha,\beta)\neq 1$). For $\alpha\geqslant 1$ and $\beta\geqslant 0$ a pair of integers, define $\mathrm{bf}_{\alpha,\beta}(\lambda)\subset\lambda$ and $\mathrm{BF}_{\alpha,\beta}(\lambda)\in\mathbb{N}_0$ by

$$\mathrm{bf}_{\alpha,\beta}(\lambda) = \big\{s\in\lambda :\ \alpha\, l(s) = \beta\, a(s)+\beta \text{ and } h(s)\equiv 0 \pmod{\alpha+\beta}\big\}$$

and

$$\mathrm{BF}_{\alpha,\beta}(\lambda) := |\mathrm{bf}_{\alpha,\beta}(\lambda)|.$$

For example, for the partition $\lambda=(7,6,4,4,2,1)$ the sets $\mathrm{bf}_{4,2}(\lambda)\subset\mathrm{bf}_{2,1}(\lambda)$ are given by coloured squares in the two diagrams below:



$\mathrm{bf}_{2,1}(\lambda)$ $\qquad\qquad$ $\mathrm{bf}_{4,2}(\lambda)$

Hence $\mathrm{BF}_{2,1}(\lambda)=5$ and $\mathrm{BF}_{4,2}(\lambda)=2$. Obviously, we further have $\mathrm{BF}_{2k,k}(\lambda)=0$ for $k\geqslant 3$.

Note that if α and β are positive coprime integers then we may drop the congruence condition on the hook-lengths of s. Indeed, $\alpha l(s)=\beta a(s)+\beta$ implies $\alpha \mid a(s)+1$ and $\beta \mid l(s)$, so that

$$h(s)=a(s)+l(s)+1=(\alpha/\beta+1)l(s)=(\alpha+\beta)\,\frac{l(s)}{\beta}\equiv 0 \pmod{\alpha+\beta}.$$

This is in accordance with $\Gamma_{\alpha+\beta}\subset T_{\alpha,\beta}$ for positive coprime α and β.

By the Bialynicki-Birula theorem [7,8], $(\mathbb{C}^2)^{[n]}_{\alpha,\beta}$ has a cellular decomposition with cells

$$C_p=\big\{z\in(\mathbb{C}^2)^{[n]}_{\alpha,\beta}:\ \lim_{t\to 0} tz=p \text{ for } t\in T_{1,\gamma}\big\},$$

where γ is sufficiently large and p is a fixed point of the $(\mathbb{C}^*)^2$ action. If p is indexed by the partition $\lambda\vdash n$, then [10]

$$\dim C_p=\mathrm{BF}_{\alpha,\beta}(\lambda).$$

This decomposition carries over mutatis mutandis to[4] $\big((\mathbb{C}^2)^{[n]}_{\alpha,\beta}\big)^{\Gamma_{\alpha+\beta}}$. Hence

$$P^{\mathrm{BM}}\Big(\big((\mathbb{C}^2)^{[n]}_{\alpha,\beta}\big)^{\Gamma_{\alpha+\beta}};z\Big)=\sum_{\lambda\vdash n} z^{\mathrm{BF}_{\alpha,\beta}(\lambda)},$$

resulting in the following partition theorem [10, Theorem 2] and [11, Corollary 1.3].

Theorem 8.7. *For integers $\alpha\geqslant 1$ and $\beta\geqslant 0$, let $r:=\alpha+\beta$. Then*

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}z^{\mathrm{BF}_{\alpha,\beta}(\lambda)}=\frac{(T^r;T^r)_\infty}{(T;T)_\infty(zT^r;T^r)_\infty}.$$

What makes this theorem difficult to prove purely combinatorially is that it is not at all clear why

$$\mathrm{BF}_{\alpha,\beta}(n)=\mathrm{BF}_{\alpha+\beta,0}(n),$$

where $\mathrm{BF}_{\alpha,\beta}(n)$ is the multiset

$$\mathrm{BF}_{\alpha,\beta}(n):=\{\mathrm{BF}_{\alpha,\beta}(\lambda):\lambda\vdash n\}.$$

In fact, if one carefully inspects the details of the proof of Theorem 8.7 in [10, Section 3] it follows that, more strongly,

$$\mathrm{BF}_{\alpha,\beta}(n;\omega)=\mathrm{BF}_{\alpha+\beta,0}(n;\omega),$$

where, for $\omega\in\mathscr{C}_r$,

$$\mathrm{BF}_{\alpha,\beta}(n;\omega):=\{\mathrm{BF}_{\alpha,\beta}(\lambda):\lambda\vdash n, (\alpha+\beta)\text{-core}(\lambda)=\omega\}.$$

For example, for all

$$(\alpha,\beta)\in\{(3,0),(2,1),(1,2)\},$$

[4]In [11] the case $\alpha=0$ is included in the statement, but we believe this to be a minor slip.

we have

$$\begin{gathered}\mathrm{BF}_{\alpha,\beta}(9;0)=\{0^{10},1^8,2^3,3\},\\ \mathrm{BF}_{\alpha,\beta}(9;(4,2))=\mathrm{BF}_{\alpha,\beta}(9;(2,2,1,1))=\{0^2,1\},\\ \mathrm{BF}_{\alpha,\beta}(9;(5,3,1))=\mathrm{BF}_{\alpha,\beta}(9;(3,2,2,1,1))=\{0\},\end{gathered}$$

and $\mathrm{BF}_{\alpha,\beta}(9;\omega)=\emptyset$ for all other $\omega\in\mathscr{C}_3$, accounting for the 30 partitions of 9. We thus have a slightly stronger result.

Proposition 8.8. *For integers $\alpha\geqslant 1$ and $\beta\geqslant 0$, let $r:=\alpha+\beta$. Then*

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r} z^{\mathrm{BF}_{\alpha,\beta}(\lambda)}=\frac{1}{(zT;T)_\infty (T;T)_\infty^{r-1}} \tag{8.6a}$$

for $\omega\in\mathscr{C}_r$, and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}z^{\mathrm{BF}_{\alpha,\beta}(\lambda)}=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (zST^r;ST^r)_\infty (ST^r;ST^r)_\infty^{r-1}}. \tag{8.6b}$$

Here (8.6b) follows from (8.6a) by the exact same reasoning which shows that (6.2) and (6.3) are equivalent, see the proof on page 14.

Since $\mathrm{bf}_{r,0}(\lambda)$ is exactly the set of bottom squares of λ whose hook-lengths are congruent to 0 modulo r, it follows that

$$\mathrm{BF}_{r,0}(\lambda)=|\mathscr{H}_r^{(\mathrm{b})}(\lambda)|.$$

The pair of identities (8.6) are thus a generalisation of (7.1). Moreover, Theorem 6.1 may be restated as the claim that, if

$$g_r(T):=\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}\prod_{s\in \mathrm{bf}_{1,0}(\lambda)}\rho\big(rh(s)\big),$$

then

$$\sum_{\substack{\lambda\in\mathscr{P}\\ r\text{-core}(\lambda)=\omega}} T^{(|\lambda|-|\omega|)/r}\prod_{s\in \mathrm{bf}_{r,0}(\lambda)}\rho\big(h(s)\big)=\frac{g_r(T)}{(T;T)_\infty^{r-1}}$$

and

$$\sum_{\lambda\in\mathscr{P}} T^{|\lambda|}S^{|\mathscr{H}_r(\lambda)|}\prod_{s\in \mathrm{bf}_{r,0}(\lambda)}\rho\big(h(s)\big)=\frac{(T^r;T^r)_\infty^r}{(T;T)_\infty (ST^r;ST^r)_\infty^{r-1}}\, g_r(ST^r),$$

for r a positive integer and ω an r-core.

Problem 8.9. *Extend the above to all* $\mathrm{bf}_{\alpha,\beta}(\lambda)$.

If such an extension exists, it is clear that $h(s)$ (which for $s\in\mathrm{bf}_{r,0}(\lambda)$ is equal to $a(s)+1$) should be replaced by a more complicated statistic on the squares of λ.

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